

Chapter 1

Polar Coordinates

1.1 Definition

Polar coordinates is an alternative way to represent points in the plane. They consist of a fixed point O , called **pole** (or **origin**), and a ray, called **polar axis**. Each point of the plane is associated to a pair of coordinates (r, θ) . The number r , called radial coordinate, is the distance from P to O ; θ is the angle between the line OP and the axis, and is called the **angular coordinate**. If $r = 0$ (any angular coordinate) then it corresponds to the origin O .

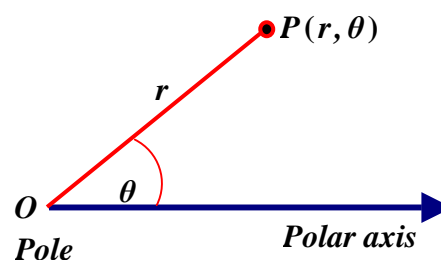


Figure 1

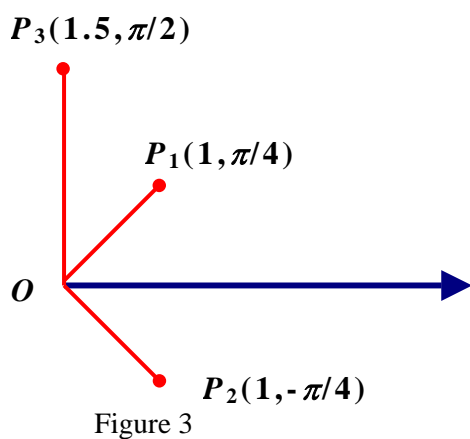


Figure 3

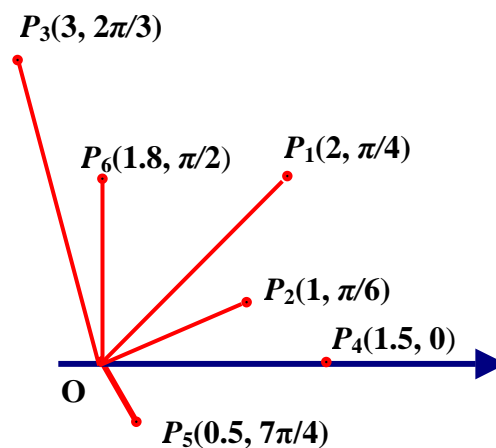
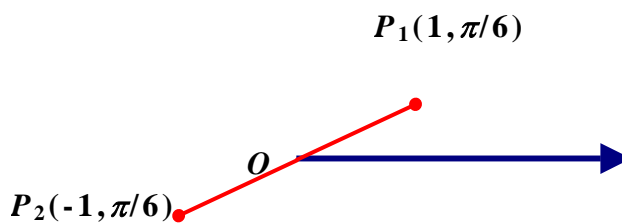


Figure 2

Figure 2 shows the representation of several points in polar coordinates. Unlike rectangular coordinates, the same point can be represented in several different ways in polar coordinates. For instance, $P(r, \theta) = P(r, \theta + 2n\pi)$. Positive values of θ are measured counterclockwise. Negative values of θ run clockwise. Thus the point P_3 on Figure 3 can alternatively be represented as $P_3 = (1.5, \pi/2)$ or $P_3 = (1.5, -3\pi/2)$.

Negative values of r are possible. Multiplying the value of r by -1 produces a symmetry with respect to the origin or a rotation of π radians. The point P_2 in the figure on the right can be represented by $P_2(-1, \pi/6)$ and also as $P_2(1, 7\pi/6)$



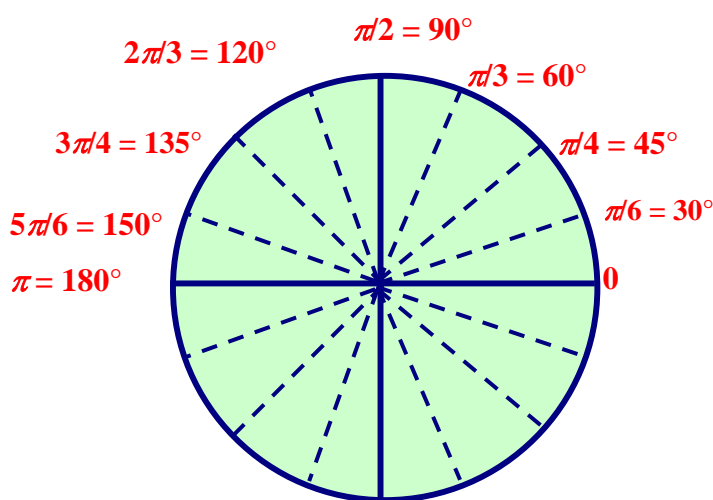
1.1.1. Relationship between polar and rectangular coordinates. Sometimes we need to change coordinates, from polar to rectangular and vice-versa. The following relationships hold, which can be easily deduced from trigonometric equations.

Changing from polar to rectangular $x = r \cos \theta, y = r \sin \theta$

Changing from rectangular to polar $r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}$

Throughout this course, angles will be measured in radians. Some conversions are shown below.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	n.d.
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$
π	0	-1	0



Examples. Converting from polar to rectangular coordinates and vice versa.

$$(r, \theta) = (1.5, \pi/6) \Rightarrow \begin{cases} x = (1.5) \cos(\pi/6) = (1.5) \frac{\sqrt{3}}{2} \approx 1.299 \\ y = (1.5) \sin(\pi/6) = (1.5) \frac{1}{2} = 0.75 \end{cases}$$

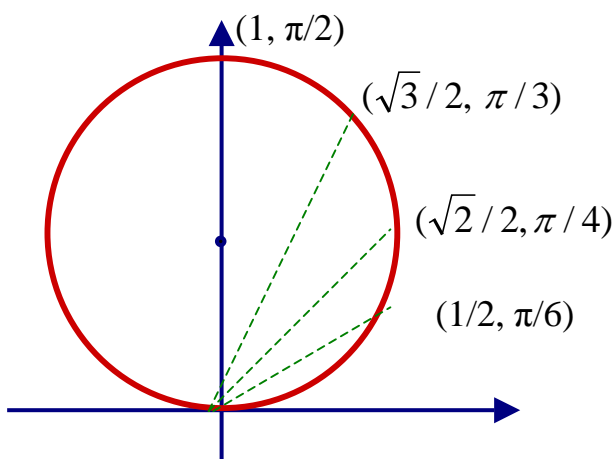
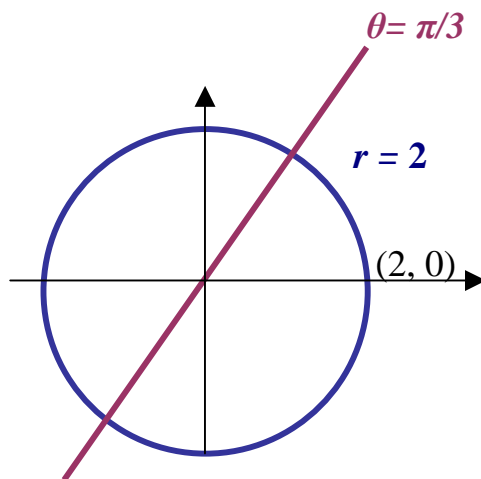
$$(x, y) = (1.7, 2.3) \Rightarrow \begin{cases} r = \sqrt{(1.7)^2 + (2.3)^2} \approx 2.86 \\ \theta = \arctan(2.3/1.7) \approx 0.93 \end{cases}$$

1.1.2. Graphs in Polar Coordinates. A function in polar coordinates takes the form $r = f(\theta)$. To each value of θ in the domain it corresponds a point $(r(\theta), \theta)$.

Examples.

$r = 2$ (circle with center at O and radius $r = 2$)

$\theta = \pi/3$ (line through the origin with slope $\pi/3$)

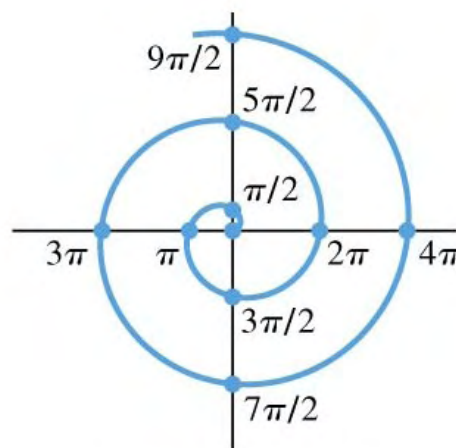


$r = \sin \theta$ (circle)

$$x^2 + y^2 = r^2 = r \times r = r \sin \theta = y$$

$$x^2 + y^2 - y = 0$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

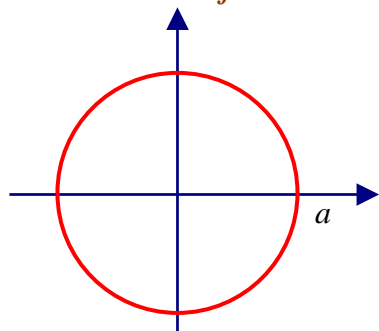


$r = \theta$ ($\theta \geq 0$)

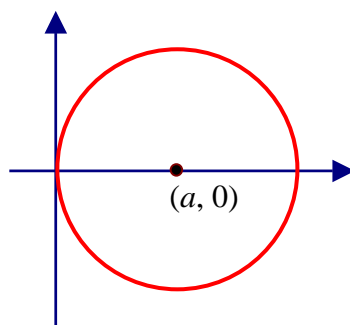
Spiral

There are some families of well-known curves. You must learn the following ones:

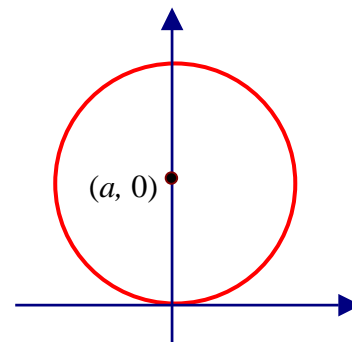
1.1.3 Families of circles. We will cover three families of circles:



(a) $r = a$. Are circles with center at the origin and radius 1.



(b) $r = 2a \cos \theta$. Are circles with center at $(a, 0)$ and radius a

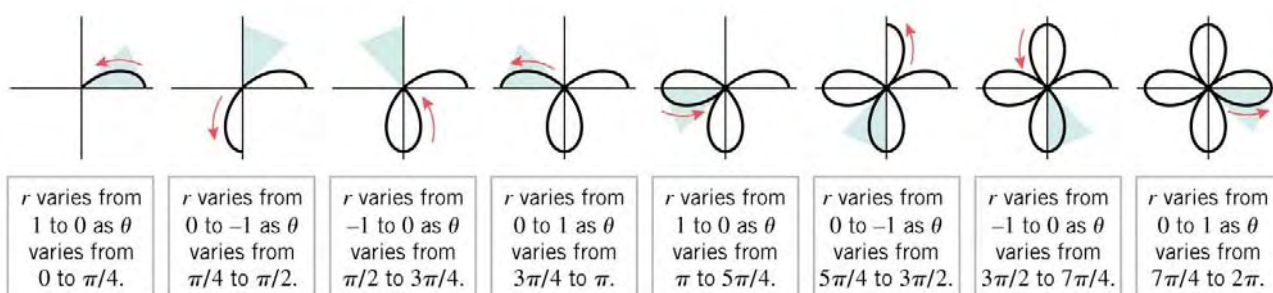


(c) $r = 2a \sin \theta$. Are circles with center at $(0, a)$ and radius a

1.1.4 Families of Rose Curves. They are of the form $r = a \sin n\theta$ and $r = a \cos n\theta$. Their shape varies according to the values of a and n .

Example. Sketch the graph of $r = \cos 2\theta$.

We proceed analyzing gradually the behavior of the functions as θ increase from 0 to 2π . The result is the figure below, a four-petal rose.



This is just one example of a larger family of flower-shaped curves called roses. They are of the form $r = a \sin n\theta$ and $r = a \cos n\theta$, with $a > 0$ and n a positive integer. Different combinations of values of a and n produce a various curves, all from the family of rose curves. The next figure illustrates several curves of the family. The values of a result in the size of the petal, whose length is exactly a . There is a pattern for the values of n : even values of n result in roses with $2n$ petals, while odd values of n produce roses with n petals. The difference between the cosine roses and the sine roses are just a rotation of $\pi/2n$.

		ROSE CURVES				
		$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$r = a \sin n\theta$						
$r = a \cos n\theta$						

You may be asked in the exams to recognize some of these curves in the exams, but not to draw graphs of them.

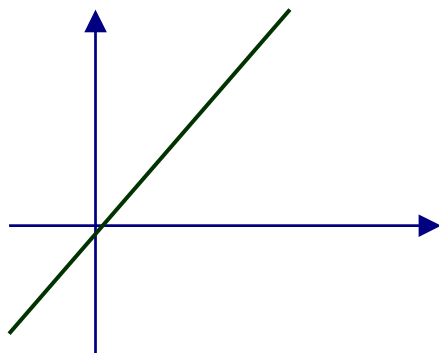
HOMEWORK (Section 1.1)

Suggested problems to solve from the book. Exercise set 11.1 (p. 732), problems 1-20. You can always verify your answer of the odd numbered problems.

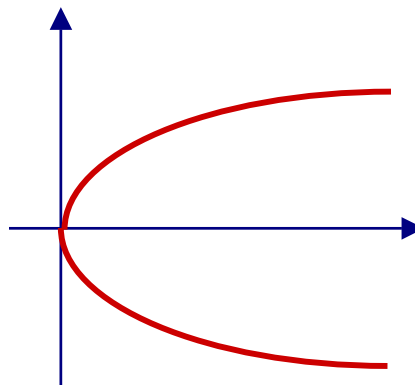
1.2 Parametric Functions

Functions of the form $x = f(t)$, $y = g(t)$, where f and g are functions of an **auxiliary** variable t , are called parametric functions. They are very useful in many applications where an explicit form does not exist.

Examples.



(a) Line through the origin
 $x = 2t$, $y = 3t$, $\Rightarrow 3x = 2y$



(b) Parabola
 $x = t^2$, $y = t$, $\Rightarrow y = t$

In these two examples, an explicit form exists and it is easy to find. Other more complicated cases are possible, where finding an explicit form may be cumbersome or even impossible. Some parametric functions may require the plotting of points one by one. The following is an example

$$\begin{cases} x = 3t^2 + \sqrt{t} + 1 \\ y = 2\sin t - t \end{cases}$$

No simple explicit form exists, so we must work with tables of values

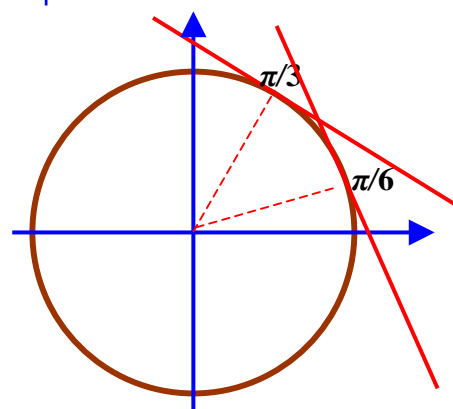
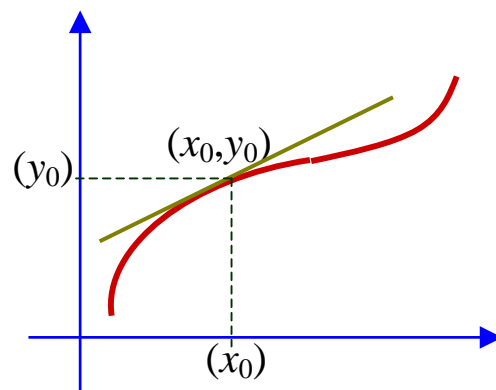
1.2.1 Tangent Lines to Parametric Functions. Given a function in parametric form $x = x(t)$, $y = y(t)$, we want to calculate the slope of the tangent line to the curve at some point $(x_0, y_0) = (x(t_0), y(t_0))$. For this, we need to find dy/dx . Assume that the curve is *smooth*, which means that both $x(t)$ and $y(t)$ have first order continuous derivatives. Then y is a differentiable function of x and we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Example. The circle of radius 1 centered at the origin.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow x^2 + y^2 = 1$$

t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\pi^2}{12} + \sqrt{\frac{\pi}{6}} + 1$	$1 - \frac{\pi}{6}$
...	(other values)	...



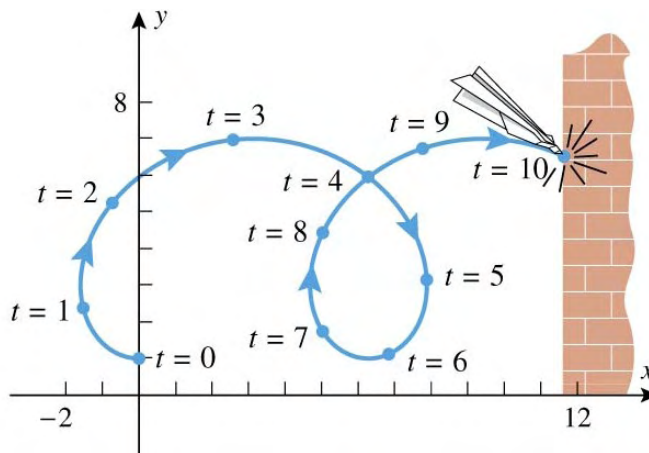
Its derivative is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{\cos t}{\sin t}$

at $t = \pi/3$, $\frac{dy}{dx} = -\frac{\cos(\pi/3)}{\sin(\pi/3)} = -\frac{1}{\sqrt{3}}$ and at $t = \pi/6$, $\frac{dy}{dx} = -\frac{\cos(\pi/6)}{\sin(\pi/6)} = -\sqrt{3}$

Example. Paper airplane (example 2, p. 736 from the book). The trajectory of a paper airplane is

$$\begin{cases} x = t - 3\sin t \\ y = 4 - 3\cos t \end{cases} \quad 0 \leq t \leq 10$$

- (a) At what times was the airplane flying horizontally?
- (b) At what times was it flying vertically?



Differentiating, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin t}{1 - 3\cos t}$

Therefore,

(a) $\frac{dy}{dx} = 0 \Rightarrow 3\sin t = 0 \Rightarrow t = 0, t = \pi, t = 2\pi, t = 3\pi$

(b) $\frac{dy}{dx} = \infty \Rightarrow 1 - 3\cos t = 0 \Rightarrow \cos t = 1/3 \Rightarrow \begin{cases} t = \arccos(1/3) \\ t = 2\pi - \arccos(1/3) \\ t = 2\pi + \arccos(1/3) \end{cases}$

1.2.2 Tangent Lines to Polar Curves. Suppose now that the function is given in polar coordinates, $r = r(\theta)$ where r is a differentiable function of θ . Applying the equations for change of coordinates

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{we get} \quad x = r(\theta) \cos \theta, \quad y = r(\theta) \sin \theta$$

differentiating

$$\frac{dx}{d\theta} = -r(\theta) \sin \theta + r'(\theta) \cos \theta = -r \sin \theta + \frac{dr}{d\theta} \cos \theta$$

$$\frac{dy}{d\theta} = r(\theta) \cos \theta + r'(\theta) \sin \theta = r \cos(\theta) + \frac{dr}{d\theta} \sin \theta$$

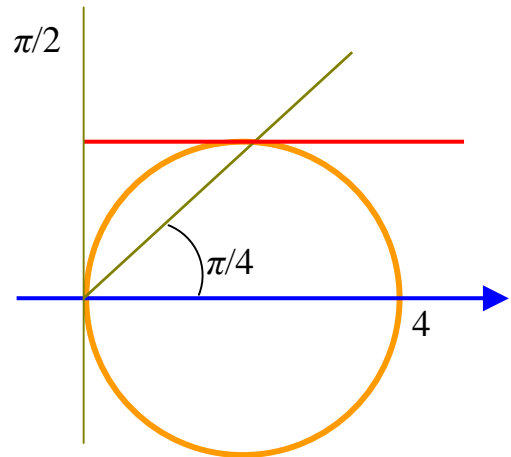
It follows that $\frac{dr}{d\theta} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}$

Example. Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Differentiating

$$\begin{aligned}\frac{dy}{dx} &= \frac{(4\cos\theta)\cos\theta + \sin\theta(-4\sin\theta)}{(-4\cos\theta)\sin\theta + \cos\theta(-4\sin\theta)} \\ &= \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = \frac{4\cos 2\theta}{-4\sin 2\theta}\end{aligned}$$

$$\text{If } \theta = \pi/4, \frac{dy}{dx} = \frac{4\cos\pi/2}{-4\sin\pi/2} = 0$$



HOMWORK (Section 1.2)

Suggested problems to solve from the book. Exercise set 11.2 (p. 740), problems 1-14 and 21-26. Find only the first derivative dy/dx , but not the second derivative, d^2/dx^2 .