Chapter 2

THE FOURIER TRANSFORM

The Fourier transform is the generalization of the Fourier series to functions not necessarily periodic.

2.1 Countable and Uncountable Sets

A set S is countable (or denumerable) if a one-to-one correspondence can be established between S and the set of natural numbers. Otherwise the set is uncountable.

Examples of countable sets are

The set of integers $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

The set of positive rational numbers $Q = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots\right\}$

Examples of uncountable sets are

The set of irrational numbers

Any bounded or unbounded interval of the real line, such as $[a, b] = \{x \in \Re : a \le x \le b\}$

We can sum the elements of a countable set. For instance, the sum of the elements of the set of rational numbers of the form $1/n^2$, $n \in \mathbb{N}$, is known to be $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. The elements of an uncountable set can't be summed. The generalization of the sum for uncountable sets is the integral. So, for the set $S = \left\{1/x^2 \,:\, x \in \Re,\ x > 1\right\}$ we can calculate the integral $\int_1^{\infty} 1/x^2 \, dx = 1$.

2.2 Complex Form of the Fourier Series

From Euler's formula, $e^{ix} = \cos x + i \sin x$, it readily follows the following

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \quad \textit{and} \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

Given a piecewise continuous, one-hand sided differentiable periodic function $f(t)$ of period 2π, its Fourier series can be written in complex form as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{int} + e^{-int}\right) + \frac{b_n}{2i}\left(e^{int} - e^{-int}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{int}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + e^{-int}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)\right] = a_0 + \sum_{n=1}^{\infty}\left(c_n e^{int} + k_n e^{-int}\right) = \sum_{n=-\infty}^{\infty}\hat{f}_n e^{int}$$

Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty}\hat{f}_n e^{int} \qquad (1)$$

where the coefficients $\hat{f}_n$ are calculated using the formulae from Fourier series theory covered in Chapter 1, which, by straightforward application of Euler's formula, omitting details, yield

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \qquad (2)$$

and this is the complex form of the Fourier Series

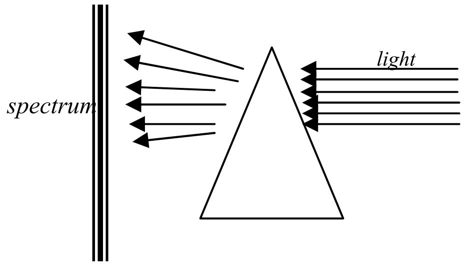
2.3 General Trigonometric Series

A pure color is a periodic electromagnetic wave of sinusoidal form

$$f(t) = \hat{f}_1(w)\cos wt + \hat{f}_2(w)\sin wt \qquad (3)$$

where w is the frequency, $1/w$ the wavelength (period), and $\hat{f}(w)$ the amplitude or the intensity of light (energy). A beam of light $f(t)$ of a given color is the superposition of an uncountable set of wavelengths (colors), each with a specific amplitude. Because we can't sum uncountable sets, the resultant $f(t)$ can be calculated as the integral of all these sinusoids.

When a beam of white light passes through a prism, the light is diffracted into the colors of the spectrum, characterized by (3), where w ranges on the interval of visible light, $[a, b]$, $w \in \mathbb{R}$, from red, the minimal visible frequency (thus the longest wavelength) through violet, the highest visible frequency (and shortest wavelength). The function $\hat{f}(w)$ is the spectrum, which is the intensity of the light at frequency w. For white light the spectrum is almost constant, $\hat{f}(w) = k$. The rainbow, therefore, which is diffracted white light, consists of infinitely many pure colors of similar intensity. In general a given color $f(t)$ is a superposition of a continuous range of wavelengths, and it is obtained integrating (3) with respect to w on the interval of visible light. Similar considerations also apply for radio frequencies and other types of electromagnetic signals.



2.4 The Fourier Transform

Now we are in condition to define the Fourier Transform

DEFINITION. Let $f(x)$ be a function satisfying

(a) $f(x)$ is piecewise continuous on $(-\infty, \infty)$

(b) $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists, that is, the integral is defined and finite

Then the Fourier transform of $f(x)$, denoted by $\hat{f}(w)$ and also by $\Im(f)$, is given by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx \qquad (4)$$

THEOREM. If $\hat{f}(w)$ is the Fourier transform of $f(x)$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\,e^{iwx}\,dw \qquad (5)$$

In this case $f(x)$ is called the inverse Fourier transform of $\hat{f}(w)$ and is denoted $\Im^{-1}\left[\hat{f}(w)\right]$

REMARKS.

- The function $f(x)$ is not required to be differentiable or periodic, as with the Fourier series.
- The Fourier transform is a generalization of the Fourier series theory to a broader class of absolutely integrable functions on the real line.
- The Fourier transform is the spectral representation of a signal with continuous infinitely many frequencies.
- Notice that formulas (4) and (5) generalize (1) and (2) covered in the Fourier series part.

2.5 Properties of the Fourier Transform

By direct use of the definition and integration it is easy to prove that the Fourier Transform has the following properties: Given two functions, $f(x)$ and $g(x)$,

1. $\mathfrak{F}(af + bg) = a\,\mathfrak{F}(f) + b\,\mathfrak{F}(g)$ a and b constant numbers
2. $\mathfrak{F}(f') = iw\,\mathfrak{F}(f)$
3. $\mathfrak{F}(f'') = -w^2\,\mathfrak{F}(f)$

4. $\mathfrak{F}[f(x-a)] = e^{-iaw}\,\mathfrak{F}(f(x))$
5. $\mathfrak{F}[f(ax)] = (1/a)\,\mathfrak{F}[f(x/a)]$

EXAMPLE. Find, by direct application of the definition (integrate), the Fourier transform of

$$f(x) = \begin{cases} k & if \quad 0 < x < a \\ 0 & otherwise \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{k}{\sqrt{2\pi}}\int_{0}^{a} e^{-iwx}dx = -\frac{k}{iw\sqrt{2\pi}}e^{-iwx}\Big|_{0}^{a} = \frac{k}{iw\sqrt{2\pi}}\left(1-e^{-iwa}\right)$$

EXAMPLE. Find the Fourier transform of

$$f(x) = \begin{cases} k & if \quad -a < x < a \\ 0 & otherwise \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{k}{\sqrt{2\pi}}\int_{-a}^{a} e^{-iwx}dx = \frac{-k}{iw\sqrt{2\pi}}e^{-iwx}\Big|_{-a}^{a} = \frac{2k}{w\sqrt{2\pi}}\left(\frac{e^{iwa}-e^{-iwa}}{2i}\right) = \sqrt{\frac{2}{\pi}}\,\frac{k\sin aw}{w}$$

EXAMPLE. Using the table, find the Fourier transform of $f(x) = \dfrac{1}{x^2+9}$

Direct observation of the table yields

$$\Im(f) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-3|w|}}{3}$$

EXAMPLE. Using the table, find the Fourier transform of $f(x) = \dfrac{1}{x^2-4x+5}$

Completing the square,

$$f(x) = \frac{1}{(x-2)^2+1}$$

. Let $g(x) = \dfrac{1}{x^2+1}$. Then $g(x-2) = \dfrac{1}{(x-2)^2+1} = f(x)$

Using property 4 above and the table of Fourier transforms, we have,

$$\mathfrak{F}(f) = \mathfrak{F}[g(x-2)] = e^{-2iw}\,\mathfrak{F}[(g(x)] = e^{-2iw}\left(\sqrt{\frac{\pi}{2}}\,\frac{e^{-|w|}}{1}\right) = \sqrt{\frac{\pi}{2}}\,e^{-2iw-|w|}$$

EXAMPLE. Find the Fourier transform of $f(x) = xe^{-3x^2}$.

First, observe that if $g(x) = -\frac{1}{6}e^{-3x^2}$ then $g'(x) = xe^{-3x^2}$

Applying property 2, $\Im(g') = iw\Im(g) = iw\Im\left(e^{-3x^2}\right)$

The table of Fourier transforms, gives us

$$\Im\left(xe^{-3x^2}\right) = iw\Im\left(-\frac{1}{6}e^{-3x^2}\right) = -\frac{iw}{6}\Im\left(e^{-3x^2}\right) = -\frac{iw}{6\sqrt{6}}e^{-w^2/12}$$

EXAMPLE. Find the Fourier transform of $f(x) = x^2e^{-x^2}$.

$$\left(e^{-x^2}\right) = (4x^2-2)e^{-x^2} \quad\Rightarrow\quad x^2e^{-x^2} = \frac{1}{4}\left(e^{-x^2}\right)'' + \frac{1}{2}e^{-x^2}.$$

From property 3 and the table of integrals, we conclude

$$\hat{f}(w) = \Im\left(x^2e^{-x^2}\right) = \frac{1}{4}\Im\left[\left(e^{-x^2}\right)''\right] + \frac{1}{2}\Im\left(e^{-x^2}\right) = \Im\left(e^{-x^2}\right)\left(\frac{-w^2}{4} + \frac{1}{2}\right) = \frac{2-w^2}{4\sqrt{2}}e^{-w^2/4}$$

2.6 Trigonometric form of the Fourier Transform

Applying Euler's formula to equations (4) and (5) we have the corresponding trigonometric version of the Fourier transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)(\cos wx - i\sin wx)\,dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)(\cos wx + i\sin wx)\,dw$$

In particular, if *f* is even, its Fourier transform is a pure cosine integral, and it is a real-valued function. If *f* is odd, the Fourier transform is a pure sine integral, and it is a pure imaginary function. The functions given in examples 2 and 5 are even. Thus in Example 2, we may also proceed in the following way

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos wx\;dx = \frac{2k}{\sqrt{2\pi}}\int_{0}^{a}\cos wx\,dx = -\frac{2k}{w\sqrt{2\pi}}\sin wx\Big|_0^a = \sqrt{\frac{2}{\pi}}\,\frac{k\;\sin aw}{w}$$

The function in Example 4 is odd, and its Fourier transform is a pure imaginary function, which can be alternatively calculated as a pure sine integral

$$\hat{f}(w) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-3x^2}\sin wx\;dx$$

The details are left to the student

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EXAMPLE. Find the Fourier transform of

$$f(x) = \begin{cases}\cos ax & if \quad -\pi < x < \pi \\ 0 & otherwise\end{cases} \qquad a \neq 0$$

Because the function is even, we get

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos wx \; dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi} \cos ax \cos wx \; dx$$

Because 2(cos ax)(cos wx) = cos $x(a+w)$ + cos $x(a-w)$, we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi} \left[\cos x(a+w) + \cos x(a-w)\right] dx = \frac{\sin \pi(a+w)}{\sqrt{2\pi}(a+w)} + \frac{\sin \pi(a-w)}{\sqrt{2\pi}(a-w)} \qquad \text{if } w \neq \pm a,$$

$$\hat{f}(a) = \hat{f}(-a) = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{\sin 2a\pi}{2a}\right)$$

2.7 Table of Fourier Transforms

	$f(x)$	$\hat{f}(x) = \Im(f)$
1	$\begin{cases} 1 & \textit{if } -b < x < b \\ 0 & \textit{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
2	$\begin{cases} x & \text{if } \ 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$e^{-iwa}\left(\frac{ai}{w} + \frac{1}{w^2}\right) - \frac{1}{w^2}$
3	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\|w\|}}{a}$
5	$\begin{cases} e^{-ax} & \text{if } \ x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
9	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\|w\| < a$; 0 if $\|w\| > a$

INTEGRAL FORMULAS

$$\int xe^{-iwx}\,dx = e^{-iwx}\left(\frac{xi}{w} + \frac{1}{w^2}\right)$$

HOMEWORK

1. Find the Fourier transform of the following functions by direct integration, without using the table.

(a) $f(x)=\begin{cases}2 & \text{if} \quad 0<x<1\\ 0 & \text{otherwise}\end{cases}$ (b) $f(x)=\begin{cases}x & \text{if} \quad 0<x<1\\ 0 & \text{otherwise}\end{cases}$

(c) $f(x)=\begin{cases}e^{-x} & \text{if} \quad 1<x<3\\ 0 & \text{otherwise}\end{cases}$

2. Find the Fourier transforms of the following functions by using the table of integrals

(a) $f(x)=\dfrac{1}{x^2+2x+10}$ (b) $f(x)=\begin{cases}e^{-2x+4} & if \;\; x>2\\ 0 & otherwise\end{cases}$

(c) $f(x)=\begin{cases}-1 & if \quad -1<x<0\\ 1 & if \quad 0<x<1\\ 0 & otherwise\end{cases}$ (d) $f(x)=\begin{cases}x & \text{if} \quad 0<x<1\\ 0 & \text{otherwise}\end{cases}$

(e) $f(x)=\dfrac{x}{x^4+2x^2+1}$ (*hint: f is the derivative of a function given in the table* (f) $f(x)=xe^{-2x^2}$

(g) $f(x)=x^2e^{-2x^2}$ (h) $f(x)=\begin{cases}x & if \;\; -\pi/2<x<\pi/2\\ 1 & if \quad \pi/2<x<\pi\\ -1 & if \quad -\pi<x<-\pi/2\\ 0 & \qquad otherwise\end{cases}$

(h) $f(x)=\begin{cases}\sin ax & if \quad -\pi<x<\pi\\ 0 & otherwise\end{cases}$ $a\neq 0$

(i) $f(x)=\dfrac{\sin 3x}{x}-\dfrac{2}{x^2+25}$ if $x>0$, $-\dfrac{2}{x^2+25}$ otherwise

SOLUTIONS

Problem 1.

(a) $f(x)=\begin{cases} 2 & \text{if } \; 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$\Im(f)=\frac{2}{\sqrt{2\pi}}\int_0^1 e^{-iwx}dx=-\frac{2}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{iw}\right]_0^1=\frac{2}{iw\sqrt{2\pi}}\left(1-e^{-iw}\right)$$

(b) $f(x)=\begin{cases} x & \text{if } \; 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

Using the integral formula above we get

$$\Im(f)=\frac{1}{\sqrt{2\pi}}\int_0^1 xe^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\left[e^{-iwx}\left(\frac{ix}{w}+\frac{1}{w^2}\right)\right]_0^1=\frac{1}{\sqrt{2\pi}}\left[e^{-iwx}\left(\frac{i}{w}+\frac{1}{w^2}\right)-\frac{1}{w^2}\right]$$

(c) $f(x)=\begin{cases} e^{-x} & \text{if } \; 1<x<3 \\ 0 & \text{otherwise} \end{cases}$

$$\frac{1}{\sqrt{2\pi}}\int_1^3 e^{-x}e^{-iwx}\,dx=\frac{1}{\sqrt{2\pi}}\int_1^3 e^{-x(iw+1)}\,dx=-\frac{1}{\sqrt{2\pi}(iw+1)}e^{-x(iw+1)}\bigg|_1^3=-\frac{e^{-3(iw+1)}-e^{-(iw+1)}}{\sqrt{2\pi}(iw+1)}$$

Problem 2. Use the table of Fourier transforms to evaluate the Fourier transforms of

(a) $f(x)=\dfrac{1}{x^2+2x+10}$

Completing the square $f(x)=\dfrac{1}{(x+1)^2+9}=g(x+1)$, with $g(x)=\dfrac{1}{x^2+3^2}$. Hence, using property 4 and table

$$\hat{f}(w)=\Im(f)=\Im[g(x+1)]=e^{iw}\Im(g)=\sqrt{\frac{\pi}{2}}\frac{e^{-3|w|+iw}}{3}$$

(b) $f(x)=\begin{cases} e^{-2x+4} & if \;\; x>2 \\ 0 & otherwise \end{cases}$

Let $g(x)=e^{-2x}\;(x>0)$ then $f(x)=e^{-2x+4}=e^{-2(x-2)}=g(x-2), \quad (x>2)$

Using again property 4 and table,

$$\hat{f}(w)=\Im(f)=\Im[g(x-2)]=e^{-2wi}\Im[g(x)]=\frac{e^{-2wi}}{\sqrt{2\pi}(2+iw)}$$

(c) Since *f* is odd, it is a pure sine transform

$$\hat{f}(w) = \frac{-i}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\sin wx\,dx = \frac{-2i}{\sqrt{2\pi}}\int_{0}^{1}\sin wx\,dx = \sqrt{\frac{2}{\pi}}\,\frac{i(\cos w-1)}{w}$$

(d) By direct integration by parts

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{0}^{1} xe^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left[-\frac{xe^{-iwx}}{iw} - \frac{e^{-iwx}}{(iw)^2}\Big|_0^1\right] = \frac{1}{\sqrt{2\pi}}\left(-\frac{e^{-iw}}{iw} + \frac{e^{-iw}}{w^2} - \frac{1}{w^2}\right)$$

(e) $f(x) = \dfrac{x}{x^4+2x^2+1}$

If $g(x) = -\dfrac{1}{2(x^2+1)}$ then $g'(x) = \dfrac{x}{(x^2+1)^2} = f(x)$. Applying property 2 together with the table, we get

$$\Im(f) = \Im(g') = iw\Im(g) = -\sqrt{\frac{\pi}{2}}\,\frac{iwe^{-|w|}}{2}$$

(f) $f(x) = xe^{-2x^2}$

We need to find *g*(*x*) such that *g*'(*x*) = *f*(*x*), in other words, we must find the antiderivative of *f*.

$$g(x) = -\frac{1}{4}e^{-2x^2}$$

Therefore, the answer is

$$-\frac{1}{4}iw\Im\left(e^{-2x^2}\right) = \frac{1}{4}iw\frac{1}{2}e^{-iw^2/8} = \frac{1}{8}iwe^{-iw^2/8}$$

(g) Is similar to Example 5. The proper adjustments are left to the student.

(h) and (h) are odd functions and the Fourier transforms can easily be obtained as a pure sine integral