Chapter 1

Linear Systems of Equations

1.1 Introduction

An equation of the form $ax + b = 0$ is called linear equation in x. The name arises from the fact that the graph of $y = ax + b$ is a straight line. In general, a linear equation on n variables $x_1, x_2, \ldots, x_n$ is an equality containing a first degree polynomial of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Whenever using few variables, we avoid the use of subscripts and use the variables x, y, z, w instead, such as $3x - y + 2z = 6$.

Non linear equations are higher degree polynomial equations or equations including more complex algebraic expressions, such as

$$3x^3 - 2xy^3 + xy^2z^3 = 1 \quad 6^{\text{th}} \text{ degree polynomial}$$

$$\log x_1 - \sqrt{x_2^2 + 1} + \frac{1}{x_3 - 2} + \sin x_4 = 0 \quad \text{general}$$

Definition. A linear system of m equations on n unknowns contains n linear equations on m variables.

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots \; a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots \; a_{2n}x_n = b_2 \\ \text{---------------------------} \\ a_{m1}x_1 + a_{m2}x_2 + \ldots \; a_{mn}x_n = b_m \end{array} \qquad (1)$$

EXAMPLE 1.

$$\begin{array}{l} 3x - y + z = 4 \\ x + 2y - z = 1 \end{array}$$

linear system of two equations on three unknowns

DEFINITION. A ***solution*** **of the system** is a set of n numbers $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, called *n-tuple*, such that, when $x_1 = s_1$, $x_2 = s_2$, …. $x_n = s_n$, all the equalities are satisfied. Because systems of equations can have more than one solution, the set of all n-tuples that solve the system is called the ***solution set***.

In example 1, the triple (1, 1, 2) is a solution, since it satisfies both equations simultaneously.

EXAMPLE 2. The system

$$\begin{aligned} x + y + z &= 1 \\ x + y - z &= -1 \end{aligned}$$

has infinitely many solutions; the solution set is (t, $-t$, 1). For this, just replace x by t, y by $-t$ and z by 1.

THEOREM. For a linear system (and only for a linear system) one of the following three alternatives is true: The system has

(a) No solution (incompatible)
(b) One (unique) solution
(c) Infinitely many solutions (indeterminate)

In other words, it cannot have exactly two or exactly three solutions. Because linear systems of equations can be solved using matrices, we must study the properties of matrices. This will be done in the next chapter.

EXAMPLE 3. The system

$$\begin{aligned} x - 2y &= 4 \\ -3x + 6y &= -10 \end{aligned}$$

has no solution. To see this, multiply the first equation by 3 and add to the second equation, and we get 0 = 2, which is inconsistent. This is an incompatible system.

The system

$$\begin{aligned} x - y &= 0; \\ x+y &= 2 \end{aligned}$$

has unique solution $(x, y) = (1, 2)$. For this, adding the equations we get $2x = 2$, that is $x = 1, y = 1$.

The system

$$\begin{aligned} 2x - y &= 4 \\ -4x + 2y &= -8 \end{aligned}$$

has infinitely many solutions. Multiplying the first equation by 2 and adding to the second, we obtain 0 = 0, which, being true, does not provide a unique solution. The solution set is (t, $2t$ –4) where t is any real number ($-\infty < t < +\infty$). For instance, for $t = 2$, the triple (2, 0) is a solution (check it!). In general, replacing x by t and y by $2t$ we get

$$\begin{aligned} 2t - (2t-4) &= 4 \\ -4t + 2(2t - 4) &= -8 \end{aligned}$$

And both equalities are true.

The system showed above in (1) can alternatively be represented in a succinct way by the array

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \qquad (2)$$

where each row of the array corresponds one equation, and each of the first n columns to each of the n variables, while the last column corresponds to the constant terms. One such representation is called *the **augmented matrix of the system***. The first n columns alone, excluding the last column of constant terms, is called *the **coefficient matrix***.

In the next chapter we expand the notion of matrix and their properties. For the moment, all we have to know is that a matrix is a rectangular array of numbers in rows and columns.

EXAMPLE 4. The system of example 1 can alternatively be written in matrix format as

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 1 & 2 & -1 & 1 \end{array}\right]$$

1.2 Solving a Linear System of Equations

Two systems are equivalent if they have the same solution set. We solve a system by transforming it into an equivalent system whose solution is easier to find. The new, equivalent system is obtained in a series of steps, by applying the following operations, called ***elementary row operations***.

1. **Multiply an equation (row) by a nonzero constant**
2. **Interchange two equations (rows)**
3. **Add a multiple of one equation (row) to another**.

The two formats given in (1) and (2) both represent the same system of equations. So the same algorithm can be used with any of them to solve the system. The advantage of using the second format is to reduce the writing of unnecessary symbols. In the example below we illustrate the solution of a system by using both approaches.

Two similar methods, ***Gauss Elimination*** method, and the ***Gauss-Jordan Elimination*** method, are available for solving a system. Both perform the same initial steps; the latter execute a few more steps in the end. In this course we will use the second approach.

EXAMPLE. Solve the system

$$\begin{array}{ll} e_1 & x+y+2z = 9 \\ e_2 & 2x+4y-3z = 1 \\ e_3 & 3x+6y-5z = 0 \end{array} \qquad \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

The elimination method consists in several steps, eliminating one variable at a time. This is equivalent, on the matrix format, to producing zeros below the diagonal principal. We perform the steps by applying elementary the row operations as listed above.

First Step Make sure that the leading term of the first row is not zero. If it is, then swap rows suitably. Then produce zeroes below the leading term of the first row.

First step

$$\begin{array}{ll} nochange & x+y+2z = 9 \\ -2e_1+e_2 & 2y-7z = -17 \\ -3e_1+e_3 & 3y-11z = -27 \end{array} \qquad \begin{array}{c} r_1 \\ -2r_1+r_2 \\ -3r_1+r_3 \end{array}\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Second step, produce zeroes below the leading term of the second row

Second step

$$\begin{array}{ll} nochange & x+y+2z = 9 \\ nochange & 2y-7z = -17 \\ -3e_2+2e_3 & -z = -3 \end{array} \qquad \begin{array}{c} r_1 \\ r_2 \\ -3r_2+2r_3 \end{array}\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Stopping here solves the system by Gauss elimination. For the Gauss-Jordan method we proceed with a couple of extra steps, now to produce zeros whenever possible above the diagonal principal. Once the leading terms of all rows have been completed with zeroes below, start getting zeroes above the leading terms.

Third step, produce a zero above the leading term of the second row.

Third step

$$\begin{array}{lrrrl} 2e_1 - e_2 & 2x & & +11z & = 35 \\ e_2 - 7e_3 & & 2y & & = 4 \\ nochange & & & -z & = -3 \end{array} \qquad \begin{array}{r} 2r_1 - r_2 \\ r_2 - 7r_3 \\ r_3 \end{array} \begin{bmatrix} 2 & 0 & 11 & 35 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Produce zeroes above the leading term of the 3rd row.

Fourth step

$$\begin{array}{lrrrl} e_1 + 11e_3 & 2x & & & = 2 \\ nochange & & 2y & & = 4 \\ nochange & & & -z & = -3 \end{array} \qquad \begin{array}{r} r_1 + 11r_3 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

Fifth step

$$\begin{array}{lrrrl} (1/2)e_1 & x & & & = 1 \\ (1/2)e_2 & & y & & = 2 \\ -e_3 & & & z & = 3 \end{array} \qquad \begin{array}{r} (1/2)r_1 \\ (1/2)r_2 \\ -r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

And we reach the solution. This system has unique solution, namely (1, 2, 3). The method on the right is equivalent to the one on the left, and it is called the Gauss-Jordan elimination procedure. It consists in transforming the coefficient matrix into an equivalent matrix in the ***reduced row-echelon*** form. We will state below the general properties of a reduced row-echelon matrix, which is also the general method of finding the general solution of a linear system of equations. Note that the coefficient matrix in the last example was reduced to the ***Identity Matrix***, which is a square matrix with 1s on the diagonal principal and zeros elsewhere. The general solution of any linear system is somewhat more complicated, and uses the concept of row-echelon matrix.

1.3 Reduced Row-Echelon Matrices

The example given above corresponds to a very special system, which has equal number of equations and unknowns, and unique solution. To solve a system in general by the Gauss-Jordan elimination method, we proceed with the augmented matrix, reducing the coefficient matrix to its reduced row-echelon form. Any matrix can be transformed into such an echelon form by performing elementary row operations. Once this task is concluded, the problem of finding the solution of the system has been achieved. The ***leading term*** of a row is its first nonzero entry.

DEFINITION. A ***reduced row-echelon*** *matrix* is a matrix with the following properties:

1. The *leading term* of each nonzero row is 1.
2. The leading term of each row is to the right of the leading term of the row above it.
3. Every leading term is the only nonzero entry of its column
4. Rows consisting entirely of zeros are grouped together at the bottom of the matrix

The following are some examples of augmented matrices for which the coefficient matrix is in reduced row-echelon form

(a) $\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 1 & 0 & 5 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions

(b) $\left[\begin{array}{ccccccc|c} 1 & 3 & 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

one solution

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

no solution

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

one solution

1.3.2 Finding the solution. Once the coefficient matrix has been reduced to its roe-echelon format, the system is solved.

Rule 1. If a zero row in the coefficient matrix corresponds to a nonzero constant, the system has no solution.

Rule 2. Otherwise, if the nonzero rows of the coefficient matrix is the identity matrix, then the system has unique solution.

Rule 3. In all other cases the system has infinitely many solutions. In this case, the row-echelon matrix will have one or more columns with no leading terms.

For instance, in the matrix of example (a), each column corresponds to one unknown. Thus there are 6 unknowns, x_1, x_2, x_3, x_4, x_5, and x_6. Columns 4th and 6th, corresponding to x_4, and x_6, have no leading term, thus x_4, and x_6 are independent of "free" variables. We write $x_4 = t$ and $x_6 = s$, where t and s are variables taking any real value. The remaining variables are dependent variables. Thus the solution is as follows

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4=t & x_5 & x_6=s & \\ 1 & 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 1 & 0 & 5 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If the column has a leading term, then its corresponding variable is dependent. If the column does not have a leading term (such as the 4th and the 6th column, then the variable is independent

$$\begin{aligned} x_1 &= 1-4x_4-2x_6 \\ x_2 &= -1-5x_4+3x_6 \\ x_3 &= 3-3x_4+x_6 \\ x_5 &= -x_6 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= 1-4t-2s \\ x_2 &= -1-5t+3s \\ x_3 &= 3-3t+s \\ x_4 &= t \\ x_5 &= -s \\ x_6 &= s \end{aligned}$$

This system has infinitely many solutions. The variables t and s are called *free or independent variables*. For each arbitrary value of *t* and *s* we have a solution. This system thus admits two independent variables and it has four dependent variables.

The solution sets corresponding to the other matrices are

(b)
$$\begin{aligned} x_1 &= 1 - 3t - 2s + r \\ x_2 &= t \\ x_3 &= -2 - 3s - 2r \\ x_4 &= 2 + 2s \\ x_5 &= s \\ x_6 &= 4r \\ x_7 &= r \end{aligned}$$

This system has three independent variables and four dependent variables.

(c)
$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 2 \end{aligned}$$

This system has unique solution

(d) This system is incompatible. The fourth row contains only zeros for the coefficient part and a nonzero on the constant terms column. It makes this system incompatible since the fourth row produces 0 = 4, and so the inconsistency. Thus it has no solution.

(e) Unique solution $(x_1, x_2, x_3, x_4) = (3. 4, -2, 2)$

1.4 Systems With Solutions and Systems With No Solution

Every matrix can be transformed into its reduced row-echelon form by means of elementary row operations. We have seen before that if a linear system has more than one solution, then it has infinitely many solutions. How does the reduced-echelon form of one such system looks like? We define in the next chapter the ***Identity Matrix*** as a square matrix containing 1s in the ***diagonal principal*** (which is the diagonal that goes from the upper left corner to the lower right corner) and 0s everywhere else.

An $n \times n$ identity matrix is of the form
$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Once the reduced-echelon matrix has been obtained the system

1. **Has Unique Solution:** If the non zero rows of the reduced-echelon matrix is the identity matrix.
2. **Is incompatible**: If a row of the coefficient reduced-echelon matrix consists entirely of zeroes but the constant term of that row is not zero.
3. **Has Infinitely many solutions:** If the system is compatible, and the nonzero rows of the reduced-echelon matrix do not produce an identity matrix.

EXAMPLES.

$$(a)\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1\\0&0&1&2\\0&0&0&0\end{array}\right)\quad \textit{unique solution}\quad \begin{array}{l}x_1=0\\x_2=-1\\x_3=2\end{array}$$

$$(\mathrm{b})\left(\begin{array}{cccc|c}1&0&0&-1&1\\0&1&0&2&0\\0&0&1&1&2\\0&0&0&0&0\end{array}\right)\quad \begin{array}{c}\text{i nfinitely}\\\text{many}\\\text{solutions}\\(1+t,\text{-}2t,\ 2\text{-}t,\ t)\end{array}\quad \begin{array}{l}x_1=1+t\\x_2=-2t\\x_3=2-t\\x_4=t\end{array}$$

$$(\mathrm{c})\left(\begin{array}{cccc|c}1&0&0&2&3\\0&1&0&0&0\\0&0&1&-1&2\\0&0&0&0&3\end{array}\right)\quad \textit{no solution}$$

$$(d)\left(\begin{array}{ccccc|c}1&0&-1&0&0&3\\0&1&5&0&-2&0\\0&0&0&1&-3&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)\quad \begin{array}{c}\textit{in finitely}\\\textit{many}\\\textit{solutions}\end{array}\quad \begin{array}{l}x_1=3+t\\x_2=-5t+2s\\x_3=t\\x_4=1++3s\\x_5=s\end{array}$$

1.5 Homogeneous Linear Systems

A linear system of equations is ***homogeneous*** if the column of constant terms is 0. In display

$$\begin{array}{lcl}a_{11}x_1+a_{12}x_2+\dots\ a_{1n}x_n&=&0\\a_{21}x_1+a_{22}x_2+\dots\ a_{2n}x_n&=&0\\\text{-----------------------------}&=&0\\a_{m1}x_1+a_{m2}x_2+\dots\ a_{mn}x_n&=&0\end{array}$$

Homogeneous systems are always consistent, since *the trivial* solution, $x_1 = 0$, $x_2 = 0$, $\dots x_n = 0$, or (0, 0, … 0) is always possible. Hence a homogeneous system either admits only the trivial solution or it has infinitely many other solutions, called ***nontrivial solutions***. In other words, homogeneous systems of equations are never incompatible. In particular, if a homogeneous system admits one nontrivial solution, then it also admits infinitely many solutions.

Homogeneous systems of equations play an important role in the development of linear algebra, so they receive special attention.

Examples. The system

$$\begin{array}{r}x+y-2z=0\\3x-2y\ -z=0\\4x-2y-2z=0\end{array}$$

Admits the nontrivial solution (1, 1, 1), therefore it has infinitely many solutions. In fact, the solution set is given by $x = t$, $y = t$, $z = t$ or (t, t, t). To verify this, just substitute x, y, and z by t.

The system

$$\begin{array}{l}x+y=0\\x-y=0\end{array}$$

admits only the trivial solution (0, 0). To see this, adding both equation gives $2x = 0$, thus $x = 0$ and $y = 0$ is only possible solution. Alternatively, the reduced row-echelon matrix of this system is the identity. The following are the steps to reduce the coefficient matrix to its row-echelon format.

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\Rightarrow\begin{array}{c}r_1\\r_1-r_2\end{array}\begin{pmatrix}1&1\\0&2\end{pmatrix}\Rightarrow\begin{array}{c}r_1-r_2/2\\r_2/2\end{array}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Note that here we omitted the column of constant terms, since it is composed only of zeroes.

HOMEWORK

Krayszig (9th ed.) p.295, #1- 16

additional problems

1. Given the following augmented matrices, determine whether the coefficient matrix is in the reduced echelon form. If it is not, answer why not (what of the 4 rules have been broken) and continue the transformation until the coefficient matrix is in the reduced echelon form. Finally, for those that are in the reduced echelon form, write the solution set,

(a) $$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) $$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 1 & 0 & 5 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) $$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 0 & -4 & -1 \\ 0 & 1 & 0 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(d) $$\left[\begin{array}{ccccc|c} 1 & 5 & 2 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 5 \\ 0 & 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(e) $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

2. Solve the following systems by the Gauss-Jordan elimination method. Determine the solution set and how many independent variables it contains

(a) $$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 & = & -3 \\ -2x_1 + x_2 & = & -5 \\ x_1 + x_2 + x_3 & = & 2 \\ 5x_2 - x_3 & = & -6 \end{array}$$

(b) $$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 & = & -3 \\ -2x_1 + x_2 & = & -5 \\ x_1 + x_2 + x_3 & = & 2 \end{array}$$

(c) $$\begin{array}{rcl} x_1 + x_2 + 2x_3 - x_4 & = & 1 \\ 2x_1 - x_2 - x_3 + 2x_4 & = & 7 \\ -x_1 + x_3 - 3x_4 & = & -2 \\ 2x_1 + 2x_3 - 2x_4 & = & 6 \end{array}$$

(d) $$\begin{array}{rcl} x_1 + 2x_2 - x_4 & = & 3 \\ x_1 + 3x_2 + x_3 + 2x_4 & = & -2 \\ 2x_1 + x_2 + 3x_3 - x_4 & = & 0 \\ 4x_1 + 6x_2 + 4x_3 & = & 1 \\ 3x_1 + 4x_2 + 4x_3 + x_4 & = & -2 \end{array}$$

(e) $$\begin{array}{rcl} x_1 + x_4 & = & -1 \\ -x_1 + x_2 + 2x_3 + x_4 & = & 1 \\ -x_1 + x_2 + 2x_3 + x_4 & = & 1 \end{array}$$

(f) $$\begin{array}{rcl} x_1 + x_2 + 2x_3 & = & 2 \\ -x_1 + x_2 - 2x_3 & = & 0 \\ 2x_1 + x_2 + 3x_3 & = & 7 \end{array}$$

(g) $$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 & = & 0 \\ 2x_1 + 2x_2 - x_3 & = & 0 \\ -x_1 - x_3 & = & 0 \\ 2x_1 + x_3 & = & 0 \end{array}$$

SOLUTION TO SELECTED PROBLEMS

2 (a)
$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= -3 \\ -2x_1 + x_2 \phantom{{}-2x_3} &= -5 \\ x_1 + x_2 + x_3 &= 2 \\ 5x_2 - x_3 &= -6 \end{aligned}$$

$$\begin{pmatrix} 1 & 3 & -2 & -3 \\ -2 & 1 & 0 & -5 \\ 1 & 1 & 1 & 2 \\ 0 & 5 & -1 & -6 \end{pmatrix} \Rightarrow \begin{array}{r} r_1 \\ 2r_1 + r_2 \\ -r_1 + r_3 \\ r_4 \end{array} \begin{pmatrix} 1 & 3 & -2 & -3 \\ 0 & 7 & -4 & -11 \\ 0 & -2 & 3 & 5 \\ 0 & 5 & -1 & -6 \end{pmatrix} \Rightarrow \begin{array}{r} r_1 \\ r_2 \\ 2r_2 + 7r_3 \\ -5r_2 + 7r_4 \end{array} \begin{pmatrix} 1 & 3 & -2 & -3 \\ 0 & 7 & -4 & -11 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 13 & 13 \end{pmatrix} \Rightarrow$$

$$\begin{array}{r} r_1 \\ r_2 \\ r_3 \\ r_3 - r_4 \end{array} \begin{pmatrix} 1 & 3 & -2 & -3 \\ 0 & 7 & -4 & -11 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} r_1 \\ r_2 \\ r_3/13 \\ r_4 \end{array} \begin{pmatrix} 1 & 3 & -2 & -3 \\ 0 & 7 & -4 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} 7r_1 - 3r_2 \\ r_2 + 4r_3 \\ r_3 \\ r_4 \end{array} \begin{pmatrix} 7 & 0 & -2 & 12 \\ 0 & 7 & 0 & -7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{array}{r} r_1 + 2r_3 \\ r_2 + 4r_3 \\ r_3 \\ r_4 \end{array} \begin{pmatrix} 7 & 0 & 0 & 14 \\ 0 & 7 & 0 & -7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} r_1/7 \\ r_2/7 \\ r_3 \\ r_4 \end{array} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because the nonzero rows of reduced row-echelon matrix is the identity, the system has unique solution, which is $(x_1, x_2, x_3) = (2. -1, 1)$