Chapter 4

EIGENVALUES AND EIGENVECTORS

All matrices in this chapter are square matrices. Eigenvalues and eigenvectors are two concepts associated with a square matrix. Because or their important applications in the study of linear algebra, we will define and study the properties of eigenvalues and eigenvectors.

4.1 Preliminaries and Definition

Given a square matrix **A** ($n\times n$) and an n-dimensional vector **x**, we are interested in solving equations of the form

$$A\mathbf{x} = \lambda\mathbf{x} \quad (1)$$

Of course, the trivial solution **x** = **0** is always possible. Therefore, we are interested in the study of the existence of nontrivial solutions and, if nontrivial solutions do exist, how many solutions exist and how to find it.

EXAMPLE 1. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

It's easy to show that $\lambda = 2$ is a solution of (1). Indeed,

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix} \quad \text{and} \quad 2\mathbf{x} = 2\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}$$

We will say that λ = 2 is an **eigenvalue** of the matrix **A**, and **x** = 2 is an eigenvector associated to the eigenvalue λ = 2. Several questions naturally arise:

(a) Are there other eigenvectors associated to the eigenvalue λ = 2?

(b) Are there other eigenvalues in addition to 2?

(c) How many solutions of (1) are possible?

(d) How to find all solutions of (1)?

It is easy to find the answer of question (a) in this example: Any multiple $a\mathbf{x}$ of **x** is also an eigenvector. The student can for this easily verify for instance that (3, 12, 3)’ and (-1, -4, -1)’ are also eigenvectos, or, with all generality, that for any real value of a, the vector (a, 4a, a)’ is an eigenvector. We will see at once the answers of questions (b) through (d)

DEFINITION. Given a square ($n \times n$) matrix **A** if there is a nonzero vector **x** in $\boldsymbol{R}^n$ and a scalar λ that solves the equation (1) then **x** is called an **eigenvector** of A and λ is called an **eigenvalue** of A.

4.2 Finding Eigenvalues and Eigenvectors

Let **I** ($n \times n$) be the identity matrix of the same size as A. Because of the fundamental property of the identity matrix **Ix** = **x**, we can write equation (1) as **Ax** = λ**Ix**. Using the properties of the algebra of matrices described in Chapter 2, equation (1) can be alternatively written as

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} \qquad (2)$$

This implies that the eigenvectors of **A** are solutions of the homogeneous equation above. As we have seen in previous chapters, for the homogeneous equation to have a nontrivial solution, the determinant of the coefficient matrix (in this case **A** – λ**I**) must be zero. That is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \qquad (3)$$

The above equation is called the **characteristic equation of A**. Let in general **A** be

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then (A – λI) is of the form

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

The determinant of this matrix is the characteristic equation, which is a polynomial of degree n in the variable λ. Thus, the roots of the characteristic equation are the eigenvalues of A. As with most of the operations we have seen in this course, finding eigenvalues and eigenvector is a tedious task involviing arduous computations, calculating determinants and reducing matrices to row-echelon format.

The student should notice that **0** in (2) is in boldface while in (3) is in regular type. The reason is very simple, in (2) it refers to the **0** vector whereas in (3), 0 is a scalar.

EXAMPLE 1 (revisited). First form the matrix (**A** – λ**I**) for the matrix of Example 1. First we must find the characteristic equation

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 2-\lambda & -1 \\ -2 & 1 & -\lambda \end{bmatrix} \quad (4)$$

Then, by straightforward manipulations we can calculate the determinant of this matrix, which results in the third degree equation

$$\lambda^3 - 3\lambda^2 + 5\lambda - 6 = 0$$

Finding roots of polynomials higher than second degree is always tricky, and most of the times only approximate values can be found, unless the polynomial has only rational roots, in which case standard methods for finding roots are available. In this example (true also for the problems in the exams) all real roots of the polynomial are rational, so we can find them by factoring the polynomial. In the next section we review the problem of finding all rational roots of a polynomial. For the moment, note that $\lambda = 2$ is one root, thus, factoring (factorization is also covered below)

$$(\lambda - 2)(\lambda^2 - \lambda + 3) = 0$$

and because the quadratic polynomial has two complex roots, we can infer that the only real value of λ that solves equation (1) is $\lambda = 2$. In other words, matrix **A** has only one real eigenvalue and two complex conjugate eigenvalues. Now, we want to find the eivenvectors of **A**. But the eigenvectors are the solution of equation (2). Knowing that $\lambda = 2$, we substitute in (4), and obtain

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & -2 \end{bmatrix}$$

All the eigenvectors of **A** are the solutions of the homogeneous system

$$(\mathbf{A} - 2\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

To solve the homogeneous system we reduce this matrix to its row-echelon form,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

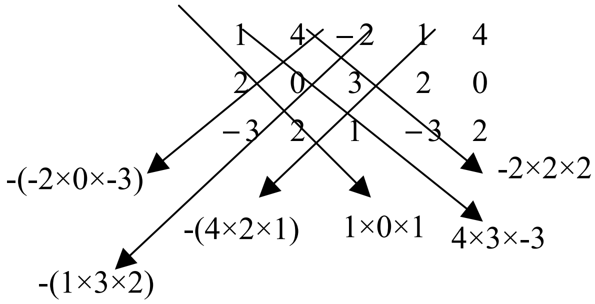
and we obtain the general solution $x_1 = t$, $x_2 = 4t$, $x_3 = t$. So, any vector of the form $(t, 4t, t)$ is an eigenvector associated with the eigenvalue $\lambda = 2$. Hence, the set of eigenvectors associated with $\lambda = 2$ form a vector space of dimension 1, the number of free variables of the solution set, also calculated as $n - r = 3 - 2$ in the last matrix.

4.3 Calculating Determinants

A good way to calculate determinants of a matrix of order 3 is described in the following example. Let the matrix **A** (4×4) be

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & 3 \\ -3 & 2 & 1 \end{bmatrix}$$

Write the first and second columns to the right of the third column, as follows



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and then multiply the diagonals, those on the orientation of the diagonal principal are positive products, and the others are negatives. The result is that

$$\det A = 1\times0\times1 + 4\times3\times(-3) + (-2)\times2\times2 - [\, (-2)\times0\times(-3) + 1\times3\times2 + 4\times2\times1] = -58$$

4.4 Triangular Matrices, Triangulation

Triangular matrices play an important role in the finding of eigenvalues.

DEFINITION. An upper triangular matrix is a matrix consisting of zeros for all the entries below its diagonal principal. That is, it is of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

The elements of the diagonal principal may or may not be zeroes.

Triangulation is an operation consisting of reducing a matrix to its upper triangular form by suitable row operations, similar, but not identical, to the ones used to transform a matrix into its reduced row-echelon format. The only valid operation of triangulation is

1. *Combine a row (not multiplied by any constant) with any row about it, multiplied by some suitable constant, so as to produce zeros on all entries below the diagonal principal.*

Thus by triangulation, the leading terms must not necessarily be one, the leading terms must not necessarily be the only nonzero number of its column, and do not interchange rows (the sign of the determinant may be affected). This procedure is similar as the one used in the previous chapters to reduce a matrix to its echelon-form.

EXAMPLE 2. Triangulate the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 0 & 3 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

The following operations triangulate **A**

$$\begin{matrix} r_1 \\ r_2 + r_1 \\ r_3 + 2r_2 \\ r_4 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 2 & 4 & 1 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_2 \\ r_3 + 2r_2 \\ r_4 - r_2 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & -4 \end{bmatrix} \Rightarrow \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 - r_3/7 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & -32/7 \end{bmatrix}$$

The last matrix on the right is a triangulation of A. The interesting feature of triangulation is exhibited in the next two theorems

THEOREM. If A is an upper triangular or lower triangular matrix, then its determinant is the product of the diagonal entries,

$$\det(A) = a_{11} \times a_{22} \times a_{33} \times \ldots \times a_{nn}$$

which, combined with the following theorem make somewhat easier the computation of the eigenvalues of a matrix.

THEOREM. If the matrix E is obtained by triangulation of a matrix A, then det(E) = det(A) = $e_{11} \times e_{22} \times e_{33} \times \ldots \times e_{nn}$, where e_{ii} are the diagonal entries of E.

EXAMPLE 2 (revisited). The determinant of the matrix **A** given in Example 2 can be calculated by multiplying the diagonal entries of its triangular form, that is,

$$\det(A) = 1 \times 2 \times 7 \times (-32/7) = -64$$

4.5 Finding Rational Roots of Polynomials

This section is a short review of elementary algebra for solving polynomial equations of order higher than 2. There is no formula for finding the roots of such polynomials similar to the quadratic formula for second-degree equations. But if there are rational roots, they can be found.

We will study complex numbers in the second part of this course. A brief introduction is given here. Because the square of a real number is always positive, $x^2 \geq 0$, the square root of a negative number, such as $\sqrt{-1}$ is not defined in the field of real numbers. Thus the field of real numbers is extended by defining an imaginary number i such that $i = \sqrt{-1}$. All complex numbers z are of the form $z = a + bi$. If $b = 0$, then the complex number is a complex-real number, and if $b \neq 0$, then z is complex-non-real number. Hence the real numbers are also a subset of the complex numbers, for which the imaginary part b =0. Given a polynomial $p(x)$ of degree n,

$$p(x) = a_n x^n + a_{n-1} x_{n-1} + \ldots + a_1 x + a_0 \quad (5)$$

the roots of the polynomial, that is, the solutions of the equation $p(x) = 0$, can be of three kinds: rational, irrational and complex.

THEOREM (*Fundamental Theorem of the Algebra*). A polynomial $p(x)$ of degree n has **exactly** n complex roots.

The theorem, however, does not specify how many roots are complex-real and how many are complex-non-real solutions. All we can say about of polynomial of degree n is that it has *at most n* real roots. Some of its real roots may be rational (of the form p/q) and others may be irrational roots (not rational). There is no formula for finding irrational roots of polynomials of degree larger than 4. But there is a standard procedure for finding the rational roots of a polynomial.

THEOREM. All rational roots of the polynomial (5) are of the form p/q, where p is a divisor of a_0 and q is a divisor of a_n.

EXAMPLE. Let $p(x) = 2x^3 + 3x^2 - 8x + 3$ The divisors of 3 are ±1, ±3, and the divisors of 2 are ±1, ±2. Therefore, if this polynomial has rational roots, they can be found among the rationals in the set $\left\{\pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{1}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm 3, \pm\frac{3}{2}, \pm 1, \pm\frac{3}{2}\right\}$. By trial and error we can thus find all the rational roots of the polynomial. The student can verify that $\frac{1}{2}, -3, 1$ are roots of $p(x)$.

Once a rational root is found, the polynomial can be reduced factorized, or reduced its degree by 1 unit, by dividing (short division) in the following way: Suppose that the root 1 is found. Then we divide the polynomial by (x- 1):

	2	3	-8	3
1		2	5	-3
	2	5	-3	0

and we can factor $p(x) = 2x^3 + 3x^2 - 8x + 3 = (x-1)(2x^2 + 5x - 3)$.

Because the roots of the second parenthesis on the right side are also roots of $p(x)$, we can continue the procedure of trying rational roots in the second degree polynomial on the right, which is shorter than trying in the third degree polynomial. Proceeding step-by-step in this way we can find all rational roots of the polynomial. In this example, there is no need of more trials, since the roots of the quadratic polynomial can be found by using the quadratic equation or by factorization. We end with

$$p(x) = 2x^3 + 3x^2 - 8x + 3 = (x-1)(x+3)(2x-1)$$

EXAMPLE. Solve the equation

$$4x^5 - 10x^4 + 17x^2 - 14x + 3 = 0$$

This equation has at most 5 roots. The divisors of $a_5 = 4$ are ± 1, ±2, ±4, and the divisors of $a_0 = 3$ are ±1, ±3. Then, if this equation has rational roots, they will be found among the rationals in the set {±1, ±1/2, ±1/4, ±3, ±3/2, ±3/4}. By trial and error, we first try 1 (the easiest to check) and it happens to be a root. So we know that the polynomial can be factored in the form

$$(x-1)(b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) = 0$$

To find the coefficients b_i we proceed with the sort division as shown below

	4	-10	0	17	-14	3
1		4	-6	-6	11	-3
	4	-6	-6	11	-3	0

So the factorization produces

$$(x-1)(4x^4 - 6x^3 - 6x^2 + 11x - 3) = 0$$

The new roots to be found are the roots of the fourth degree polynomial. So we must try new roots, and we try 1 again, and it happens to be a root, so that 1 is a double root of the original polynomial. We divide again by the same procedure

	4	-6	-6	11	-3
1		4	-2	-8	3
	4	-2	-8	3	0

So the equation becomes

$$(x-1)^2 (4x^3 - 2x^2 - 8x + 3) = 0$$

And continue in this way trying rational roots from the set above on the third degree polynomial. We try 1 again, but this time it doesn't work. We try ½ and 3, but they don't work. Finally, when we try 3/2 and it works. So we reduce once again the third degree polynomial

	4	-2	-8	3
3/2		6	6	-3
	4	4	-2	0

So, finally we get

$$(x-1)^2 (2x-3)(2x^2 + 2x - 1) = 0$$

The last two roots to be found are not rational, but we can find them by using the quadratic formula, and they are $\frac{-1+\sqrt{3}}{2}$ and $\frac{-1-\sqrt{3}}{2}$

As an exercise, for homework, try to find all rational roots of the following polynomials (keep in mind that only the rational roots can be found in this way, irrational roots and complex roots cannot be found by this technique).

HOMEWORK. Factor completely the following polynomials

(a) $x^3 - x^2 - x - 2$ (b) $x^3 - x^2 - 3x + 2$ (c) $x^3 + 3x^2 - x - 3$ (d) $x^4 + x^3 - 3x^2 - 4x - 4$

4.6 Back to Eigenvalue Problems

Using the simplified techniques for calculating determinants, and the method for finding roots of polynomials, we can now go back to fincing eigenvalues.

EXAMPLE 3. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix} = -\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0$$

To solve the cubic equation we try rational roots among the set {±1, ±2, ±4}, and we found that 4 solves the characteristic polynomial. We reduce the grade as before, after multipying the whole equation by -1

	1	-8	17	-4
4		4	-16	4
	1	-4	1	0

and get

$$(\lambda-4)(\lambda^2 - 4\lambda + 1) = 0$$

Hence the roots of the characteristic equation are 4, $2+\sqrt{3}$, and $2-\sqrt{3}$

THEOREM. A square matrix A is nonsingular if and only if $\lambda = 0$ is not an eigenvalue of A.

The if and only if implies the double statement (1) if $\lambda = 0$ is an eigenvalue of A then A is singular; and (2) If A is nonsingular then all of its eigenvalues are different from zero.

4.7 Eigenspaces

Because the characteristic equation of a matrix A of order *n* is a polynomial of degree *n*, the maximum number of roots I *n*, and therefore the maximum number of eigenvalues of A is also *n*. Because the characteristic equation may have repeated roots, the number of eigenvalues is actually less than or equal to *n*. Complex roots are possible, but will not be discussed in this part of the course, since the part corresponding to complex analysis has not yet be covered.

EXAMPLE 3. Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ -3/4 & 1 & 3 \\ 1/4 & 0 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -3/4 & 1-\lambda & 3 \\ 1/4 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda - 1 = -(\lambda-1)^3 = 0$$

which has a triple root $\lambda = 1$.

EXAMPLE 4. Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, which, after factorization, becomes $(\lambda-1)(\lambda-2)^2 = 0$. Hence it has two real eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 1$.

Let λ be an eigenvalue of a square matrix A. An *eigenvalue associated with* λ is a vector **v** satisfying $(A - \lambda I)\mathbf{v} = 0$.

THEOREM. The set of all eigenvectors associated with a particular eigenvalue λ form a vector space, called **the eigenspace associated with** λ.

THEOREM. The dimension of the eigenspace associated with some eigenvalue λ is equal to the order of multiplicity of the eigenvalue.

In Example 4, we have two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 1$, with multiplicities 2 and 1 respectively. Thus the dimension of the eigenspace associated with λ_1 is 2 and the dimension of the eigenspace associated with λ_2 is 1. In Example 3, the dimension of the eigenspace associated with the only eigenvector $\lambda = 1$ is 3, its order of multiplicity.

HOMEWORK

Kreyszig, p. 338, #1-25