SIIT MAS210 MIDTERM EXAM 1ST SEMESTER 2006 DR. RUBEN

Solution

Problem 1. Let Ax = **0** be a linear system with 5 equations and 4 unknowns. If rank(A) = 3, then
(a) The system has unique solution
(b) If S(A) is the vector space of solutions, then dim(S(A)) = 4.
(c) det(A) = 0
(d) The reduced row-echelon matrix of A has 2 zero rows.
(e) The system is incompatible

Problem 2. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be five vectors in R^4. Then
(a) They form a basis for R^4.
(b) They are linearly independent
(c) They are linearly dependent
(d) They must span R^4
(d) We cannot know whether they are a basis in R^4 unless we know all the vectors.

Problem 3. Let A be a 4×4 matrix, with det(A) = -17. Then
(a) The homogeneous system **Ax = 0** admits the solution (-17, -17, -17, -17)
(b) The matrix A is singular
(c) The homogeneous system admits the unique solution (0, 0, 0, 0)
(d) The system is incompatible
(e) The system admits infinitely many solutions

Problem 4. Let V be a vector space with dim(V) = 4 and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be four vectors in V.
(a) If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, then they form a basis for V
(b) The 4 vectors cannot be a basis for V
(c) The 4 vectors cannot be linearly independent
(d) The 4 vectors cannot span V
(e) The matrix whose columns are these four vectors must have rank less than 4

Problem 5. Let A be a 5×4 matrix of rank 2
(a) The inverse A^{-1} of A has also rank 2
(b) The reduced row-echelon form of A has exactly 3 zero rows
(c) The reduced row-echelon form of A has exactly 2 zero rows
(d) The homogeneous system Ax = 0 admits solution unique
(e) Is S(A) is the vector space of solutions of the homogeneous system **Ax = 0** then dim(S(A)) = 5

Problem 6. Solve the system

$$\begin{aligned} x + y - z &= 4 \\ -x + y + 2z &= 4 \\ x + 3y \quad &= 12 \end{aligned}$$

(If it is incompatible, write 'incompatible'. If it has infinitely many solutions, find the solution set.)

answer:

$$x = \frac{3}{2}t$$

$$y = 4 - \frac{t}{2}$$

$$z = t$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ -1 & 1 & 2 & 4 \\ 1 & 3 & 0 & 12 \end{array}\right] \Rightarrow \begin{array}{c} r_1 \\ r_1 + r_2 \\ -r_1 + r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 2 & 1 & 8 \\ 0 & 2 & 1 & 8 \end{array}\right] \Rightarrow \begin{array}{c} r_1 \\ r_1/2 \\ r_2 - r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 1/2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\begin{array}{c} r_1 - r_2 \\ r_1/2 \\ r_2 - r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & 1/2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Problem 7. Let A be the coefficient matrix of Problem 6
(a) Determine rank (A)
(b) What is the maximum number of independent column vectors of A?

answer:

(a) rank(A) = 2
(b) No. = 2

Problem 8. Let A be the coefficient matrix of Problem 6
(a) find det(A)
(b) Find A^{-1} if it exists. Otherwise, write singular.

answers

(a) det(A) = 0
(b) A^{-1} = singular

Problem 9. Given the vectors $\mathbf{u}_1 = (1, -1, 0)$, $\mathbf{u}_2 = (1, 1, 2)$, $\mathbf{u}_3 = (0, 1, 1)$
(a) Are they linearly independent (yes/no, explain)
(b) Do they span R^3? (yes/no, explain)

answers

(a) (y/n) No, they are linearly dependent
(b) (y/n) No, we need 3 linearly independent vectors to span R^3

(a) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_1 + r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_2/2 \\ r_2 - r_3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Part (b) answer follows directly from (a). You can also work the long way by setting the system

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

whose solution is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & (a-b)/2 \\ 0 & 1 & 1/2 & (a+b)/2 \\ 0 & 0 & 0 & c-a-b \end{array}\right]$$

If $c - a - b \neq 0$ the system has no solution, which means that the vectors of the form $(a, b, a+b)$ are not spanned by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

Problem 10. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ and let S(A) be the vector space of solutions of the homogeneous system Ax = 0.
(a) Find a nonzero vector of S(A)
(b) Find the dimension of S(A)

answers:
(a) (3/2, -1/2, 1)
(b) dim(S(A) = $n - r$ = 3 – 2 = 1

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_1 + r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_2/2 \\ r_2 - r_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} r_1 - r_2 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

solution set: $x = 3t/2$, $y = -t/2$, $z = t$. With $t = 1$

Problem 11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$. Calculate $A\,A^T$

answer:

$$\begin{bmatrix} 14 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 1 \\ 1 & 2 \end{bmatrix}$$