

SIIT MAS117 - SUMMER 2007 - MIDTERM EXAM - Dr. Ruben

Instructions

1. Examination contains 12 multiple choice questions and all are compulsory
2. Calculator, textbooks and lecture notes are not allowed in the examination room.
3. Circle the correct answer. Showing work is not required. Each correct answer is 8 points; each incorrect answer is -2 points. A double mark in one question invalidates the answer.

Answers

Problem 1	(a)	(b)	(c)	(d)	(e)
Problem 2	(a)	(b)	(c)	(d)	(e)
Problem 3	(a)	(b)	(c)	(d)	(e)
Problem 4	(a)	(b)	(c)	(d)	(e)
Problem 5	(a)	(b)	(c)	(d)	(e)
Problem 6	(a)	(b)	(c)	(d)	(e)
Problem 7	(a)	(b)	(c)	(d)	(e)
Problem 8	(a)	(b)	(c)	(d)	(e)
Problem 9	(a)	(b)	(c)	(d)	(e)
Problem 10	(a)	(b)	(c)	(d)	(e)
Problem 11	(a)	(b)	(c)	(d)	(e)
Problem 12	(a)	(b)	(c)	(d)	(e)

Problem 1. Given the parametric equations $x(t) = t + \cos t$; $y(t) = t + \sin t$, the slope of the tangent line to this curve at $t = \pi/6$ is

- (a) $\frac{2\pi + \sqrt{3}}{9}$; (b) $\frac{\sqrt{3} - 1}{2}$; (c) $\frac{1 + 3\pi}{1 - 2\pi}$; (d) $2 + \sqrt{3}$; (e) -1

The correct answer is (d)

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{1 + \cos t}{1 - \sin t} \quad \text{at } t = \pi/6, \quad \frac{dy}{dx} = \frac{1 + \cos(\pi/6)}{1 - \sin(\pi/6)} = \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{2 + \sqrt{3}}{\frac{1}{2}} = 2 + \sqrt{3}$$

Problem 2. The angle between the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is

- (a) $5\pi/6$, (b) $2\pi/3$; (c) 0; (d) $\pi/2$; (e) π

The correct answer is (d)

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1 \times 1 + 1 \times 1 + 1 \times (-2)}{\sqrt{3} \sqrt{6}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Problem 3. The point P_0 of intersection of the plane $2x + y - z = 2$ with line passing through the point $P(1, 1, -3)$ parallel to the vector $\mathbf{u}: (1, -1, 0)$ is

- (a) $(-1, 0, -3)$ (b) $(-3, 5, -3)$ (c) $(1, 1, 1)$ (d) $(1, -1, 0)$ (e) $(5, 3, -3)$

The correct answer is (b)

The parametric equations of the line are

$$\begin{aligned} x &= 1 + t \\ y &= 1 - t \\ z &= -3 \end{aligned}$$

Substituting these values of x , y , and z in the equation of the plane

$2(1 + t) + 1 - t + 3 = 2$, that is, $t = -4$. Substituting this value of t on the equations of x , y , and z , we get $(-3, 5, -3)$

Problem 4. The equation of the plane passing through the point $P_0(1, 3, -1)$ normal to the vector $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is

- (a) $3x - 2y + z = -4$ (b) $x + 3y - z = 0$ (c) $3x^2 - 2y + z^2 = -4$ (d) $3x - 4y + 2z = 1$ (e) $x - y + z = -1$

The correct answer is (a)

Direct application of the formula given on page 18 yields

$$3(x - 1) - 2(y - 3) + (z + 1) = 0.$$

Simplifying, we get

$$3x - 2y + z = -4$$

Problem 5. Evaluate $\lim_{x \rightarrow (2,1)} \frac{x^2 - 2xy + 2x - 4y}{x^2 - 4y^2}$

- (a) ∞ (b) 5 (c) 1 (d) does not exist (e) $\frac{0}{0}$

The correct answer is (c)

Factoring and simplifying

$$\lim_{x \rightarrow (2,1)} \frac{x^2 - 2xy + 2x - 4y}{x^2 - 4y^2} = \lim_{x \rightarrow (2,1)} \frac{x(x - 2y) + 2(x - 2y)}{(x + 2y)(x - 2y)} = \lim_{x \rightarrow (2,1)} \frac{(x + 2)(x - 2y)}{(x + 2y)(x - 2y)} =$$

$$\lim_{x \rightarrow (2,1)} \frac{x + 2}{x + 2y} = \frac{4}{4} = 1$$

Problem 6. Evaluate $\lim_{x \rightarrow (4,12)} \frac{3x - y}{\sqrt{4x} - \sqrt{x + y}}$

- a) ∞ (b) 2 (c) $\frac{0}{0}$ (d) does not exist (e) 8

The correct answer is (e)

Multiplying by the conjugate

$$\begin{aligned}\lim_{x \rightarrow (4,12)} \frac{3x-y}{\sqrt{4x}-\sqrt{x+y}} &= \lim_{x \rightarrow (4,12)} \frac{(3x-y)(\sqrt{4x}+\sqrt{x+y})}{(\sqrt{4x}-\sqrt{x+y})(\sqrt{4x}+\sqrt{x+y})} = \\ \lim_{x \rightarrow (4,12)} \frac{(3x-y)(\sqrt{4x}+\sqrt{x+y})}{(4x-(x+y))} &= \lim_{x \rightarrow (4,12)} \frac{(3x-y)(\sqrt{4x}+\sqrt{x+y})}{3x-y} = \\ \lim_{x \rightarrow (4,12)} \sqrt{4x}+\sqrt{x+y} &= 8\end{aligned}$$

Problem 7. The gradient of $f(x, y) = x \sin(x + y^2)$ at the point $(-1, -1)$ is

- (a) $\nabla f(-1, -1) = \mathbf{i} + \mathbf{j}$ (b) $\nabla f(-1, -1) = -\mathbf{i} + 2\mathbf{j}$ (c) $\nabla f(-1, -1) = 2\mathbf{i} - \mathbf{j}$
(d) $\nabla f(-1, -1) = 2\mathbf{i} - 2\mathbf{j}$ (e) $\nabla f(-1, -1) = \mathbf{i} + 5\mathbf{j}$

The correct answer is (b)

$$\begin{aligned}f_x &= \sin(x+y^2) + x \cos(x+y^2), & f_x(-1, -1) &= -1 \\ f_y &= 2xy \cos(x+y^2), & f_y(-1, -1) &= 2\end{aligned}$$

Direct application of the formula gives $\nabla f(-1, -1) = -\mathbf{i} + 2\mathbf{j}$

Problem 8. The directional derivative of $f(x, y) = \ln(1+x+y)$, in the direction of $\mathbf{u} = \mathbf{i} + \mathbf{j}$ at the point $(1, 0)$ is

- (a) $2\mathbf{i} + \mathbf{j}$ (b) $\ln(1+\mathbf{i})$ (c) $1/\sqrt{2}$ (d) -7 (e) 0

The correct answer is (c)

$$f_x = \frac{1}{1+x+y}, \quad f_x(1, 0) = \frac{1}{2} \quad f_y = \frac{1}{1+x+y}, \quad f_y(1, 0) = \frac{1}{2}$$

The unit vector in the direction of \mathbf{u} is $\mathbf{v} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. Applying the formula of the directional derivative,

$$D_{\mathbf{u}}f(1, 0) = f_x(1, 0)v_1 + f_y(1, 0)v_2 = \frac{1}{2} \frac{1}{\sqrt{2}} + \frac{1}{2} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Problem 9. The critical points of $f(x, y) = x^3 - 3xy + 3y^2 + 17$ are

- (a) $(3, -3), (1, 0)$ (b) $(-1, 1), (1, 2)$ (c) no critical points (d) $(0, 0), (1/2, 1/4)$
(e) $(5, 3), (1, -2)$

The correct answer is (d)

$$f_x = 3x^2 - 3y = 0 \quad f_y = -3x + 6y = 0 \quad \Rightarrow \quad x = 2y$$

Substituting in the first equation, $3(2y)^2 - 3y = 0 \Rightarrow 3y(4y - 1) = 0$

$y = 0 \Rightarrow x = 0, \Rightarrow (0, 0)$ is a c.p

$y = 1/4 \Rightarrow x = 1/2 \Rightarrow (1/2, 1/4)$ is a c.p

Problem 10. Find relative extrema and saddle points of $f(x, y) = 3x^2 + 3xy + y^3 - 4$

(a) rel. max. at (0,0) and rel. min. at (1, -1) (b) rel. min. at (-1/4, 1/2) and saddle pt. at (0,0)

(c) saddle point at (0,0) (d) rel. max. at (1, -1) (e) rel. min. at (3, -2)

The correct answer is (b)

$$f_x = 6x + 3y = 0 \Rightarrow y = -2x$$

$$f_y = 3x + 3y^2 = 0$$

Substituting in the second equation, $3x + 3(-2x)^2 = 0 \Rightarrow x(4x + 1) = 0$

$x = 0 \Rightarrow y = 0, \Rightarrow (0, 0)$ is a c.p

$x = -1/4 \Rightarrow y = 1/2 \Rightarrow (-1/4, 1/2)$ is a c.p

$$f_{xx} = 6; f_{yy} = 6y; f_{xy} = 3$$

$D(0, 0) = (6)(0) - 9 < 0 \Rightarrow$ saddle point

$D(-1/4, 1/2) = (6)(3) - 9 > 0$, and $f_{xx} > 0 \Rightarrow$ relative minimum.

Problem 11. Find absolute extrema of $f(x, y) = xy - x - 3y$ in the triangle with vertices (0, 0), (0, 4), (5, 0)

(a) Min. = 1 Max. = 3 (b) Min. = -12 Max. = 0 (c) Min. = -1 Max. = 2 (d) Min. = 0 Max. = 8

(e) Min. = 11 Max. = 27

The correct answer is (b)

$$f_x = y - 1 \Rightarrow y = 1$$

$$f_y = x - 3 \Rightarrow x = 3 \text{ crit. point} = (3, 1)$$

along $y = 0, 0 \leq x \leq 5, f(x, 0) = -x, f' = -1$ therefore, no c.p

along $x = 0, 0 \leq y \leq 4, f(0, y) = -3y, f' = -3$ therefore, no c.p

along the hypotenuse $y = -(4/5)x + 4, f(x, y) = x(-(4/5)x + 4) - x - 3(-(4/5)x + 4),$

$$f(x, y) = (-4/5)x^2 + (32/5)x - 12, f'(x) = (-8/5)x + 32/5 = 0 \Rightarrow x = 4, y = 4/5$$

including the vertices, we have the following points

	(0, 0)	(0, 4)	(5, 0)	(3, 1)	(4, 4/5)
$f(x, y)$	0	-12	-5	-3	-16/5

Minimum = -12, Maximum = 0

Problem 12. Find the constraint maximum and constraint minimum of $f(x, y) = \underline{xy}$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 4$

- (a) Min = 1 & Max = 2 (b) Min = $-\sqrt{2}$ & Max = $\sqrt{2}$ (c) Min = -11 & Max = -2
 (d) Min = -1 & Max = 12 (e) Min = 1 & no maximum

The correct answer is (b)

$$\begin{aligned} f_x = \lambda g_x &\Rightarrow y = \lambda(2x) \\ f_y = \lambda g_y &\Rightarrow x = \lambda(4y) \\ g(x, y) &\Rightarrow x^2 + 2y^2 = 4 \end{aligned}$$

substituting $y = 2\lambda x$ in the 2nd equation $x = 4\lambda(2\lambda x) \Rightarrow x = 8\lambda^2 x \Rightarrow x(1 - 8\lambda^2) = 0 \Rightarrow x = 0$ & $\lambda = \pm 1/\sqrt{8}$

If $x = 0$, then $y = 0$ but this contradicts the third equation, so no solution.

if $\lambda = 1/\sqrt{8}$ then $y = x/\sqrt{2} \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \pm 1 \Rightarrow (\sqrt{2}, 1), (-\sqrt{2}, -1)$

$\lambda = -1/\sqrt{8}$ then $y = -x/\sqrt{2} \Rightarrow x = \pm 1/\sqrt{2} \Rightarrow y = \pm 1 \Rightarrow (-\sqrt{2}, 1), (\sqrt{2}, -1)$

All the solutions of the system, there fore, are $(0, 1)$, $(1/\sqrt{2}, 0)$, and $(-1/\sqrt{2}, 0)$

$f(x, y)$	$(\sqrt{2}, 1)$	$(-\sqrt{2}, -1)$	$(-\sqrt{2}, 1)$	$(\sqrt{2}, -1)$
	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$

Min = $-\sqrt{2}$, Max = $\sqrt{2}$