Chapter 4

SOLVING EQUATIONS

4.1 Solving Polynomial Equations by Iteration

Iteration method is used to find roots of polynomials of degree larger than 2. Let $f(\mathrm{x})$ be a polynomial of degree larger than 2. We are interested in solving the equation

$$f(\mathrm{x}) = 0$$

We transform the equation algebraically into an equation of the form $x = g(x)$. Then we choose an initial point x_0, and compute

$$x_1 = g(x_0)$$

$$x_2 = g(x_1)$$

and continue by an iterative process to calculate

$$x_{n+1} = g(x_n)$$

until achieving the require accuracy. This process is called *fixed point iteration*. The name has no relationship whatsoever with the definition of fixed point representation of numbers given in the previous section.

EXAMPLE . Solve the equation $$f(x) = x^4 + 3x - 1 = 0 \qquad (6)$$

We can write $$x = \frac{1 - x^4}{3} = g(x) \qquad (7)$$

Then we pick $x_0 = 0$, and we get

$$x_1 = g(x_0) = g(0) = 0.33333333333333$$

Next,

$$x_2 = g(x_1) = g(0.33333333333333) = 0.3291810699588$$

Continuing in this way the iterative process, we obtain

$$x_3 = 0.32941759557524 \quad x_4 = 0.32940809603398 \quad x_5 = 0.32940854878868$$

The correct value of the root, with 14 significant digits is 0.32940852819255, that is, x_5 yields a value of the root with a precision of 7 correct decimals. We can stop when the difference $|x_n - x_{n-1}|$ is less than a given positive number β

REMARKS. The selection of the initial point is important to insure the convergence to the root. For instance, suppose that we chose $x_0 = 3$ rather than 1. Then the iterative process diverges to infinity.

There is no unique way to solve problems of this type. For instance, dividing equation (1) by x we get

$$x^3 + 3 - \frac{1}{x} = 0 \Rightarrow x = \frac{1}{x^3 + 3}. \tag{8}$$

Repeating the iterative process for this new equation, starting again with $x_0 = 1$, we obtain the same value for the root. A third way to solve the equation is

$$x = \sqrt[4]{1 - 3x} \tag{9}$$

even though this last is less interesting, since it can lead to imaginary solutions. Also

$$x = \frac{1 - 3x}{x^3} \tag{10}$$

is a valid equation.

The speed of the convergence to the value of the root depends of two things: a good choice of the initial point and a good choice of the equation $x = g(x)$.

Choosing the initial point. The initial point must be chosen to be a point near a root. Because $f(0) = -1$, $f(1) = 3$, and the function is continuous, we know that there is a root between 0 and 1. This is why the point 0 was chosen as initial point. In fact, any point in the interval [0, 1] would produce a similar result. By the graphing theory of functions covered in Mathematics I, we can always have some rough idea of the location of the roots of the equation.

The calculations in this example were performed using MATLAB. In format long, MATLAB provides results with 16 significant digits.

In the previous example we have seen that finding an algorithm that converges quickly to a root depends of the choice of the equation $x = g(x)$ and also of the initial point. Convergence to a root is not always guaranteed. The following theorem provides sufficient conditions for convergence.

THEOREM. Let s be a solution of $x = g(x)$. If g has continuous derivative on an interval J containing s, and if $|g'(x)| \le K < 1$ in J, then the iteration process defined above converges to some point in J.

EXAMPLE . Given the polynomial $p(x) = x^5 - x^3 + 4x - 3$, because $p(0) = -3$ and $p(1) = 1$, we know that p has a root in the interval (0, 1). Find it with an approximation of 6 decimals.

The equation $x = g(x) = \dfrac{3 - x^5 + x^3}{4}$ satisfies the assumptions of the theorem on the interval [0,1]. Indeed, $|g'(x)| = |-(1.25)x^4 + (0.75)x^2| \le 0.5 < 1$ on the interval [0,1]. Starting with $x_0 = 1$, we obtain,

$$x_1 = 0.750000 \quad x_2 = 0.796142, \quad x_3 = 0.796193$$

The actual root, with 6 significant digits is

$$s = 0.796192$$

So the third iteration produces a solution with 5 correct decimals, and an error bound equal to $(0.5)\times 10^{-5}$. The ninth iteration produces s up to all 15 digits provided by MATLAB. By standard methods on continuous functions, it is easy to see that all the other four roots of $p(x)$ are complex roots.

4.2 Solving Functional Equations, Taylor's Formula

Taylor's and MacLaurin's formulae are used to approach a function by polynomials. It is useful to evaluate functions. Let $f(x)$ is sufficiently differentiable on certain interval containing the point a, then f can be expressed by Taylor's formula as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is an intermediate point between x and a. If we expand up to the n-th derivative, neglecting the last term, the error is given by

$$|\varepsilon| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \right| < \frac{M}{(n+1)!}(x-a)^{n+1}$$

where M is a bound for the $(n+1)$-th derivative of $f(x)$ on the interval $[a, x]$. If $a = 0$ the expansion is called MacLaurin series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

EXAMPLE. Find the MacLaurin's expansion of $f(x) = e^x$ up to the fifth derivative.

First, calculate the derivatives of f.

	f	f'	f''	f'''	$f^{(4)}$	$f^{(5)}$
$f^{(n)}(x)$	e^x	e^x	e^x	e^x	e^x	e^x
$f^{(n)}(0)$	1	1	1	1	1	1

Applying MacLaurin's formula $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \ldots..$

EXAMPLE. Find the MacLaurin's expansion of $f(x) = \sin x$.

	f	f'	f''	f'''	$f^{(4)}$	$f^{(5)}$
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$
$f^{(n)}(0)$	0	1	0	-1	0	1

Therefore, the MacLaurin's expansion of $\sin x$ is

$$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots\ldots$$

EXAMPLE. Using MacLaurin's expansion of $f(x) = e^x$ to approximate the value of $\sqrt[3]{e}$ (expand up to the fifthe derivative).

Because $\sqrt[3]{e} = e^{1/3}$, considering the function $f(x) = e^x$ and expanding around the point $a = 0$ and with the value $x = 1/3$ in MacLaurin's formula, expanding up to the 5th derivative using the previous example above

$$\sqrt[3]{e} \approx 1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3 3!} + \frac{1}{3^4 4!} + \frac{1}{3^5 5!} \approx 1.395610$$

The error is bounded by $\frac{1}{3^6 6!} < 2\times10^{-6}$. The correct value, is 1.395612…..

Some algorithms may be inefficient if the convergence is slow.

EXAMPLE. Because $\tan(\pi/4) = 1$, then $\pi = 4\times\text{Arctan } 1$ (the inverse tangent function). The MacLaurin series of the inverse tangent function, is

$$f(x) = \text{Arctan } x = x - 1/3\, x^3 + 1/5\, x^5 - 1/7\, x^7 + \ldots\ldots$$

Thus

$$\pi = 4 \text{ Arctang } x \approx 4\,(1 - 1/3 + 1/5 - \ldots..)$$

Expanding this series up to the 19th derivative we obtain $\pi \approx 3.04$, an unacceptable value.

4.3 The Newton-Raphson Method

Is another iteration method used to solve functional equations of the form $f(x) = 0$. In section 4.1 we have seen a method to find roots of polynomial equations. The method in this section applies to any functional equation. If f has continuous derivative f', then

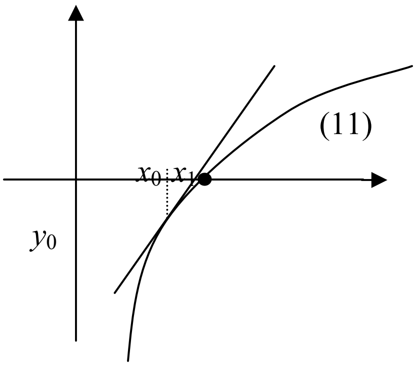
$$\frac{dy}{dx}(x_0) = f'(x_0) \Rightarrow dy = f'(x_0)\,dx \Rightarrow$$
$$y_1 - y_0 = f'(x_0)(x_1 - x_0) \Rightarrow$$
$$y_1 = f(x_0) + f'(x_0)(x_1 - x_0)$$

and, because for small values of dy we have $dy \approx \Delta y$, we can approximate the value of the solution $y_1 = 0$.

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

and then by iteration

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}} \tag{11}$$



EXAMPLE. Set up an iteration using the Newton-Raphson method for computing the natural logarithm of numbers, and apply it to calculate ln (0.5).

In general, suppose that we want to calculate ln c for a given value of c (in this example, c = 0.5). Let ln $c = x$. It follows that $c = e^x$, therefore $e^x - c = 0$, so we set the function $f(x) = e^x - c$ and calculate the root of it, $f(x)=0$. In this example we have

$$\ln 0.5 = x$$

we want to calculate the value of x. For this, write

$$e^x = 0.5 \;\Rightarrow\; e^x - 0.5 = 0.$$

Making $f(x) = e^x - 0.5$, we must find the root of $f(x)$ using the Newton-Raphson. For this $f'(x) = e^x$.

$$x_n = x_{n-1} - \frac{e^{x_{n-1}} - 0.5}{e^{x_{n-1}}}$$

Applying (11), the iteration is

$$x_n = x_{n-1} - 1 + (0.5)e^{-x_{n-1}}$$

Starting with $x_0 = 0$, we obtain (choosing the right initial point is crucial, it will be given to you in the exam for tne Newton-Raphson method)

x_1 = -0.50000000000000 x_2 = -0.67563936464994
x_3 = -0.69299480927629 x_4 = -0.69314716895203
x_5 = -0.69314718055995

The value of x_5 is the correct value up to all 14 decimals. It is crucial that the starting point, x_0, be sufficiently close of the actual solution, or the iteration may diverge to infinity. In the exam, you will be asked to stop when the difference between $|x_n - x_{n-1}|$ is less than a given number (10^{-e}).

EXAMPLE. Find all the real roots of $x^3 + 2x - 2 = 0$ with an error bound of 10^{-6} by using the Newton-Raphson method.

Because $f'(x) = 3x^2 + 2 > 0$ for all real x, and since $\lim_{x \to -\infty} f(x) = -\infty$ *and* $\lim_{x \to +\infty} f(x) = +\infty$, the function has only one real root. And $f(0) = -2$, $f(1) = 1$ implies that the real root is in the interval (0, 1). Newton's algorithm yields

$$x_n = x_{n-1} - \frac{x_{n-1}^3 + 2x_{n-1} - 2}{3x_{n-1}^2 + 2} = \frac{2x_{n-1}^3 + 2}{3x_{n-1}^2 + 2}$$

Starting from $x_0 = 0.5$, we get (details left to the student to complete).

4.4 Speed of Convergence

We have seen before that some algorithms are efficient, which means that in a few steps can be obtained a good approximation. Others are not. The speed of convergence is related to the *order of an iteration.*

Let an iteration be of the form $x_{n+1} = g(x_n)$ and let s be a solution of $x = g(x)$. Because the error is $\varepsilon = s - x_n$, then $x_n = s - \varepsilon$. Expanding g in his Taylor's series around the point s, assuming that g is sufficiently differentiable, we get

$$x_{n+1} = g(x_n) = g(s) + \frac{g'(s)}{1!}(x_n - s) + \frac{g''(s)}{2!}(x_n - s)^2 + \cdots$$
$$= g(s) + \frac{g'(s)}{1!}\varepsilon + \frac{g''(s)}{2!}\varepsilon^2 + \cdots$$

if $\varepsilon < 1$, then $\varepsilon > \varepsilon^2 > \varepsilon^3$ …. , that is, the terms of the Taylor series become smaller.

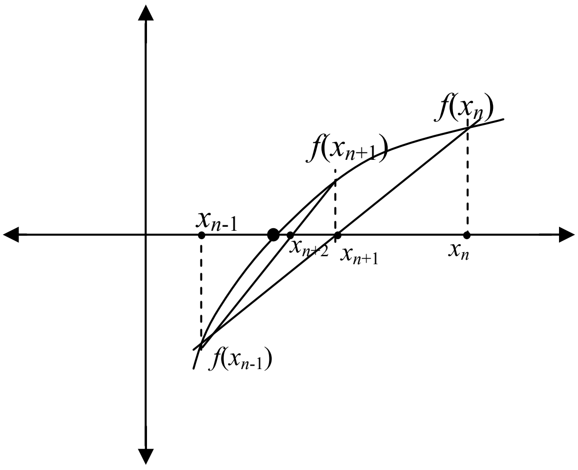
DEFINITION. *If an iteration is such that* $g'(s) = g''(s) = g^{n-1}(s) = 0$, *and* $g^{(n)}(s) \neq 0$, *then the order of the iteration is n. In other words, the order of the iteration is the first nonzero derivative of* $g(x)$.

As the order of the iteration is larger, then the speed of convergence is faster. To see this, subtract $s = g(s)$ in both sides of the last equation, and assume that the iteration is of order k. It follows that

$$-\varepsilon_{n+1} = x_{n+1} - s = g(x_n) - s = \frac{g^{(k)}(s)}{k!}\varepsilon_n^k + \frac{g^{(k+1)}(s)}{k!}\varepsilon_n^{k+1} + \cdots + |\varepsilon_{n+1}| < \frac{M}{k!}\varepsilon_n^k \qquad (12)$$

and the speed of convergence is larger as k becomes larger.

Newton's method is, in general, of second order. Indeed, $g(x) = x - \frac{f(x)}{f'(x)}$. Differentiating and evaluating at $x = 0$, $g'(s) = \frac{f(s)f''(s)}{\left(f'(s)\right)^2}$ because $f(s) = 0$. Differentiating again, $g''(s) = \frac{f''(s)}{f'(s)}$, which in general is not zero. If $f''(s) \approx 0$, then $g''(s)$ is large, and because of (12), the error may be also large.



EXAMPLE. Apply Newton's method to solve $f(x) = 0$ where $f(x) = e^{(x-1)^2} - 2x + x^2$.

We know that one root is $x = 1$. But because the derivative

$$f'(x) = 2(x-1)\left[e^{(x-1)^2} - 1\right]$$

is zero at $x = 1$, applying iteration (11) produces a divergent sequence, even if starting with points very close to 1. In this case, f' converges very fast to zero as x approaches 1 (the order of convergence of f' is 3). But with the function $f(x) = e^{x-1} - x$, which is in similar conditions as the one above, that is, $f(1) = f'(1) = 0$, (11) produces convergence to the solution $x = 1$, since the order of convergence of $f'(x) = e^{x-1} - 1$ is much slower (order 1). The convergence, however, is pretty slow, and to obtain a good approximation it is necessary to perform more than 20 iterations.

4.5 Secant Method for Solving Equations

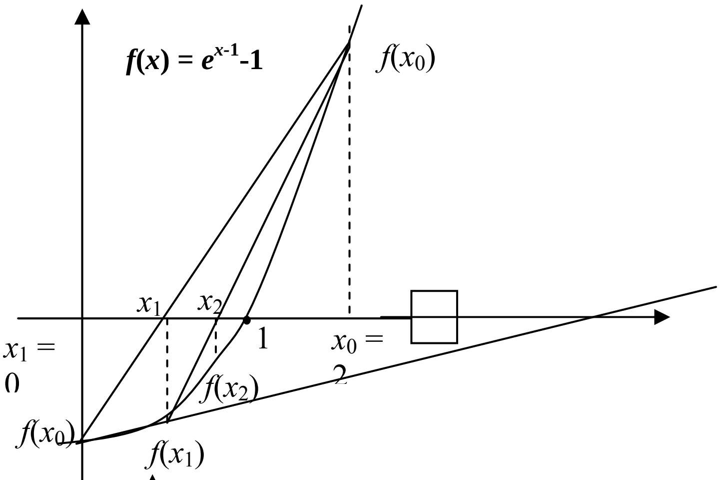
Another method, similar to the previous one, is the method of secants. Suppose that we know that the root is between two points, x_0 and x_1. We take the secant that joints the points ($x_0, f(x_0)$) and (x_1, $f(x_1)$), which intersects the x–axis at a point x_2. We continue the process iteratively, always joining the secant between the last two points found. By simple algebraic manipulations, taking the secant between the points ($x_{n-1}, f(x_{n-1})$) and ($x_n, f(x_n)$), and intercepting with the x-axis yields a new point x_{n+1} given by the equation

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

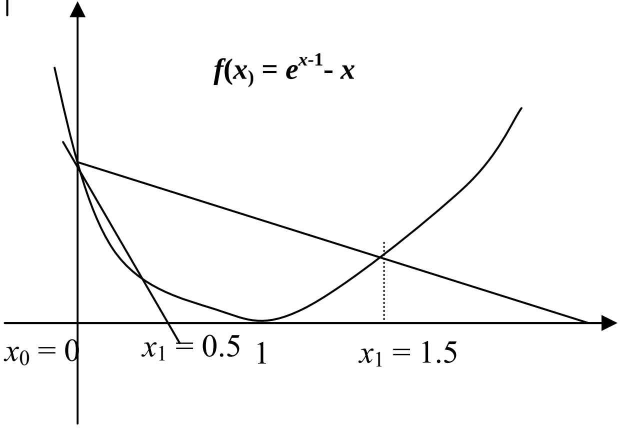
EXAMPLE . The function $f(x) = e^{x-1} - 1$ is zero at x = 1. The first and second derivatives are always positive. $f(0) < 0$ and $f(1) > 1$. We pick $x_0 = 2$ and $x_1 = 0$. The iteration produces the following result

n	x_n	$f(x_n)$
1	0.53788284273999	-0.37005146742556
2	0.79696916802073	-0.18374692566023
3	0.91318887099416	-0.08314975430148
4	0.96335346187496	-0.03598318167161
5	0.98461697707659	-0.01526530859996
6	0.99355825987227	-0.00642103659923
7	0.99887317434704	0.99730522369965
8	0.99952890005699	-0.00269114864999
9	0.99980305866255	-0.00112619102333
10	0.99991767208317	-4.709889928579125e-004

Because the first and second derivatives are positives, the general shape of the graph is known, and we pick the initial points each ate a different side of the root. Notice also that, in this case, the quotient $\frac{f''(x)}{f'(x)} > 0$, so if we apply Newton's method, we should pick the initial point on the right of the root, and not on the left, to insure the convergence in the case of Newton's method.



A combination of the secant method and Newton's method is often applied. Notice that the tangent intercept the x-axis on the left of the root, and the secant on the right. So, the midpoint between these to points can be used as x_n.
If we want to apply the secant method to the function in Example 9, because the derivative is zero at the root, when applying the secant method, we must pick the two initial points on the same side of the root, as we can see in the next graph.



4.6 Lagrange Interpolation

Given a function $f(x)$ and a set of points x_0, x_1, …. x_n , taken inside some interval J, there is a polynomial $p(x)$ of degree n such that $f(x_i) - p(x_i) = 0$ at all these $n+1$ points. That polynomial is know to approach the function inside J. Because the format of that polynomial is long to write (it is in the book), we provide here an example with $n = 3$. The generalization to more points is straightforward. Let x_0, x_1, x_2, x_3, be four points in an interval J where the function $f(x)$ is differentiable $n+1$ times. Then the Lagrange interpolating polynomial that coincides with f at all these 4 points is

$$\begin{aligned} p(x) = {} & f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ & + f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \end{aligned}$$

HOMEWORK

1. Find the number of significant digits of the following numbers

(a) 185.00101 (b) 0.00134510 (c) 0.00014500 (d) 1.00000

2. Represent the following numbers in floating point scheme using the given number of significant digits and find bounds for the error β and the relative error β_r.

(a) 175.031 6 significant digits (b) 0.000012540000 7 significant digits

(c) 1000000000 9 significant digits (d) 1000000000 5 significant digits

3. Given $x_1 = 21.1358$, $x_2 = 2.142999$, $x_3 = 0.001243780$, $x_4 = 122.0101010101$
Rounding the numbers to 4 significant digits, let $S = x_1 + x_2 + x_3 + x_4$. Find the error bound β for S.

4. Find the roots of $x^2 + 6x + 1 = 0$ with an approximation or 5 correct decimals using the iteration procedure covered in section 3. (hint: first solve the quadratic equation and find, using a calculator, the roots with an approximation of at least 6 decimals. Then choose the correct starting points. For the root between –5 and –6, divide the equation by x^2 and use $x = -6x^2/(1+x^{2-})$

5. Let $f(x) = x^4 - 8x^3 - 14x^2 - 9x - 15$.

(a) Evaluate $f(x)$ at the points -1.32, -3.12, 2.21, -1.98, -2.731, -.97, 1.13
(b) Find one root of $f(x)$ by using the method of iteration. Utilize the result of (a) to determine a suitable initial point. Stop iteration when $| x_n - x_{n-1} | < 10^{-4}$.

6. Use the MacLaurin's expansion of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

to approximate the value of sin(0.5). Expand up toe the 5th derivative.

7. (a) Find the Taylor's expansion around $a = 1$ of $f(x) = \ln x$, expand up to the 6th derivative.
(b) Use the result on (a) to approximate the value of ln 1.3.

8. (a) Find the MacLaurin expansion of $f(x) = e^x$ up to the fourth derivative.
(b) Using the result of (a), to approximate the value of $\sqrt[5]{e}$ (expand the series up to the fourth derivative.)

9. Find the Taylor's expansion of $f(x) = \ln x$ around the point $x = 1$.

10. (a) Find MacLaurin's expansion of $f(x) = \cos x$ up to the seventh derivative
(b) Use MacLaurin's expansion of $f(x) = \cos x$ to approximate the value of cos(0.23)

11. Set up an iteration using the Newton-Raphson method to approximate the value of $\sqrt[3]{x}$ and compute $\sqrt[3]{5}$. Stop your iterative process when $| x_n - x_{n-1} | < 10^{-4}$.

12. Set up an iteration using the Newton-Raphson method to approximate the value of ln(1.35). Stop the iterative process when $|x_n - x_{n-1}| < 10^{-4}$.

13. Which of the following functions are not suitable for applying Newton's method.

(a) $f(x) = \ln\left(\dfrac{1}{x-0.9}\right) - x^2 + 2x$ on the solution near $x = 2$

(b) $f(x) = \sin(x - \pi) + 1$ on the solution near $x = \pi/2$

(c) $f(x) = e^{2x-1} - 1$ on the solution near $x = 1/2$

14. Apply the method of the secant with 3 iterations to calculate the root of $f(x) = x^3 - 3x + 2.1$ on the solution near $x = 1$

15. Given the function $f(x) = e^x$ and the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, construct the Lagrange interpolating polynomial through these points.

SOLUTION

3. Given $x_1 = 21.1358$, $x_2 = 2.142999$, $x_3 = 0.001243780$, $x_4 = 122.0101010101$
Rounding the numbers to 4 significant digits, let $S = x_1 + x_2 + x_3 + x_4$. Find the error bound β for S.

Rounding to 4 significant digits
$x_1 = 0.2114E02$, $x_2 = 0.2143E01$, $x_3 = 0.1244E\text{-}02$, $x_4 = 0.1220E03$

$$\beta_1 = \frac{1}{2}10^{-2}, \quad \beta_2 = \frac{1}{2}10^{-3}, \quad \beta_3 = \frac{1}{2}10^{-6}, \quad \beta_4 = \frac{1}{2}10^{-1}$$

Hence the error bound for the sum is the sum of the error bounds

$$\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{1}{2}\left(10^{-2} + 10^{-3} + 10^{-6} + 10^{-1}\right) = \frac{1}{2} \times 0.111001$$

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Coded Instructions Used With MATLAB To Solve Some Examples

Example 3

```
>>  format long
>>  x = 0
>> for i = 1:5, x = (1-x^4)/3, end
```

Example 4

```
>> x =  1
>> for i = 1:5, x = (3 - x^5 + x^3)/4, end
```

Example 7

```
>> x =  0
>>  for i = 1:5, x = x - 1 + 0.5*exp(-x) , end
```

Example 10

```
>> x2 = 2
>>  f2 = exp(1) - 1
>>  x1 = 0
>> f1 = exp(-1) - 1
>> f1 = exp(-1) - 1
>> for i = 1:10, x1 = x1 - f1*(x2-x1)/(f2-f1),
   f1 = exp(x1-1)- 1, end
```