Chapter 2

MATRICES AND DETERMINANTS

The study of matrices is an important tool to solve linear systems of equations. The calculations required to solving systems are long and tedious. As the number of equations and unknowns increases, also the amount of computations increases exponentially. To solve a system of 10 equations and ten unknowns, for instance, may require thousands of multiplications and additions. Problems that arise in practical studies require sometimes solving systems of 100×100, requiring millions of calculations. So, a computer is required in fact to find the solution. In this course we are going to work with small systems only.

2.1 Definitions and Notation

The properties of linear systems of equations can be studied from the rich realm of matrix models, which provide concise ways to study problems of existence and uniqueness of solutions.

DEFINITION. *An **m×n matrix** is a rectangular array of numbers in m rows and n columns. This is an m by n matrix. It is denoted as*

$$\mathbf{A}\,(m\times n)=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{or also by } \; a_{ij}\,(m\times n)$$

The first number, m in this case, is always the number of rows and the second number, n indicates the number of columns.

DEFINITIONS. (a) *The size of a matrix is the number of rows and columns.* A *matrix* A(m×n) *has m rows and n columns.*

(b) *A column vector* **u** (n×1) *is a matrix of n rows and* 1 *column. A row vector* **v** (1×m) *is a matrix with* 1 *row and m columns. The entries of a vector are called components.*

Thus, a 4 by 3 matrix is of different size than a 3 by 4 matrix. Matrices are denoted in capital letters and vectors in boldface low case. The term (m×n) is read "m by n" and not m times n. So a (5×4) matrix and a (4×5) have different sizes.

DEFINITION. *The **zero matrix** is a matrix consisting entirely of zeros. The zero vector is a vector consisting entirely or zeros.*

$\mathbf{0} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$; column vector, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_4 \end{pmatrix}$; row vector $\mathbf{v} = (v_1, v_2, \dots\ v_n)$

EXAMPLE. Row vector 1× $\mathbf{u} = (1, -2, 0)$. Column vector 3×1 $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

2.2 Operations with Matrices

Matrices are elements of a new Matrix Space. Several operations are possible with matrices. In a similar way as we do with real numbers, the ensemble of operations together with their properties comprise the *matrix algebra,* which portray some differences with respect to the usual algebra of real numbers. We review briefly some aspects of it.

Equality. Two matrices are equal if they have same size (same number of rows and same number of columns) and corresponding entries are equal.

Addition. If A and B have same size, then the sun **A** + **B** is a matrix obtained by adding the corresponding entries of each matrix. Thus the addition of matrices is possible only between matrices of same size.

Scalar Multiplication. If **A** is a matrix and *c* is a scalar (real numbers are called *scalars*), then the product *c***A** is a matrix of the same size than A, obtained by multiplying each entry of **A** by the real number *c*. Thus scalar multiplication is always possible.

Product of two matrices. The product of two matrices **A** and **B**, **A**×**B** is possible only if the number of columns of **A** equals the number of rows of **B**. Let **A** be $(m\times p)$ and **B** $(p\times n)$. Then the product **AB** is an $(m\times n)$ matrix **C**. The entry c_{ij} of C is obtained by multiplying the *i*-th row of **A** by the *j*-th column of **B**. The product is better illustrated by examples. So, read the book if you have doubts on how to multiply two matrices.

Transposition. Given a matrix **A** $(m\times n)$, its transpose, $\mathbf{A}^T$ (also denoted as **A'** by many authors) is a matrix $(n\times m)$ obtained by interchanging rows and columns. Thus, the rows of A^T are equal to the columns of A. The transpose of any matrix always exists.

EXAMPLE. Given the matrices $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & -2 \end{pmatrix}$ then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & -1 \end{pmatrix} \text{ and } 2(\mathbf{A} + \mathbf{B}) = \begin{pmatrix} 6 & 4 & 12 \\ 2 & 4 & -2 \end{pmatrix}$$

EXAMPLE. The matrices A and B in the previous example cannot be multiplied, since the number *n* of columns of A is 3 and the number of rows *m* of B is 2. The matrices

$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ -1 & 2 & 0 & 1 \end{pmatrix}$ on the other hand, can be multiplied, since $m = n = 3$ and the result is the 2×4 matrix

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ -1 & 1 & -1 & -2 \end{pmatrix}$$

The entry c_{ij} (row *i*, column *j*) is obtained by multiplying the *i*th row of **A** by the *j*th column of **B**.

For instance, the element $c_{12} = 2$, corresponding to the first row second column of *C* is obtained by multiplying the first row of **A** with the second column of **B**: (1. 2. 0)×(0, 1, 2)

$$1\times 0 + 2\times 1 + 0\times 2 = 2$$

EXAMPLE. The transpose of A in the previous example is the (3×2) matrix

$$\mathbf{A}' = \mathbf{A}^{T} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix}$$

In particular, the transpose of a row vector is a column vector and vice versa.

EXAMPLE. If $\mathbf{u} = (1, 3, 0)$ then $\mathbf{u}' = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and if $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$ then $\mathbf{v}' = (1, 0, 5)$

Often you will see a vector written as (*a*. *b*, *c*)' and not as in column notation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This is only for editing purposes, because the latter takes to much space in the paper.
A brief summary of the properties of operations follows. We assume in all cases that the matrices involved are compatible, that is, their sizes are such that the operation is possible.

$A + B = B + A$	commutative, sum	$A + \mathbf{0} = A$
$(A + B) + C = A + (B + C)$	associative sum	$1\,A = A$ (1 scalar)
$(AB)\,C = A\,(BC)$	associative product	$(k\,A)^T = k\,A^T$
$k\,A = A\,k$ (k scalar)	commutative scalar product	$(A + B)^T = A^T + B^T$
$k\,(AB) = (kA)\,B = A\,(k\,B)$	associative scalar product	$(A^T)^T = A$
$A\,(B + C) = AB + AC$	distributive	$(AB)^T = A^T B^T$
$(A + B)\,C = AC + BC$	distributive	

Some of the properties of the product for real numbers do not stand for matrices. For instance, **A**×**B** is not necessarily equal to **B**×**A**. The equality **A**×**B** = **0** does not necessarily imply that either **A** = **0** or **B** = **0**. The cancellation law for real numbers is not satisfied, and the equality A×C = A×D does not necessarily imply **C** = **D**.

2.2.1 Multiplication of a matrix by a vector. The product of a matrix **A** (*m*×*n*) by a column vector **u** (*n*×1) is another vector **v**(*n*×1).

EXAMPLE.

$$\underset{3\times 3}{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & -2 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix}}$$

2.3 Square Matrices, Identity, Inverse

Square matrices play a fundamental role in linear algebra. We will study square matrices in detail.

DEFINITION. A ***square matrix*** ($n \times n$) is a matrix with equal number of rows and columns.

DEFINITION. *The **Identity Matrix** of Order n,* **I**($n \times n$) *is a square matrix consisting of* 1*s in the principal diagonal and* 0*s everywhere else. It has the property that, for any matrix* $A(n \times n)$ *, the relations* $\mathbf{A} \times \mathbf{I} = \mathbf{I} \times \mathbf{A} = \mathbf{A}$ hold.

Thus, the identity matrix in the algebra of square matrices plays the same role as the number 1 in the algebra or real numbers.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Can we divide matrices? We can divide square matrices in a similar way we can divide numbers. When we divide, for instance 6 ÷ 3, in fact we are multiplying the number 6 by the inverse of the number 3, that is, 6×(1/3). In the same way, we can divide two square matrices of same size *n*, by multiplying by the inverse, that is

$$\mathbf{A} \times (\mathbf{B}^{-1}),$$

Where $\mathbf{B}^{-1}$ means the inverse of **B**.

DEFINITION. *Given a square matrix* A ($n \times n$), ***the inverse of* A**, if it exists, *is another matrix* ($n \times n$), denoted $\mathbf{A}^{-1}$, *such that* $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$.

Note that the inverse is defined only for square matrices. Not every matrix **A** has an inverse. If it does, we say that A is *invertible* or ***nonsingular***. If the inverse of **A** does not exist, we say that **A** is ***singular***. There is a main difference between matrices and numbers: The inverse of every nonzero number always exists, while the inverse of a square matrix does not necessarily exists.

THEOREM. The inverse of a matrix, if it exists, is unique.

EXAMPLE. Verify that $\mathbf{A}^{-1} = \begin{pmatrix} 0 & -1 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$ is the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

Multiplying these matrices we readily obtain $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$

2.4 Finding the Inverse

Evaluating the inverse of a square matrix is equivalent to solving a system of equations. Let **A** be the matrix of the previous example. We want to find its inverse $\mathbf{A}^{-1}$ so write four unknowns for each of its entries

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

To find the values of the unknowns apply the condition of the inverse

$$\mathbf{AA}^{-1} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}+2x_{21} & x_{12}+2x_{22} \\ -x_{11} & -x_{12} \end{pmatrix}.$$

Solving the system

$$\begin{aligned} &x_{11} + 2x_{21} = 1 \\ &x_{12} + 2x_{22} = 0 \\ &-x_{11} = 0 \\ &-x_{12} = 1 \end{aligned}$$

yields $x_{11} = 0$, $x_{12} = -1$, $x_{21} = ½$, $x_{22} = ½$. Hence, if the system has one solution, then the matrix **A** is invertible. If the system is inconsistent, the matrix A is not invertible or singular. And if the system has unique solution, then the inverse exists and is unique. Because of the theorem stated above, which was not proven, this system cannot have infinitely many solutions.

THEOREM. If A and B are nonsingular (invertible) matrices of same size, then their product is also a nonsingular, and $(A\,B)^{-1} = A^{-1}\,B^{-1}$.

Because the inverse of a matrix can be found by solving a system of equations, we can use an analogous of the Gauss-Jordan procedure to find the inverse of a matrix A. For this we form a kind of augmented matrix, joining I (the identity matrix) to the right or A. That is, we produce a matrix of the form

$$[\,\mathbf{A} \mid \mathbf{I}\,]$$

and by elementary row operations we transform the matrix A into its reduced row-echelon form. If on the left side we get the identity, then A is nonsingular and its inverse is the matrix standing on the right. In other words, A is invertible, after applying the Gauus-Jordan procedure, we get the identity on the left side and the inverse is the matrix on the right side

$$[\,I \mid A^{-1}\,]$$

EXAMPLE. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & 0 \\ 1 & -2 & 4 \end{bmatrix}$$

Form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{array}\right]$$

transform the left side into its reduced row-echelon format

$$\begin{matrix} r_1 \\ 2r_1 - r_2 \\ r_1 - r_3 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 4 & 2 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & -1 \end{array}\right] \Rightarrow \begin{matrix} r_1 - r_2 \\ r_2 \\ r_2 - r_3 \end{matrix}\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & -1 & 0 \\ 0 & 3 & 4 & 2 & 1 & 0 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right] \Rightarrow$$

$$\begin{matrix} 3r_1 - r_3 \\ 3r_2 - 2r_3 \\ r_3 \end{matrix} \left[\begin{array}{ccc|ccc} 9 & 0 & 0 & 2 & -4 & -1 \\ 0 & 9 & 0 & 4 & 1 & -2 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right] \Rightarrow \begin{matrix} (1/9)r_1 \\ (1/9)r_2 \\ (1/6)r_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/9 & -4/9 & -1/9 \\ 0 & 1 & 0 & 4/9 & 1/9 & -2/9 \\ 0 & 0 & 1 & 1/6 & 1/6 & 1/6 \end{array}\right]$$

The matrix on the right side is, therefore, A^{-1}. The inverse exists if and only if the reduced row-echelon form of A is the identity, as in the case of the example.

THEOREM. If a square matrix admits an inverse, then transforming it to its reduced row-echelon form yields the identity matrix.

Another form to express the result is

THEOREM. A reduced row-echelon form of a square matrix either contains rows of zeros or it is the identity. In the former case the matrix is singular. In the latter case it is nonsingular.

2.5 Application of Matrices to Solving Linear Systems

In chapter one we have seen how to use matrices to solve linear systems. In this section we will deepen these concepts. The properties of matrices can easily be transported to the existence and uniqueness of solutions.

We recall that the product of an $(m\times n)$ matrix by an $(n\times 1)$ column vector is an $(m\times 1)$ column vector.

EXAMPLE.

$$\underset{(4\times 3)}{\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}} \underset{(3\times 1)}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} = \underset{(4\times 1)}{\begin{pmatrix} 5 \\ 3 \\ 2 \\ 4 \end{pmatrix}}$$

Let the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\ a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\ a_{2n}x_n &= b_2 \\ \text{---------------------------} &= \text{--} \\ a_{m1}x_1 + a_{m2}x_2 + \dots\ a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

Let **A** be the coefficient matrix, **x** the vector of unknowns, and **b** the vector of constants,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The system can be written alternatively in matrix format.

$$\mathbf{Ax} = \mathbf{b}, \quad \text{that is,} \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{2}$$

where $\mathbf{A}_{(m \times n)}$ is the coefficient matrix, $\boldsymbol{x}_{(n)}$ is the n-dimensional column vector of unknowns, and $\boldsymbol{b}_{(m)}$ is the m-dimensional column vector of the constant terms.

EXAMPLE 1. The system

$$\begin{aligned} x + y + z &= 1 \\ y + z &= -1 \\ x + 2y &= 0 \end{aligned}$$

for instance, can be written $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, that is, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x}$ is the vector of unknowns, $\mathbf{A}$ is the coefficient matrix, and $\mathbf{b}$ is the vector of constant terms.

DEFINITION. *If a matrix **B** is the result of transformation of the rows of another matrix **A** by means of elementary row operations, then **A** and **B** are said to be **row equivalent**.*

THEOREM. If **A** and **B** are the coefficient matrices of two systems, and if they are row equivalent matrices, then the systems have same solution set.

The Gauss-Jordan elimination method for solving the system consists of transforming the coefficient matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

into its reduced row-echelon form by means of suitable linear combinations of rows. Because the original coefficient matrix and its reduced row-echelon form are equivalent, then both matrices have same solution set. Let us review the Gauss-Jordan method for solving systems

EXAMPLE. Let the system

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

The augmented row-echelon matrix is (check it!)

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 1 \\ 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution set of the system is given by

$x_1 = 1 + x_3 - 3x_4$ or, alternatively by $x_1 = 1 + t - 3s$

$x_2 = 4 - 2x_3 + x_4$ $\qquad x_2 = 4 - 2t + s$

$x_3 = t$

$x_4 = s$

The variables t and s are *free variables*, or *independent variables*, since they can take any arbitrary value. For each value of t and each value or s we have a different solution. The solution set of system above is said to have dimension 2, which is the number of free variables.

EXAMPLE. Find the solution set of $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 2 & -2 & 5 & 4 \end{array}\right]$

$$\begin{array}{c} r_1 \\ r_1 + r_2 \\ r_2 - r_3 \\ -2r_3 + r_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow \begin{array}{c} r_1 - 2r_3 \\ r_2 - 3r_3 \\ r_3 \\ -r_3 + r_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & -1 & 0 & -8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -7 + t \\ x_2 = -8 + t \\ x_3 = t \\ x_4 = 4 \end{array}$$

It has only one independent variable, t.

EXAMPLE. Solve the system $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 2 & 1 & 4 & -1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$

$$\begin{array}{c} r_1 \\ -2r_1 + r_2 \\ r_1 - r_3 \\ r_4 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 3 & -2 & -5 \\ 0 & -3 & 3 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \Rightarrow \begin{array}{c} r_1 \\ r_2 \\ r_2 + r_3 \\ r_4 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right] \Rightarrow \begin{array}{c} 3r_1 + r_2 \\ r_2 \\ r_3 \\ r_3 - r_4 \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 7 & 1 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\begin{array}{c} r_1 - r_3 \\ r_2 + 2r_3 \\ r_3 \\ r_4 \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 22 \\ 0 & 3 & 0 & -11 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} r_1/3 \\ r_2/3 \\ r_3 \\ r_4 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 22/3 \\ 0 & 1 & 0 & -11/3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 22/3 \\ x_2 = -11/3 \\ x_3 = -3 \end{array}$$

EXAMPLE. Solve the system $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & -2 & 1 & 2 \\ 2 & 1 & 3 & 2 \end{array}\right]$

$$\begin{matrix} r_1 \\ r_1 - r_2 \\ -2r_1 + r_3 \end{matrix}\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & -5 & -1 & 0 \end{array}\right] \Rightarrow \begin{matrix} r_1 \\ r_2 \\ r_1 + r_3 \end{matrix}\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We stop here. The system is incompatible,

THEOREM. Let n be the number of unknowns and r the number of nonzero rows of the reduced row-echelon matrix. Then the following statement hold

1. If a row of zeros in the coefficient matrix contains a number different from zero in the constant term, the system is inconsistent.

Otherwise, if there are no rows of zeros in the coefficient matrix, or if all the rows of zeros in the coefficient matrix correspond to zeros in the constant terms column, then

2. If $r < n$¸ then the system has infinitely many solutions, with $n - r$ being the number of independent variables.

3. If $r = n$, then the system has solution unique.

So in Example 1, $n = 4$ and $r = 2$. The system has 2 independent variables, t and s. In Example 2, $n = 4$ and $r = 1$. The system has 1 independent variable. In Example 3, $n = 3$ and $r = 3$. The system has unique solution (notice that in this case, the reduced row-echelon matrix, excluding the zero rows, is the identity matrix I. In Example 4 the system is inconsistent.

The equation linear $ax = b$, as we see in high-school, is solved by multiplying both sides of the equation by a^{-1}, that is, $a^{-1}ax = a^{-1}b \Rightarrow x = a^{-1}b$ or, $x = b/a$. Square systems can be solved by using the same procedure. Suppose now that the system in (1) has same number of equations and unknowns. Writing it in matrix form as in (2), the corresponding coefficient matrix A is square. Suppose that A in (2) is nonsingular; then the solution of the system can be obtained by finding the inverse of A in the following way

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Because the inverse of a nonzero matrix does not always exist, systems of equations do not necessarily have solution. If the matrix **A** is singular, then the system is incompatible. The existence of solutions of linear system with equal number of equations and unknowns thus can be study using the properties of matrices and the existence of an inverse.

THEOREM. For a linear system **Ax** = **b** with equal number of equations and unknowns (so **A** is square), all of the following statements are equivalent.

1. The system has unique solution
2. **A** is nonsingular (it has an inverse)
3. The reduced row-echelon form of **A** is the identity.
4. The solution of the system is the vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
5. The homogeneous system **Ax** = **0** has the trivial solution as unique solution.

EXAMPLE. Let the system of example 1,

$$\begin{aligned} x + y + z &= 1 \\ y + z &= -1 \\ x + 2y \quad &= 0 \end{aligned}$$

Solving it by the Gauss-Jordan elimination method

$$\left(\begin{array}{ccc|c}1&1&1&1\\0&1&1&-1\\1&2&0&0\end{array}\right)\Rightarrow\begin{array}{c}r_1\\r_2\\r_1-r_3\end{array}\left(\begin{array}{ccc|c}1&1&1&1\\0&1&1&-1\\0&-1&1&1\end{array}\right)\Rightarrow\begin{array}{c}r_1\\r_2\\r_2+r_3\end{array}\left(\begin{array}{ccc|c}1&1&1&1\\0&1&1&-1\\0&0&2&0\end{array}\right)\Rightarrow\begin{array}{c}r_1-r_2\\2r_2-r_3\\r_3\end{array}\left(\begin{array}{ccc|c}1&0&0&2\\0&2&0&-2\\0&0&2&0\end{array}\right)$$

$$\Rightarrow\begin{array}{c}r_1\\r_2/2_3\\r_3/2\end{array}\left(\begin{array}{ccc|c}1&0&0&2\\0&1&0&-1\\0&0&1&0\end{array}\right)$$

and the answer is $(x, y, z) = (2, -1, 0)$ unique solution. We can solve it alternatively by using the inverse. Let us now calculate the inverse $\mathbf{A}^{-1}$. As before, if $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Calculating the inverse of $\mathbf{A}$,

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\1&2&0&0&0&1\end{array}\right)\Rightarrow\begin{array}{c}r_1\\r_2\\r_1-r_3\end{array}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\0&-1&1&1&0&-1\end{array}\right)\Rightarrow\begin{array}{c}r_1\\r_2\\r_2+r_3\end{array}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\0&0&2&1&1&-1\end{array}\right)\Rightarrow$$

$$\begin{array}{c}r_1-r_2\\2r_2-r_3\\r_3\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&2&0&-1&1&1\\0&0&2&1&1&-1\end{array}\right)\Rightarrow\begin{array}{c}r_1\\r_2/2_3\\r_3/2\end{array}\left(\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&-1/2&1/2&1/2\\0&0&1&1/2&1/2&-1/2\end{array}\right)$$

Hence,
$$\mathbf{A}^{-1}=\begin{pmatrix}1&-1&0\\-1/2&1/2&1/2\\1/2&1/2&-1/2\end{pmatrix}$$

The solution of the system can also be obtained by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, that is

$$\begin{pmatrix}1&-1&0\\-1/2&1/2&1/2\\1/2&1/2&-1/2\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

2.6 Linear Combinations of Vectors

Because we will deal with general systems with any number of equations and unknowns, not necessarily equal, we must study properties of matrices in general, not just square matrices. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$, be n m-dimensional vectors and let $c_1, c_2, \ldots c_n$, be n real numbers.

DEFINITION. *A* ***linear combination*** *of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$, *is given by*

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n$$

The ***trivial linear combination*** *happens when all coefficients* c_i *are 0. Otherwise, if not all* c_i *are zero, then we have a* ***nontrivial linear combination***.

Given the n column vectors of size m, $\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{m1} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{m2} \end{bmatrix}$, …. , $\mathbf{u}_n = \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{mn} \end{bmatrix}$

a linear combination of the vectors is given by

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = \begin{bmatrix} c_1u_{11} + c_2u_{12} + \ldots + c_nu_{1n} \\ c_1u_{21} + c_2u_{22} + \ldots + c_nu_{2n} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ c_1u_{m1} + c_2u_{m2} + \ldots + c_nu_{mn} \end{bmatrix}$$

If all the coefficients c_i are zero, this is called the *trivial* linear combination. If not all coefficients c_i are zero, then it is a *nontrivial* linear combination. Thus a system of equations can be presented as a linear combination of column vectors, each vector corresponding to one column of the coefficient matrix. The system in (2) can then be written

$$x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + \ldots + x_n\boldsymbol{u}_n = \boldsymbol{b},$$

where the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are the columns of the coefficient matrix A.

With row vectors, $\boldsymbol{u} = (u_1, u_2, ..., u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots v_n)$, $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ linear combinations take the form

$$a\,\boldsymbol{u} + b\,\boldsymbol{v} + c\,\boldsymbol{w} = (a\,u_1 + b\,w_1 + c\,w_1, a\,u_2 + b\,w_2 + c\,w_2, a\,u_3 + b\,w_3 + c\,w_3)$$

2.7 Linear Independence

The existence and uniqueness of solutions of linear systems is closely related with the notion of linear independence of vectors. We introduce this section by solving two similar problems

EXAMPLE. Given the vectors

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ -4 \end{bmatrix}, \boldsymbol{a}_4 = \begin{bmatrix} -4 \\ -2 \\ 1 \\ 4 \\ 17 \end{bmatrix}$$

The trivial linear combination always produces **0** as result

$$0\times\boldsymbol{a}_1 + 0\times\boldsymbol{a}_2 + 0\times\boldsymbol{a}_3 + 0\times\boldsymbol{a}_4 = \mathbf{0}$$

(zero in boldface means the 5-dimensional zero vector). Is there a nontrivial linear combination that yields **0?** In other words, can we find four numbers x_1, x_2, x_3, x_4, not all zero, such that

$$x_1\times\boldsymbol{a}_1 + x_2\times\boldsymbol{a}_2 + x_3\times\boldsymbol{a}_3 + x_4\times\boldsymbol{a}_4 = \mathbf{0}$$

The answer is, sometimes yes, sometimes not. In this example the answer is yes, the reader can verify that with $(x_1, x_2, x_3, x_4) = (2, 1, -3, -1)$

$$2\boldsymbol{a}_1 + \boldsymbol{a}_2 - 3\boldsymbol{a}_3 - \boldsymbol{a}_4 = \mathbf{0}.$$

The questions that naturally arises is whether, given a set of vectors, there always exists a nontrivial combination equal to zero, and if so, how to find them. Solving the homogeneous system

$$x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + x_3\,\boldsymbol{a}_3 + x_4\,\boldsymbol{a}_4 = \mathbf{0}$$

if the trivial solution is the only solution of the system, then there are no nontrivial linear combinations equal to zero. If a nontrivial solution exists then there are also infinitely many linear combinations equal to zero.

EXAMPLE 1. For the vectors of the previous example

$$\begin{bmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & 1 & -2 \\ -1 & 0 & -1 & 1 \\ 2 & 0 & 0 & 4 \\ 3 & -1 & -4 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where each column in the coefficient matrix is one of the four given vectors. This is a homogenous system. The reduced row-echelon matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

so the solution set is $x_1 = -2t$, $x_2 = -t$, $x_3 = 3t$, $x_4 = t$. The linear combination presented in (b) corresponds to $t = -1$. Because if a homogeneous system has one nontrivial solution, it has infinitely many solutions, then, if there is a nontrivial linear combination producing the vector **0**, there are also infinitely many trivial combinations.

EXAMPLE 2. Given the vectors

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \; \boldsymbol{a}_4 = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

find a nontrivial linear combination such that $c_1\,\boldsymbol{a}_1 + c_2\,\boldsymbol{a}_2 + c_3\,\boldsymbol{a}_3 + c_4\,\boldsymbol{a}_4 = \mathbf{0}$ (observe that using boldface **0** indicates the zero vector and not the scalar vector). Proceeding as before, we must solve the homogenous system

$$\begin{bmatrix} 1 & 1 & 3 & 5 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 2 & 4 \\ 0 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The reduced row-echelon form of the coefficient matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which has unique solution, the trivial linear combination. Therefore, it does not exist a nontrivial linear combination of the vectors that is equal to zero. In summary, for the vectors of Example 1 we found infinitely many linear combinations equal to zero, and for Example 2 there exists no nontrivial linear combination equal to zero. We say that the vectors of Example 1 are *linearly dependent* while the vectors in Example 2 are *linearly independent*.

DEFINITION. *A set of n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \boldsymbol{a}_n$ of the same size is* ***linearly independent (l i)*** *if there is no nontrivial linear combination such that*

$$c_1\, \boldsymbol{a}_1 + c_2\, \boldsymbol{a}_2 + \ldots + c_n\, \boldsymbol{a}_n = \mathbf{0}$$

If there exists a nontrivial linear combination such that the sum is zero, then the vectors are said to be ***linearly dependent*** (*ld*)

In other words, the equality above is satisfied if only if $c_1 = 0$ $c_2 = 0, \ldots, c_n = 0$.

Given k vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_k$, the problem of determining whether they form a linearly independent or linearly dependent system is equivalent to solving the homogeneous linear system of equations $x_1\, \boldsymbol{v}_1 + x_2\, \boldsymbol{v}_2 + \ldots + x_k\, \boldsymbol{v}_k = 0$. If it has unique solution, the trivial solution, then they are linearly independent; they are dependent if the homogeneous system admits infinitely many solutions.

2.8 Rank of a Matrix

The existence of unique solution of systems can be set in terms of the rank of the coefficient matrix.

DEFINITION. *The* ***rank*** *of a matrix is the maximum number of linearly independent column vectors of the matrix and it is denoted as* ***rank* A**. Some authors denote it as rank(A).

THEOREM. The maximum number of linearly independent column vectors of a matrix is equal to the maximum number of linearly independent row vectors.

So the rank of a matrix can also be defined as the maximum number of linearly independent row vectors. Because the column vectors of a matrix A are the row vectors of its transpose, A^T, the rank of a matrix is equal to the rank of its transpose. The rank of a matrix cannot be larger than the number of rows and cannot be larger than the number of columns. We conclude that the rank of a matrix ($m \times n$) is "at most" the smaller of m and n. That is

$$\text{rank } A_{(m\times n)} \le \min(m, n), \tag{4}$$

where min means minimum, or the smallest of both numbers. So, the rank of a 5 by 3 matrix, for instance, can be 1, 2, or 3.

THEOREM. If **A** and **B** are two row equivalent matrices, then rank A = rank B.

THEOREM. The rank of a matrix is equal to the number of nonzero rows of its reduced row-echelon matrix.

The rank of a matrix is equal to the rank of its reduced row-echelon matrix. And the rank of a row-echelon matrix is equal to the number of nonzero rows. Thus, the rank of the matrix in Example 1 above is 3 and the rank of the matrix in Example 2 is also 3.

Rather than proving the theorem, let's illustrate it with an example. Consider the row-echelon matrix given in (3). It has 5 row vectors, only 3 of which, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, are nonzero vectors. Let us try to find a nontrivial linear combination such that

$$c_1\,\boldsymbol{r}_1 + c_2\,\boldsymbol{r}_2 + c_3\,\boldsymbol{r}_3 = 0 \tag{5}$$

where $\boldsymbol{r}_1 = (1, 0, 0, 2)$, $\boldsymbol{r}_2 = (0, 1, 0, 1)$, $\boldsymbol{r}_3 = (0, 0, 1, -3)$. Because the first component of $\boldsymbol{r}_1$ is the only nonzero among the first components, then c_1 must be equal zero. Because the second component of $\boldsymbol{r}_2$ is the only second nonzero component, then c_2 must be equal to zero, and by the same token, c_3 must also be zero, thus the only linear combination of the rows of the matrix in (3) that satisfies (4) is the trivial linear combination, which means that the three rows of the matrix are linearly independent, and therefore its rank is 3.

Returning to a homogeneous system with m equations and n unknowns, we have seen that either it has one solution, the trivial solution, or infinitely many solutions.

THEOREM. Let **A** be the coefficient matrix of a homogeneous system. If rank $\mathbf{A} = n$, where n is the number of unknowns, then the system has unique solution, the trivial solution. If rank $\mathbf{A} < n$, then there are infinitely many solutions in addition to the trivial solution.

Now, suppose that the number of equations m is less than the number of unknowns n, that is $m < n$. From (4) it follows that rank $A \le m < n$, and therefore $r = \text{rank } A < n$ and the last theorem implies that that the system has infinitely many solutions. The dimension of the solution set (the number of free variables) is exactly $n - r$.

2.9 Linearly Independence and Rank

The above discussion lead to the following summary result. Let $\mathbf{u}_1$, $\mathbf{u}_2$, … $\mathbf{u}_n$ be a set of n vectors, each with m components (m-dimensional vectors). Form the matrix **A** ($n\times m$) whose columns are the n vectors $\mathbf{u}_i$. Then, if rank $\mathbf{A} = n$, or, equivalently, if the number of nonzero rows in the reduced row-echelon form of **A** is equal to n, then the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has unique solution, and the vectors are linearly independent. But if rank $\mathbf{A} < n$, then the vectors are linearly dependent, and any nontrivial solution of the homogeneous system provides a nontrivial linear combination of the vectors equal to zero.

2.10 Determinants

Given a square matrix **A**, the determinant of **A**, denoted det (**A)** or |**A**|, is a number associated with the matrix. The procedure to calculate the determinant of a matrix is somewhat cumbersome and the computations tedious. We start illustrating the procedure with a (2×2) matrix. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \text{ Then } \det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}.$$

So to indicate that it is a determinant, we replace the brackets by bars.

EXAMPLE. Calculate $\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix}$. Applying the last formula, 1×(-2) - 2×3 = -8.

Crammer's rule for linear systems of 2 equations on two unknowns. Given the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

It's solution, using determinants, is

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

EXAMPLE. Solve the system $x + 2y = 1$, $x - y = -2$ using Cramer's rule.

$$x = \frac{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{(1)(-1)-(2)(-2)}{(1)(-1)-(2)(1)} = \frac{3}{-3} = -1 \qquad y = \frac{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{(1)(-2)-(1)(1)}{(1)(-1)-(2)(1)} = \frac{-3}{-3} = 1$$

While Crammer's rule is not an efficient method of solving systems, it provides an illustrative account of the uses of determinants. Here, if the determinant of the coefficient matrix zero, then the system is incompatible. As we will see, if the matrix is singular, then its determinant is zero.

For a third order square matrix, its determinant is calculated as follows (expanded by the first row)

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

The terms C_{ij} are called minors or cofactors. The cofactor C_{ij} of an entry a_{ij} is the determinant of the reduced matrix obtained by suppressing the i-th and the j-th column of the original matrix and being multiplied by $(-1)^{i+j}$. So if $i+j$ is even the cofactor is positive, if the sum is odd the cofactor is negative.

EXAMPLE. Calculate $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \\ 2 & 1 & 1 \end{vmatrix} = (1)\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} + (2)\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} =$

(1)[(3)(4)-(4)(1) + …]+(1)[]

DEFINITION. For a square matrix A of order n in general, its determinant, det A is given by the following recurrent formula

$$\det \mathbf{A} = a_{11}\, \boldsymbol{C}_{11} - a_{12}\, \boldsymbol{C}_{12} + \ldots. + (-1)^{1+n}\, a_{1n}\, \boldsymbol{C}_{1n}$$

The first expression on the last equality consists of expanding the determinant by the first column; the second expression expands by the first row. It can be proven that both expressions yield the same number. The determinant, in fact, can be calculated by expanding by any row or by any column.

The formula above implies that, to calculate a determinant of a matrix or order n, we have first to calculate n cofactors, which are determinants of order n-1. In turn, to calculate each of the cofactors, we have to calculate n-2 cofactors of order n-2, and so one. This means that calculation of determinants is an arduous, often unbearable task. As n increases, the amount of calculations grows exponentially. Fast computers nowadays can perform in relatively short of time the calculation of determinants of high order, higher than n = 100, which are not unusual in many applications. Fifteen years ago, however, that was an unfeasible job, and scientists have to resort to numerical analysis to find approximate solutions.

THEOREM, Crammer's rule. The solution of the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where A is an n-order square matrix (same number of equations and unknowns), is

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \ldots., \quad x_n = \frac{D_n}{D},$$

where D = det A and D_k is the determinant of the matrix obtained by replacing the k-th column of A by the column of the constant terms, $\boldsymbol{b}$.

THEOREM. For a square matrix **A** of order n the following three conditions are equivalent.

1. **A** is nonsingular
2. det A ≠ 0
3. rank A = n
4. $r = n$, where r is the number of nonzero rows of the reduced row-echelon matrix
5. The reduced row-echelon matrix is the identity **I**.
6. The homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.
7. The column vectors of **A** are linearly independent
8. The row vectors of **A** are linearly independent

Conversely,

THEOREM. For a square matrix A of order n the following three conditions are equivalent.

1. A is singular
2. det A = 0
3. $r < n$, where r is the number of nonzero row vectors of the reduced row-echelon matrix
4. The reduced row-echelon matrix is not the identity **I**.
5. rank A < n
6. The homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions.
7. The column vectors of **A** are linearly dependent
8. The row vectors of **A** are linearly dependent

THEOREM. For any two square matrices A and B of the same size,

$$\det(AB) = \det(BA) = \det A \det B$$

The inverse of a nonsingular matrix can be computed by using determinants

THEOREM. If A ($n\times n$) is nonsingular, then its inverse can be calculated by

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2N} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ij} are the cofactors explained above

SUMMARY OF CHAPTER 2

Linear Systems

A*x* = ***b***, A ($n\times m$), n unknowns and m equations. If r is the number of nonzero rows of B, the reduced row-echelon matrix of A, then

1. If B has a row of zeros in the coefficient matrix, and a nonzero number in the constant terms column, then the system is incompatible.

Otherwise, if no zero row in the coefficient matrix is followed by a nonzero constant term, then

2. If $r = n$ then the system has unique solution and the reduced row-echelon form of A, excluding the zero rows, is the identity.
3. If $r < n$ then the system has infinitely many solutions, with $n - r$ equal to the number of independent variables of the solution set

1. If $r < n$ then the system has infinitely many solutions, with $n - r$ equal to the number of independent variables of the solution set

Homogeneous linear systems

A*x* = **0**. The system is always consistent, having the trivial solution. If it admits a nontrivial solution, then it has infinitely many solutions.

Square linear systems

A*x* = ***b***, A ($n\times n$), n unknowns and n equations.
If A is nonsingular
1. It has unique solution
2. **A** is nonsingular
3. The solution is $A^{-1}\boldsymbol{b}$
4. The reduced row-echelon form of A is the identity
5. $\det A \neq 0$
6. rank A = n

Linear combinations of vectors.
Given n column vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_n$ with m components each, let A be ($m\times n$) matrix where the n columns of A are each of the n vectors $\boldsymbol{v}_i$, they are linearly independent if any of the following holds
1. A linear combination $c_1\,\boldsymbol{v}_1 + c_2\,\boldsymbol{v}_2 + \ldots c_n\,\boldsymbol{v}_n = \mathbf{0}$ if and only if all coefficients $c_i = 0$
2. rank A= min (n, m)
3. The homogeneous system A***x*** = **0** has unique solution (the trivial solution)

Given n row m-dimensional vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$, let A be ($n\times m$) matrix where the n rows of A are the n vectors $\boldsymbol{v}_i$. They are linearly independent if any of the following holds
1. A linear combination $c_1\,\boldsymbol{v}_1 + c_2\,\boldsymbol{v}_2 + \ldots c_n\,\boldsymbol{v}_n = \mathbf{0}$ if and only if all coefficients $c_i = 0$
2. rank A= min (n, m)
3. The homogeneous system **A*x*** = **0** has unique solution (the trivial solution)

Rank of a matrix.
Given an ($m\times n$) matrix A, rank A is
1. the maximum number of linearly independent column vectors and also maximum number of linearly independent row vectors.
2. the number of nonzero rows of the reduced row-echelon form of A

HOMEWORK

Kreyzsig, p. 277, # 1-8; p. 286, # 1-11; p. 301, # 1-12; p. 314, #5 – 10, 18-20; p. 322, #1-12

Additional problems

1. Given the matrices $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 4 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

and the vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \boldsymbol{v} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \quad \boldsymbol{r} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Determine which of the following operations are possible. If it is possible, perform it.

(a) A + B (b) B + C (c) 3A – 2B (d) AB (e) AC
(f) AD (g) $A^T B$ (h) BC (i) CB (j) A$\boldsymbol{u}$
(k) A$\boldsymbol{v}$ (l) $\boldsymbol{u} \bullet \boldsymbol{v}$ (dot product or inner product) (m) $\boldsymbol{u}^T \bullet \boldsymbol{v}$ (n) $\boldsymbol{u} + \boldsymbol{v}$

2. Given the vectors $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{a}_4 = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$

(a) Find $\boldsymbol{u}$ as the result of the linear combination $\boldsymbol{u} = 3\,\boldsymbol{a}_1 - \boldsymbol{a}_2 + 2\,\boldsymbol{a}_3$

(b) Find a non trivial linear combination such that $c_1\,\boldsymbol{a}_1 + c_2\,\boldsymbol{a}_2 + c_4\,\boldsymbol{a}_4 = 0$

(c) Find a non trivial linear combination such that $c_1\,\boldsymbol{a}_1 + c_2\,\boldsymbol{a}_2 + c_3\,\boldsymbol{a}_3 = \mathbf{0}$

3. Find A^{-1} if A is

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

4. Which of the matrices A in Problem 3 above are singular?

5. Calculate det A for each of the matrices A in Problem 3 above

6. Solve the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ for each of the matrices A of Problem 3 above.

7. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = (1, 3, -1)$ and A is each of the matrices in Problem 3.

8. For the following sets of vectors, determine whether they are *l.i.* or *l.d.*
(a) $\boldsymbol{v}_1 = (1, -1, -1, 1)^T$, $\boldsymbol{v}_2 = (1, 2, 1, -3)^T$, $\boldsymbol{v}_3 = (2, 1, 0, -2)^T$.
(b) $\boldsymbol{u}_1 = (1, 3, 2, 4)$, $\boldsymbol{u}_2 = (-1,42, 1, -1)$, $\boldsymbol{u}_3 = (0, 2, -1, -1)$.

9. Let A be the matrix obtained from Problem 8 part (a), where the columns are the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ and let B be the matrix obtained from part (b) where the rows are the vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$, find rank A and rank B

10. For the vectors of Problem 8, part (a) and part (b), find a nontrivial linear combination equal to zero: $c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + c_3 \boldsymbol{v}_3 = 0$ and $c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + c_3 \boldsymbol{u}_3 = 0$.

11. Find the rank of the following matrices

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad B=\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 5 \end{bmatrix},\quad D=C^T,\quad E=\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 3 & 8 \\ 1 & 5 & 11 \end{bmatrix},\quad F=\begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & 1 & -5 & 0 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$ 12.

Solve the homogeneous system H$\boldsymbol{x} = \mathbf{0}$, where H is each of the matrices of Problem 11.

13. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$ be the set of vectors defined by the column vectors each of the matrices of Problem 1. For each of these sets of vectors, determine whether they are linearly dependent or linearly independent. If they are linearly dependent, find a nontrivial linear combination of them equal to zero.