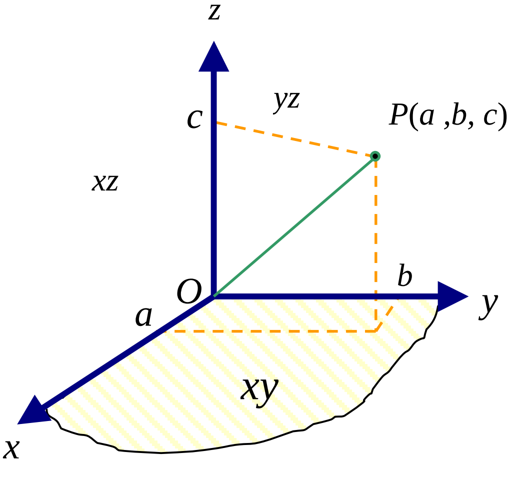
# Chapter 2

# Three Dimensional Space

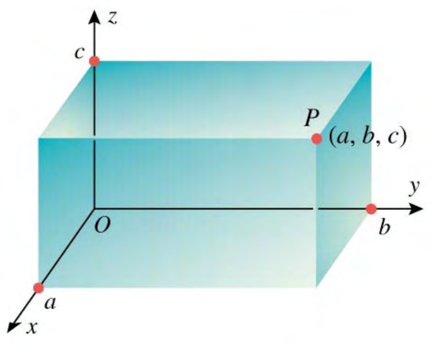
## 2.1. Rectangular Coordinates

The three dimensional spaces is a generalization of the two-dimensional space covered in Math I. Much of the concepts in Math II are generalizations of the material of Math I. Each point $P$ in the three-dimensional space is associated to an unique ordered triple $P(a, b, c)$; $a, b,$ and $c$ are the coordinates of the point: $a$ is the *x-coordinate; b* is the *y-coordinate;* and $c$ is the *z-coordinate.*




We have three coordinate axis, the *x-axis*, the *y-axis*, and the *z-axis* intercepting at the origin, the point *O*. Because two intercepting lines in the three-dimensional space define a plane, the three pairs of coordinate axis generate three coordinate planes, the *xy-plane*, the *xz-plane*, and the *yz-plane*.

The three-dimensional space is divided by the three coordinate planes into eight regions, called octants. The octant you see in front in these graphs is the first octant, on which the orientation of the coordinate axis $x \to y$, $y \to z$, and $z \to x$ occurs in the counter clock orientation.



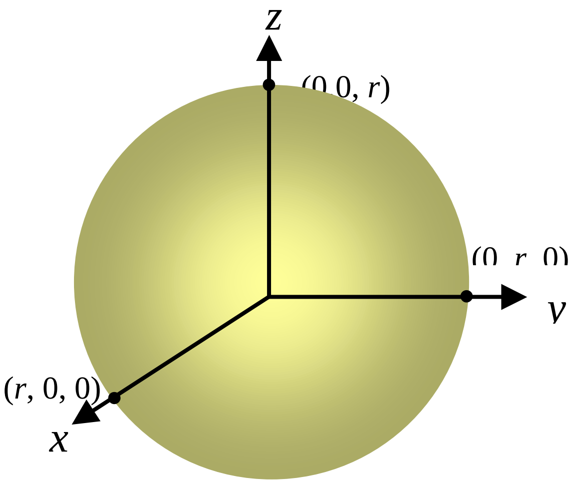

By the same token, higher order dimensional spaces can also be defined, and are subject to much study in physics and other sciences. A point $P$ in a four-dimensional space, for instance, is given by a quadruple $(x, y, z, w)$. In general, an $n$-dimensional space consists of $n$-tuples of the form $(x_1, x_2, \dots x_n)$, but points in more than three dimensions cannot be represented graphically, since we live in a three-dimensional space. The need for further dimensions arises in problems in which other variables, such as time $t$, play a role in the equations. The two-dimensional spaces is denoted $R^2$, the three-dimensional space is denoted $R^3$, and the $n$-dimensional spaces is denoted $R^n$.

**2.1.1 Distance between two points**. A fundamental equation is the distance between two points, which is the basis of all future analysis. Given two points, $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the distance between $P_1$ and $P_2$ is defined by

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Straightforward use of the Pythagorean theorem can show that this definition corresponds to the intuitive notion of distance.

**2.1.2 Spherical surfaces.** Next we find the equation of some well-known surfaces. We start by the sphere.




**Example**. Find the equation of the sphere of center at the origin $O$ and radius $r$.

Because the sphere is, by definition, the set of all points whose distance to the center is equal to $r$, applying the formula above to a generic point $P$ of coordinates $(x, y, z)$, we have

$$d(P,O) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = r$$

Simplifying and squaring both sides of the equation we obtain the equation

$$\boxed{x^2 + y^2 + z^2 = r^2}$$

**Example**. Find the equation of the sphere of center $M(x_0, y_0, z_0)$ and radius $r$.

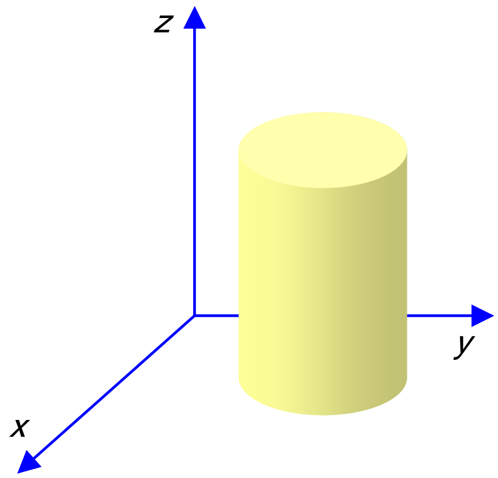
By a similar procedure, replacing in the previous example the coordinates of the origin by the coordinates of $M$, we readily obtain

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

In general, any equation of the form

$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$ is a sphere of center $M(-A/2, -B/2, -C/2)$ and radius

$$r = \frac{\sqrt{A^2 + B^2 + C^2 - 4D}}{2}$$




provided that the radicand is positive. If the radicand is zero, the sphere is reduced to a point, and if it is negative, the equation has no solution.

**2.1.3. Cylindrical surfaces**. An equation of he form

$$(x - a)^2 + (y - b)^2 = r^2$$

produces a vertical cylindrical surface, the intersection with the *xy* plane being a circle with center at $(a,b)$ and radius $r$.

The fact that the variable $z$ is absent from the equation means that the equation is satisfied for any value of $z$, and therefore it is a vertical cylinder. Any equation of the form $f(x, y) = 0$ in the three-dimensional space represents a vertical cylindrical surface. In general, any equation in the three-dimensional space that contains only two of the variables $x$, $y$, and $z$, represents a cylindrical surface in the direction of the missing coordinate.

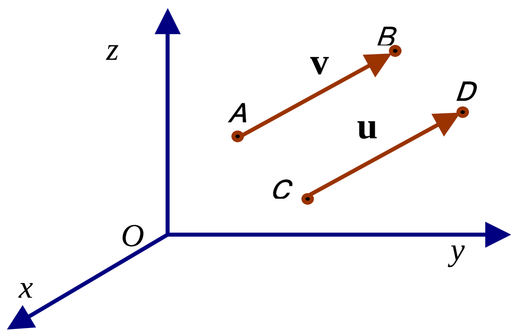
## HOMEWORK (Section 2.1)

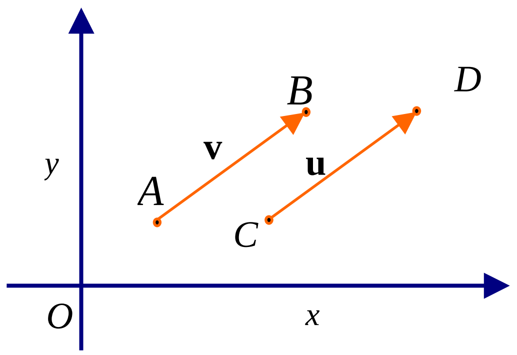
Suggested problems to solve from the book. Exercise set 12.1 (p. 794), problems 1-28.

# 2.2 Vectors

Vector spaces are studied in detail in Math III. In this course we will cover only a particular case of vectors in two and three dimensions.

**Definition**. A Vector Space contains ***vectors*** and ***scalars***. The latter for us will be real numbers. A vector in this course will be an oriented segment in the two-dimensional space or in the three-dimensional space. They can be represented by arrows or by two points *AB*, the first being the initial point and the second the end point.




Given two points, $A$ and $B$, the vector $\vec{AB}$ is the oriented segment from $A$ to $B$. $A$ is the ***initial point*** and $B$ is the ***terminal point.*** Thus $\vec{AB}$ and $\vec{BA}$ are different vectors. In fact, $\vec{AB} = -\vec{BA}$ A vector has two characteristics, direction and length, the latter is called the ***norm*** of the vector, and denoted by $||\mathbf{v}||$. Two vectors are equal if they have same length and direction. Thus the vectors $u = \vec{AB}$ and $v = \vec{CD}$ in the graph are equal. The zero vector is a vector of norm 0. If $A=B$ then $\vec{AB} = 0$. We have

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

Vectors play an important role in physics. Namely, a force can be conveniently represented by a vector. A Force has three characteristics as vectors do; direction, strength (the norm of the vector) and point of application (the initial point of the vector).

**2.2.1 Representation of vectors.** A vector in $R^2$ is given by a pair of coordinates, $\mathbf{v} = (v_1, v_2)$, and in $R^3$, by a triple $\mathbf{v} = (v_1, v_2, v_3)$ (keep in mind that any other arrow with same direction and length as $\mathbf{v}$ is the same vector). We use bold face letters to denote vectors, while normal letters denote scalars (real numbers).

**2.2.2 Addition of Vectors.** Is defined as follows. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in either $R^2$ or $R^3$,

$$\text{In } R^2 \quad \mathbf{v} = (v_1, v_2),\ \ \mathbf{w} = (w_1, w_2)$$

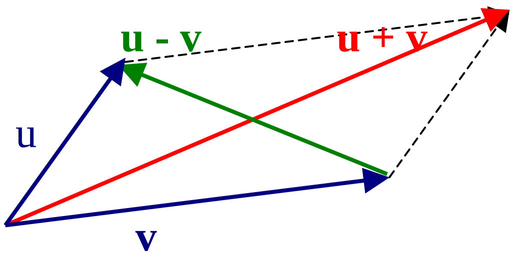
$$\text{In } R^3 \quad \mathbf{v} = (v_1,\ v_2,\ v_3),\ \ \mathbf{w} = (w_1,\ w_2, v_3),$$

Then

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2) \quad \text{or} \quad \mathbf{v} + \mathbf{w} = (v_1 + w_1,\ v_2 + w_2,\ v_3 + w_3)$$

The norm of a vector is given by

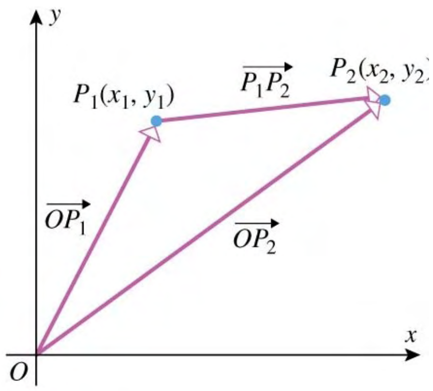
$$\|v\| = \sqrt{v_1^2 + v_2^2} \quad \text{or} \quad \|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$




By the parallelogram rule the sum and difference of two vectors is represented by the diagonals of the parallelogram they form.

Triangular inequality.

$\| \mathbf{u} + \mathbf{v} \| \le \| \mathbf{u} \| + \| \mathbf{v} \|$, and $\| \mathbf{u} + \mathbf{v} \| = \| \mathbf{u} \| + \| \mathbf{v} \|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ have same direction

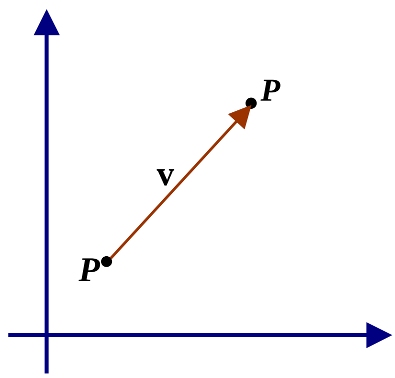



A vector can also be represented by two points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points. To find the coordinates $(v_1, v_2)$ of the vector $\vec{P_1P_2}$ we proceed as follows

$$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = (x_1, y_1) - (x_2, y_2) = (x_2 - x_1, y_2 - y_1)$$

Similarly, in three dimensions,

$$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$



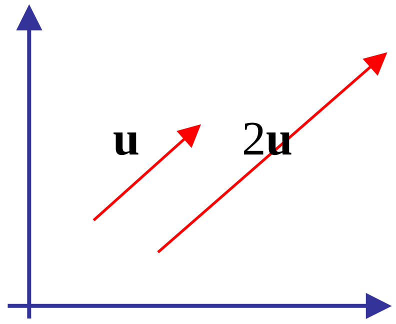

We can make use of a very convenient definition, the difference of two points as a vector.

$\mathbf{v} = \vec{P_1P_2} = P_2 - P_1$ and therefore $P_2 = P_1 + \mathbf{v}$ defines the addition of a vector plus a point. We will make use of this convention later.

**2.2.3 Scalar multiplication of vectors.** There are several types of multiplications of vectors. The following is the scalar multiplication, by which a vector **v** is multiplied by an scalar (a number) *k*.

**Definition**. Given a vector $\mathbf{v} = (v_1, v_2, v_3)$ and a scalar *k*, the scalar multiplication $k\mathbf{v}$ is defined as

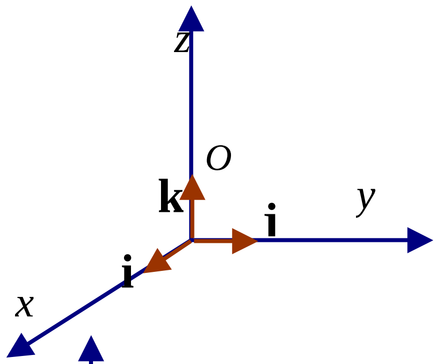
$$k\mathbf{v} = (kv_1, kv_2, kv_3)$$




Multiplying a vector **v** by a scalar *k* produces a vector with same direction as **v** and norm( length) *k* times the length of **v**. Hence,

$$\| k\,\mathbf{v} \| = |k|\,\| \mathbf{v} \|$$

A vector of norm 1 is called **unit vector**. Given a vector **v**, the vector $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector with the same direction as **v**.




The three unit vectors along the three axis, *x*, *y*, and *z* play an important role and are represented by **i, j,** and **k** respectively. Their coordinates are **i** = (1, 0, 0), **j** = (0, 1, 0), **k** = (0, 0, 1)

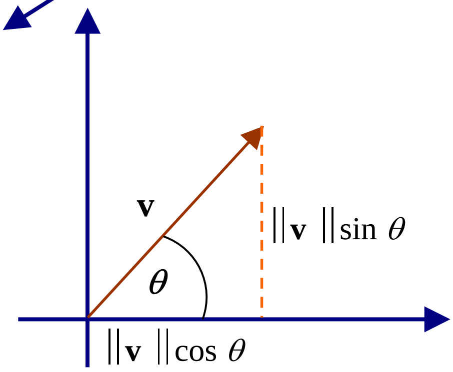
The sum and scalar multiplication of vectors satisfy most of the usual properties of the operations, such as commutative, associative, distributive, etc. For a list of all the properties see Theorem 12.2.6 of the book (p. 800).

A vector can also be represented by its length and the angle it forms with the *x*-axis. If the angle between **v** and the *x*-axis is $\theta$, then we can write

$$\mathbf{v} = \|\mathbf{v}\|(\cos\theta, \sin\theta)$$

or

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}$$




# HOMEWORK (Section 2.2)

Suggested problems to solve from the book. Exercise set 12.2 (p. 805), problems 1-30.