Chapter 3

VECTOR SPACES

The notions covered in the previous chapters can be generalized to more comprehensive, abstract spaces, called *Vector Spaces*. Applications of vector spaces can be found in all areas of science.

3.1 Definition and Properties

A vector space is, in sum, a set of abstract things, called vectors. They cannot be described in detail. Vectors can be of many different kinds, but all vector spaces share the same properties. Hence a vector space is defined rather by its properties than by the substance of its elements. Matrices are vectors in the vector space of matrices. Continuous functions are vectors in the vector space of continuous functions. Order pairs of numbers, (a, b) are vectors in their correspondent vector space. The definition given below, exhibits the properties of vector spaces. Anything possessing these properties is called a vector space.

3.1 Definition. A vector space **V** is a nonempty set of objects, *vectors* and *scalars,* with two operations defined, the addition of vectors, and scalar multiplication of a scalar by a vector, with the following properties:

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	commutative
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ $c\,(d\,\mathbf{u}) = (cd)\mathbf{u}$	associative
$c\,(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ $(c + d)\,\mathbf{u} = c\mathbf{u} + c\mathbf{v}$	distributive
there is a vector **0** such that, for every vector **u**, we have $\mathbf{u} + \mathbf{0} = \mathbf{u}$	existence of 0
for each vector **u** there is a vector **-u** such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	inverse
any linear combination $c_1\mathbf{u} + c_2\mathbf{v}$ of two vectors in **V** is also in **V**	linearity

The last property in particular means that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_k$ are vectors in V, then for any linear combination

$$\boldsymbol{v} = c_1\,\boldsymbol{v}_1 + c_2\,\boldsymbol{v}_2 + \ldots + c_k\,\boldsymbol{v}_k$$

the vector $\boldsymbol{v}$ is also in the vector space V.

Notice that the word vector now refers to any type of mathematical entity holding the properties described above. The column vectors and row vectors defined in the previous chapters are particular cases of vectors. We will see several more general examples.

To prove that a given a set V it is a vector space we must prove that it satisfies all the properties above. But this is rather a long, tedious task. In fact, the two properties the student should verify in the exam when asked to prove that a given set is a vector space are the existence of the zero vector and the linearity property, all the other properties being true most of the times.

EXAMPLE 1. The space R^3 is the vector space of triplets $\mathbf{u} = (x, y, z)$ as defined in the previous chapter. Indeed, the zero vector is $0 = (0, 0, 0)$, and linear combinations of triplets result in another triplet which belongs to R3. In General, the space R^n of n-tuplets $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ is a vector space.

EXAMPLE 2. Let **V** be the space of triplets $\mathbf{u} = (u_1, u_2, u_3)$ with second component equal to twice the first component, in other words, $\mathbf{u} = (u_1, 2u_1, u_3)$. The zero vector, $\mathbf{0} = (0, 0, 0)$ clearly belongs to V. To prove that it is a vector space, we must prove that any linear combination of vectors of **V** is also in **V**. Let $\mathbf{u} = (u_1, 2u_1, u_3)$ and $\mathbf{v} = (v_1, 2v_1, v_3)$ and c_1 and c_2 any two scalars. Then $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v} = (u_1 + v_1, 2(u_1 + v_1), u_3 + v_3)$, and so the vector **w** is also in **V.**

EXAMPLE 3. Let V be the vector space of triplets such that the third component is the sum of the first and the second component, that is, vectors of the form $\mathbf{u} = (x, y, x+y)$. The student can easily verify that the zero vector belongs to V and that the linearity property holds.

EXAMPLE 4. The vector space U of triplets $\mathbf{u} = (u_1, u_2, u_3)$ whose third component is equal to 0, that is, $\mathbf{u} = (u_1, u_2, 0)$ is a vector space. It is very simple to verify that the linear combination of vectors of U is also in U.

EXAMPLE 5. An example of a set that is not a vector space. Let **U** be the set of triples $\mathbf{u} = (u_1, u_2, u_3)$ with third component equal to 1, that is, $\mathbf{u} = (u_1, u_2, 1)$. Then U is not a vector space. Indeed, $3\mathbf{u} = (3u_1, 3u_2, 3)$, which is not a vector in **U**, and the zero vector doesn't belong to **U**.

EXAMPLE 6. The set of polynomials $p(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n$ of degree less than or equal to n is a vector space, since any linear combination of polynomials of degree n or less is another polynomial of degree n or less.

EXAMPLE 7. The space of matrices $(m\times n)$ is a vector space, since linear combinations of these matrices gives a matrix of the same type.

EXAMPLE 8. Let $C[0, 1]$ be the space of continuous functions in the closed interval $[0, 1]$. Because linear combinations of continuous functions in a given interval result in a function which is continuous on the interval, the set V is a vector space.

EXAMPLE 9. Let A be an $(n\times m)$ matrix, and let $S(\text{A})$ be the set of solutions of the homogeneous system $\text{A}\boldsymbol{x} = \mathbf{0}$. Then $S(\text{A})$ is a vector space. Indeed, the zero vector, which is the trivial solution, is in $S(\text{A})$. Also, any linear combination of solutions of the system is also a solution. To see this, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two different solutions, that is, $\text{A}\boldsymbol{x} = \mathbf{0}$ and $\text{A}\boldsymbol{y} = \mathbf{0}$. Let $z = a\boldsymbol{x} + b\boldsymbol{y}$ be a linear combination of them. Then, using the properties of the operations with matrices seen in the previous chapter, we have

$$\text{A}\boldsymbol{z} = \text{A}(a\boldsymbol{x} + b\boldsymbol{y}) = \text{A}(a\boldsymbol{x}) + \text{A}(b\boldsymbol{y}) = a\text{A}\boldsymbol{x} + b\text{A}\boldsymbol{y} = \mathbf{0}$$

It follow that $\boldsymbol{z}$ is also a solution of the homogeneous system. We have seen that a homogeneous system either has unique, trivial solution, or it has infinitely many solutions, which include the trivial solution. If the system has unique solution, the vector space $S(\text{A})$ has only one vector, the zero vector. We will see later that the zero vector itself makes a vector space, called the trivial vector space.

3.2 Dimension and Basis

Dimension and basis are two crucial concepts in the development of vector spaces.

DEFINITION. *The maximum number of linear independent vectors of a vector space V is called the dimension of V and is denoted by dim V.*

EXAMPLE 1 (revisited). We can prove that d = dim (R^3) = 3. The proof is divided in two parts; we first prove that $d \geq 3$ and then we prove that $d \leq 3$.

(a) Because the three vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) are *l.i.* (verify it), then d must be 3 or more.

(b) We have seen before that any set of four or more three dimensional vectors is *l.i.*, therefore d must be less than four.

In general, we have that $\dim(R^n) = n$.

EXAMPLE 2 (revisited). In a similar way, we can prove that d = dim(V) = 2. Indeed because the vectors (1, 2, 0) and (0, 0, 1) are in V and are *l.i.*, then $d \geq 2$. On the other hand any three vectors in V are *l.d.* and then $d \leq 2$. To prove this last statement, take any three vectors of V, $\mathbf{x}= (x_1, 2x_1, x_3)$, $\mathbf{y} = (y_1, 2y_1, y_3)$, $\mathbf{z} = (z_1, 2z_1, z_3)$, and verify that the rank of the matrix whose columns are the vectors **x**, **y**, **z** has rank equal to 2.

In a similar way it is easy to prove that the vector spaces defined in in examples 3 and 4, have dimension 2.

DEFINITION. *Let* V *be a vector space of dimension n. Any set of n linearly independent vectors of* V, $\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n$ *is called a basis of* V.

EXAMPLE 1 (revisited). A basis for R^3 is e_1= (1, 0, 0), e_2= (0, 1, 0), e_3= (0, 0, 1) because $\dim(R^3)$ = 3 and they are *l.i.* Any vector (*x, y, z*) in R^3.

3.3 Span of a Set of Vectors

A set of vectors generate a vector space by means of all the infinite possible linear combinations with them.

DEFINITION. Given a set S of r vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r$, the set of all linear combinations of the vectors in S is a vector space called the space spanned by S, and denoted **span** S.

EXAMPLE. The set of vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (1, 0, 1)$, $\mathbf{v}_4 = (0, 1, 1)$ span R^3. Indeed, any vector $\mathbf{w} = (x, y, z)$ of R^3 can be written as a linear combination $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ of these vectors. For this, solve the system

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x \\ 1 & 1 & 0 & 1 & y \\ 1 & 0 & 1 & 1 & z \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x \\ 0 & 0 & 1 & -1 & x-y \\ 0 & 1 & 0 & -1 & x-z \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & -1 & x-z \\ 0 & 0 & 1 & -1 & x-y \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & z \\ 0 & 1 & 0 & -1 & x-z \\ 0 & 0 & 1 & -1 & x-y \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & z \\ 0 & 1 & 0 & -1 & x-z \\ 0 & 0 & 1 & -1 & x-y \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -x+y+z \\ 0 & 1 & 0 & -1 & x-z \\ 0 & 0 & 1 & -1 & x-y \end{array}\right]$$

and we obtain the solution $c_1 = -2t - x + y + z$, $c_2 = t + x - z$, $c_3 = t + x - y$, $c_1 = t$.

The following property of a basis is fundamental.

THEOREM. *A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n$ of a vector space V is a basis of V if the following two conditions hold*:

1. The set $\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n$ is linearly independent
2. They span the vector space (which means that every vector of V can be written as a linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n$.

EXAMPLE 1. (revisited). The set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is not a basis, of R^3. Property 2 of the last theorem is true, as we have seen, but property 1 is not true, since they are *l.d.*, because the rank of the matrix is 3 and n, the number of vectors, is 4. The first three vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, do form a basis, since they are linearly independent, and the dimension of R^3 is 3.

EXAMPLE 2. (revisited). A basis for the space V in Example 2 is $e_1 = (1, 2, 0)$, $e_2 = (0, 0, 1)$. To see this, notice first that they belong to V, and 1) they are *l.i.*, and 2) any vector of V can be written as e linear combination of $x_1e_1 + x_3e_2$, and therefore they span the space V (verify both properties).

EXAMPLE 3. (revisited). A basis for the space V of Example 3 is $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (0, 1, 1)$ (verify this assertion).

EXAMPLE 4.(revisited). Similarly a basis for this space is $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, which is of dimension 2. They belong to V, are *l.i.* and they span the space.

EXAMPLE 6. (revisited). A basis for the set V of polynomials of degree less than or equal to n is $p_0 = 1$, $p_1 = x$, $p_2 = x^2$, …, $p_n = x^n$. It is obvious that they are *l.i.* and they span the space, since any polynomial

$$p(x) = a_0\ a_1x + \ldots\ a_{n-1}x^{n-1} + a_n x^n \text{ can be written as } a_0p_0 + a_1 p_1 + \ldots a_{n-1} p_{n-1} + a_n p_n .$$

The dimension of V is $n+1$.

THEOREM. Every vector space has a basis, which is not necessarily unique.

We have seen before that the vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) form a basis for R^3. It is left to the reader to verify that the set (1, 1, 0), (1, 0, 1), (0, 1, 1) is another basis for R^3.

Another basis for the vector space of example 2, different than the one given above, is, (1, 2, 1), (1, 2, 0) (verify it).

EXAMPLE. The set (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) on R^3 is not a basis. They do span R^3, however, they are nor a *l.i.* set of vectors. Any three of these four vectors, on the other hand, are a basis in R^3.

The set (1, 2, 1), (2, 3, 4) is not a basis for R^3 because it does not span R^3. Indeed, the vector (1, 1, 1) for instance, can't be written as a linear combination of these two vectors (verify it).

The set (1, 3, 4), (0, 1, 1), (-1, 1, 0), (1, 1, 2) spans the vector space V of Example 4, but these vectors are not *l.i.*, therefore they are not a basis for the space.

3.4 Subspaces

If W is a subset of a vector space V, which with the property that W is a vector space itself, then W is a subspace.

DEFINITION. We say that *W is a subspace of a vector space V if the following two conditions hold:*

1. W is a subset of V
2. W is itself a vector space.

For instance, the set V given in Example 2 is a subspace of R^3 because it is a subset of R^3 and it is itself a vector space, but the set V given in example 4, which is a subset of R^3, is not a subspace, since it is not a vector space.

The subsets of R^3 given in examples 3, and 4 are all different subspaces of R^3. But the subset V of R^3 is not a subspace, since it is not a vector space.

THEOREM. If W is a subspace of V, then dimension W $\leq$ dimension V, where the equality is possible only if W = V.

THEOREM. If a vector space has dimension *n*, any set of *m* vectors, with $m > n$ is linearly dependent.

THEOREM. If a vector space V has dimension *n*, any set of *n* linearly independent vectors of V form a basis for V.

THEOREM. Given a set *S* of *r* vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, … $\mathbf{v}_r$, the set of all linear combinations of the vectors in *S* form a vector space V of dimension at most *r*, that is, dimension V $\leq r$.

3.5 The Zero Vector Space 0

The space consisting of one single vector, the zero-vector is a vector space, since it satisfies all the properties of a vector space (verify this assertion). It is called the trivial vector space. The vector space **0** is a subspace of any vector space. The dimension of the zero vector space is zero. there is no interest in the study of this space, but it must be included among the vectors spaces for consistency.

3.6 Finite and Infinite Dimensional Vector Spaces

The dimension of a vector space has been defined as the number of vectors of a basis. How about if the basis contains infinitely many vectors?

DEFINITION. *A vector space with a finite number of vectors in its basis is called finite-dimensional. Otherwise, it is called infinite-dimensional.*

EXAMPLE 8 (revisited). The space of continuous functions on the interval [0, 1], ***C***[0, 1] is an infinite-dimensional vector space. Indeed, because polynomials are continuous functions, and since the set of polynomials of degree less than or equal n has dimension $n+1$, it follows that the dimension of this space, is larger than any given integer, therefore can't be a finite integer.

EXAMPLE 9. (revisited). The vector space $S(\mathrm{A})$ of solutions of the homogeneous system $\mathrm{A}\boldsymbol{x} = \mathbf{0}$ is either the zero vector space, if the homogeneous system has the trivial solution as the unique solution, or it has infinitely many vector, otherwise. In the latter case, the dimension of $S(\mathrm{A})$ is the number of free variables of the solution set.

3.7 Summary Theorems on Vector Spaces

We present in this section a number of theorems related to basis, dimension, and linear independence. Some of the results here may be redundant in the sense that they have been presented before.

THEOREM. If $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots. \boldsymbol{v}_n\}$ is a basis for a vector space V, then every vector $\boldsymbol{v}$ of V can be represented by a linear combination $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots. + c_n\boldsymbol{v}_n$ in exactly one way.

THEOREM. Basis for a vector space are not unique.

THEOREM. All basis of a given vector space V have same number of vectors, equal to the dimension of V

THEOREM. Let V be a finite-dimensional vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n\}$ be any basis.
(a) If a set of vectors has more than n vectors, then the vectors are linearly dependent.
(b) If a set of vectors has fewer than n vectors then they do not span V.

THEOREM. If W is a subspace of V then dim(W) ≤ dim(V). Moreover, if dim(W) - dim(V) then W = V.

THEOREM. Let $\mathcal{D} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots. \mathbf{v}_n\}$ be any set of n vectors and let $\mathcal{S}(\mathcal{B})$ be the vector space spanned by $\mathcal{B}$. Then the dimension of $\mathcal{S}(\mathcal{B})$ satisfied dim $\mathcal{S}(\mathcal{B}) \le n$. Moreover, if the vectors of the set $\mathcal{S}$ are linearly independent, then dim $\mathcal{S}(\mathcal{B}) = n$.

3.8 Row Space, Column Space and Null Space

Let A be an $(n \times m)$ matrix. Each row of A is an n-dimensional row vector (n components), and each column is an m-dimensional column vector. The rows of A span a subspace of R^n (which eventually may be equal to R^n), and the columns of A span a subspace of R^m (eventually equal to R^m).

DEFINITION. *The vector space spanned by the m rows of A is called the* ***Row Space*** *of A, is denoted by $\mathcal{R}(A)$. The vector space spanned by the n columns of A is called the* ***Column Space*** *of A, and is denoted by $\mathcal{C}(A)$. The vector space of solutions of the homogeneous system A**x** = **0** is called the* ***Null Space*** *of A, and is denoted by $\mathcal{S}(A)$*
Thus the row space of A is a subspace of R^n and the column space of A is a subspace or R^m and the null space of A is a subspace of R^n. This in particular implies that (1) dim $\mathcal{R}$(A) ≤ n; (2) dim $\mathcal{C}$(A) ≤ m; and (3) dim $\mathcal{S}$(A) ≤ n. In fact more is true.

THEOREM. For ant $(n \times m)$ matrix A the following relations hold
(a) dim $\mathcal{C}$(A) = dim $\mathcal{R}$(A) ≤ min (n, m)
(b) if r =dim $\mathcal{C}$(A) = dim $\mathcal{R}$(A), then r = rank A
(d)dim $\mathcal{S}$(A) = $n - r$.

EXAMPLE. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

And let $\mathcal{S}$(A) be the null space of A, that is, the vector space whose elements are solutions of the homogeneous system A**x** = **0**. Let us find vectors of $\mathcal{S}$(A). For this, solve the system. The reduced row-echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, rank(B) = 3 and the dimension of $\mathcal{S}$(A) = 4 – 3 = 1. Because the solution set is $x_1 = 3t$, $x_2 = -3t$, $x_3 = t$, $x_4 = t$, the vectors of $\mathcal{S}$(A) all have the form ($3t$, $-3t$, t, t) for arbitrary values of t. Te system has only one free variable, which is also the dimension of $\mathcal{S}$(A).

THEOREM. Let A be an ($n \times m$) matrix and let E be its reduced row-echelon matrix. Then the row space of A is equal to the row space of E. Moreover, the null space of A is equal to the null space of E.

Because the nonzero row vectors of E are linearly independent, they form a basis for the row space of E, and therefore, they are also a basis for the row space of A.

Keeping in mind that the null space of A, $\mathcal{S}$ (A) is the vector space of all solutions of the homogeneous system A**x** = **0**, which is a subspace of R^n, and therefore has dimension less than or equal to n, we have

DEFINITION. *The dimension of $\mathcal{S}$ (A) is called the* ***nullity*** *of A and is denoted as* ***nullity(A)***.
The theorems above imply in particular the following important equality. For any an ($n \times m$) matrix A we have
rank(A) + nullity(A) = n

THEOREM. A linear system A**x** = **b** is consistent if and only if **b** belongs to $\mathcal{C}$(A), in other words, the system is consistent if only if **b** can be obtained as a linear combination of the column vectors of A.

All of the above can be expressed perhaps in a simple way. The homogeneous system A**x** = **0** is a linear system with n unknowns and m equations. The solution set has r = rank(A) dependent variables and $n - r$ = nullity(A) free or independent variables, and of course, we have $r + (n - r) = n$.

HOMEWORK

Kreyszig, p.329, #1-12; p. 301, #13-20

additional problems

1. Let V be the set of quadruplets of the form **v** = (v_1, 0, v_3, v_1+v_3), that is, the second component is zero and the fourth component is the sum of the first and the third.
   (a) Prove that V is a subspace of R^4.
   (b) Find a basis for this vector space
   (c) What is the dimension of V?

2. (a) Prove that the set of vectors $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (1, 0, 0, 1)$, $\mathbf{v}_3 = (1, 0, 1, 0)$, $\mathbf{v}_4 = (0, 1, 0, 1)$ is a basis of R^4. Given the vector $\mathbf{u} = (1, 2, -3, 1)$

(b) Find a linear combination of the vectors $\mathbf{u}_i$ of the basis that is equal to $\mathbf{u}$.

3. (a) Prove that the vectors

$$\mathbf{v}_1 = (1, -1, 0, 0),\ \mathbf{v}_2 = (1, 0, 0, -1),\ \mathbf{v}_3 = (1, 0, 2, 0),\ \mathbf{v}_4 = (1, -1, -2, -1)$$

do not form a basis for R^4.

(b) Let V be the space spanned by the four vectors of part (a). What is the dimension of V?

(c) Find a basis for V.

4. Let A be the (4×4) matrix whose columns are the 4 vectors $\mathbf{v}_1$ to $\mathbf{v}_4$ of problem 2(a), and let B be the (4×4) matrix whose columns are the 4 vectors $\mathbf{v}_1$ to $\mathbf{v}_4$ of problem 3(a).

   (a) Find rank A and rank B.
   (b) Calculate det B.
   (c) Solve the homogeneous systems $A\boldsymbol{x} = 0$ and $B\boldsymbol{x} = 0$.
   (d) Find B^{-1}.

5. Let V the vector space spanned by the vectors $\mathbf{v}_1 = (1, 1, 0, 0)$, $\mathbf{v}_2 = (1, -1, 1, 1)$, $\mathbf{v}_3 = (0, 2, -1, -1)$, $\mathbf{v}_4 = (3, 1, 1, 1)$. Find

   (a) The dimension of V
   (b) A basis for V
   (c) Let W be the subspace of V that have fourth component equal to zero. Find the dimension of W

6. Let A be the matrix below and let S(A) be the null space of A.

$$A = \begin{bmatrix} 2 & -1 & 3 & -3 & 0 \\ 1 & 0 & 2 & -2 & 1 \\ -1 & 2 & 1 & 4 & 0 \\ 2 & 1 & 6 & -1 & 1 \\ 3 & -1 & 5 & -5 & 1 \end{bmatrix}$$

   (a) Find the dimension of S(A)
   (b) Find a base for S(A)

7. Do the set of vectors $\mathbf{v}_1 = (17, 3, -11, 4)$, $\mathbf{v}_2 = (26, -17, 0.5\sqrt{7})$, $\mathbf{v}_3 = (11/17, -101, 33, 10)$ span R^3?

8. Are the vectors $\mathbf{v}_1 = (117, -14, 41)$, $\mathbf{v}_2 = (-117, 0.35, \sqrt{7})$, $\mathbf{v}_3 = (11/17, -10, 303)$, $\mathbf{v}_4 = (11/121, -11, 101)$ linearly independent?