# Chapter 1

# FOURIER SERIES

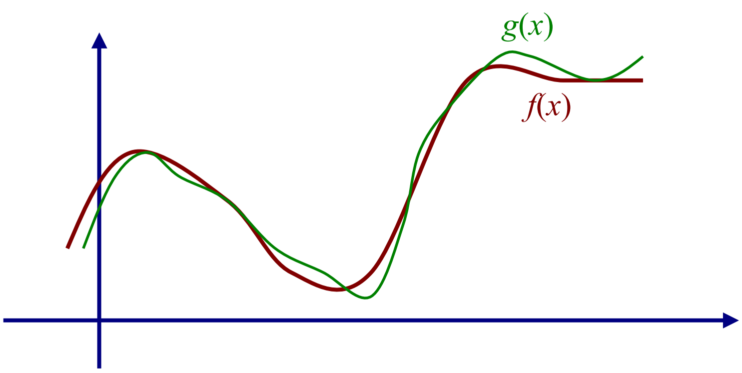
## 1.1 Introduction

One important area of Numerical Analysis deals with the approximation of functions by polynomials or by series of well-known functions. Several methods of approximation of functions are available. In Mathematics I you have seen the approximation of a function by a Taylor Polynomial. Later on in this course you will se the Lagrange's Interpolation Theorem, which approximates functions by interpolating polynomials, who coincide with the function at a set of given points. Fourier series deals with periodic functions and how to approximate them by a trigonometric series. The first two chapters of this second part are concerned with Fourier Series and with the Fourier Transform. The rest of the course will be consecrated to the study of techniques that facilitate the calculation of long tedious numerical computations.

**1.1.1 Approximation**. Sometimes calculating the values of a function $f(x)$ can lead to long, tedious calculations. Suppose that the values of $f$ can be approximated by values of another function, $g(x)$, which is not equal to $f(x)$ but approximates it, and is such that we can easily find the values of $g$. Suppose that we can allowed a small error, $\varepsilon > 0$ in our calculations, and let $g(x)$ such that

$$| f(x) - g(x) | < \varepsilon$$

that is, the error we make by using the values of $g$ rather than $f$ is less than $\varepsilon$.




There are several ways of approximating functions, among them

(1) By Polynomials. This is Taylor's polynomials, studied in MATH 1. Let $f(x)$ be a function. We want to find a polynomial $p(x)$ that approximates it

$$f(x) \approx p(x)$$

(2) By Fourier Series. The function $f(x)$ is approximated by a Fourier Series $F(x)$ such that

$$f(x) \approx F(x)$$

where the sign means "approximately".

## 1.2 Periodic Functions

Periodic functions repeat their behavior on intervals of equal size. Thus the study of a periodic function along the real axis can be restricted to its study on one period interval, since in the rest of the line just the function repeats itself.

**DEFINITION.** A function $f(x)$ is called ***periodic*** if it is defined for all real $x$, except perhaps at some given points, and is such that for some number $p$, called ***period***, $f$ satisfies

$$f(x + p) = f(x)$$

for all $x$.

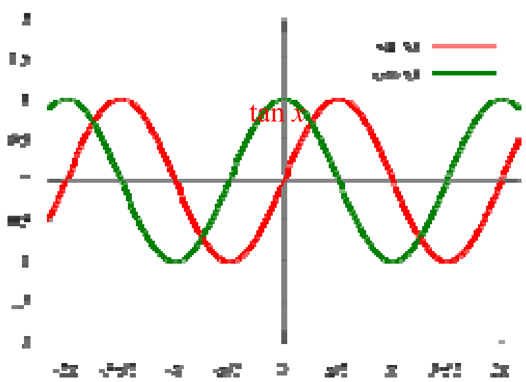
So,

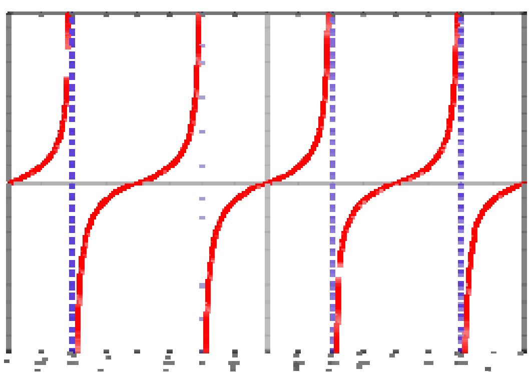
$$f(x) = f(x + p) = f(x + 2p) = f(x + 3p) \;.....$$

$$f(x) = f(x - p) = f(x - 2p) = f(x - 3p) \;.....$$

If a function is periodic, the period is not unique. A function periodic of period $p$ is also periodic of period $2p$, $3p$, etc.

**EXAMPLE.** The function $f(x) = \sin x$ is periodic of period $p = 2\pi$, because $\sin(x) = \sin(x+2\pi)$ for all real values of $x$. Also, $\sin(x+4\pi) = \sin(x)$, so $p = 2\pi, 4\pi, \;....\; 2n\pi$ are also periods of the function. The function $f(x) = \cos x$ is also periodic of period $2\pi$, and $f(x) = \tan x$ is periodic of period $p = \pi$, and it is not defined at the points $k(\pi/2)$ for any integer $k$.

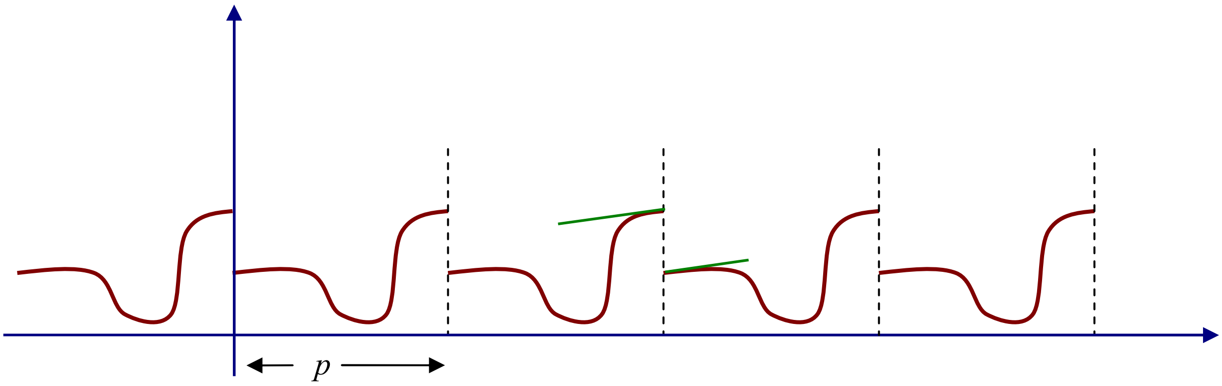


Hence, if $f(x)$ is periodic of period $p$, we have

$$f(x + np) = f(x)$$

for any integer $n$. In general, the graph of a periodic functions looks like this




# 1.3 General Trigonometric Series

A series, as defined in Mathematics I, is the summation of infinitely many terms. Let $a_n$ be an infinite sequence of numbers.

$$a_1,\ a_2,\ a_3,\ ....a_n,......$$

Then, by definition, the series S, which is the sum of the infinitely many elements of the sequence, is given by

$$S = a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

A trigonometric series is a series whose elements are functions of sin $x$ and cos $x$.

**EXAMPLE**. Let $a_n = \frac{1}{n^2}$, that is $a_n = 1, \frac{1}{2^2}, \frac{1}{3^2}, .....$ then $S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + .... = \frac{\pi^2}{6}$

$a_n$ is the sequence, and $S$ consists of adding all infinite terms of the sequence together.

**DEFINITION**. Given two sequences $a_n$ and $b_n$, a trigonometric series whose coefficients are the elements of the sequences is given by

$$F(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ... + a_n \cos nx + b_n \sin nx + ... \quad (1)$$

A trigonometric series is therefore given once the two sequences, $a_n$ and $b_n$ are given. The trigonometric series above has period $2\pi$; a more general form of trigonometric series of any period will be studied in the next section. Periodic functions can be approximated by partial sums of Fourier series.

# 1.4 Functions of Period $2\pi$

We will see in this section how to approximate a function of period $2\pi$ by a trigonometric series of the form (1)

As a reminder, a function is piecewise continuous if it is continuous if it is continuous on intervals, except perhaps at the border points of the intervals, but it is continuous from the left at the right endpoint and continuous from the right at the left end point of each interval. And a function is one-hand side differentiable, if the derivative from the left and the derivative from the right exist at any point. The example on the graph above, shows a piecewise on-hand sided differentiable function.

**THEOREM**. Let $f$ be piecewise continuous, one-hand side differentiable, and periodic of period $2\pi$, then $f(x)$ can be represented by a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

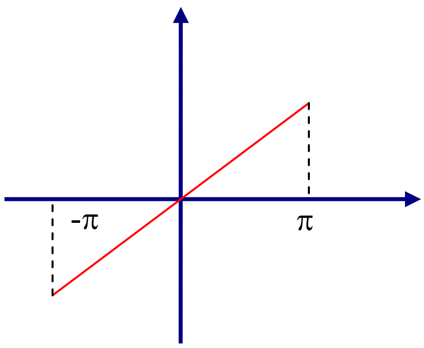
where the coefficients $a_n$ and $b_n$ are calculated as

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\quad n = 1, 2, \cdots \tag{3}$$

According to the definition of series given above, (2) is equivalent to

$$\lim_{N\to\infty}\left( f(x) - a_0 - \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)\right) = 0$$

**EXAMPLE**. Let $f(x) = x$ if $-\pi \le x < \pi$, and $f(x+2\pi) = f(x)$.




The coefficients $a_n$ and $b_n$ are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \left.\frac{x^2}{4\pi}\right|_{-\pi}^{\pi} = 0 \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx\,.$$

Integrating by parts,

$$\int x\cos nx\,dx = \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$$

which yields

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left(\frac{\pi\sin n\pi}{n} + \frac{\pi\sin(-n\pi)}{n}\right) + \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{\cos(-n\pi)}{n^2}\right) = 0$$

Because $\sin x = -\sin(-x)$ and $\cos x = \cos(-x)$.

On the other hand, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$

$$\int x\sin x\,dx = -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}$$

Therefore

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right|\right)_{-\pi}^{\pi} = -\frac{2\cos n\pi}{n\pi}$$

because $\sin n\pi = 0$ for all $n$ and $\cos n\pi = \begin{cases} -1, & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

$$b_n = -\frac{\cos n\pi}{n\pi} = \begin{cases} 2/n\pi & n \text{ odd} \\ -2/n\pi & n \text{ even} \end{cases}$$

$$f(x) = \frac{2}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - ....\right)$$

# 1.5 Even and Odd Functions

As a reminder, a function $f(x)$ is said to be ***even*** if $f(x) = f(-x)$ for all $x$ and $f(x)$ is ***odd*** if $f(x) = -f(-x)$ for all $x$. We can use this result to simplify our calculations if a function is either even or odd. We have the following

$$g(x)h(x) \begin{cases} \textbf{even} & \text{if both g and h are even or both are odd} \\ \textbf{odd} & \text{if one is even and the other is odd.} \end{cases}$$

$\sin x$ is odd, $\cos x$ is even $x^n$ is even if $n$ is even and *odd* if $n$ is odd

A general rule, useful to simplify calculations is

$$\int_{-a}^{a} f(x)\,dx = 0 \quad f(x)\ odd$$
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad f(x)\ even$$

RULE. If $f(x)$ is even, then all terms $b_n$ are zero, and if $f(x)$ is odd, all terms $a_n$ are zero, and we have

$$f \text{ even} \quad \Rightarrow \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

this means that, if $f$ is even, all terms $b_n$ are zero.

$$f \text{ odd} \quad \Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

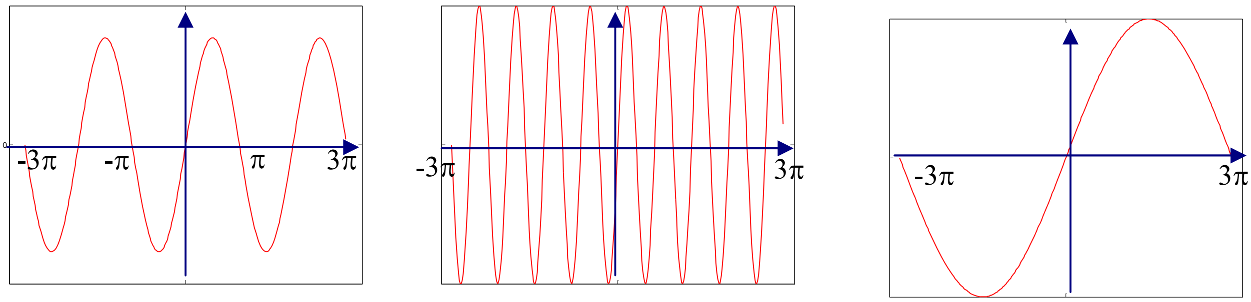
this means that, if $f$ is odd, all terms $a_n$ (including $a_0$, are zero)

# 1.6 Functions of Any Period 2L

More generally, Let now $f(x)$ be periodic of period $2L$. We can approximate it by a Fourier series of period $2L$.

**THEOREM.** If $f(x)$ is periodic of period $p$ and $k$ is a real number then the function $g(x) = f(kx)$ is periodic of period $p/k$

**EXAMPLE.** The function $f(x) = \sin x$ is periodic of period $2\pi$. Then $g(x) = \sin 3x$ is periodic of period $2\pi/3$ and $h(x) = \sin(x/3)$ is periodic of period $6\pi$.




(a) $f(x) = \sin x$
$p = 2\pi$

(b) $g(x) = \sin 3x$
$p = 2\pi/3$

(c) $h(x) = \sin(x/3)$
$p = 6\pi$

If $f$ is periodic of any period $2L$, then the function

$$g(x) = f(Lx/\pi)$$

is periodic of period $2\pi$. Substituting in the last theorem, becomes

**EXAMPLE.** The function $f(x) = \sin(x/3)$ is periodic of period $6\pi$, because $f(x) = f(x+6\pi)$, Indeed

$$\sin\left(\frac{x}{3}\right) = \sin\left(\frac{x+6\pi}{3}\right) = \sin\left(\frac{x}{3} + 2\pi\right)$$

**THEOREM.** Let $f$ be piecewise continuous, one-hand sided differentiable, and periodic of period $2L$, then $f(x)$ can be represented by a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (4)$$

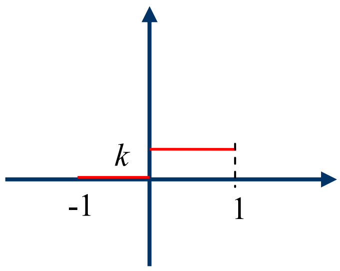
with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \cdots \qquad (5)$$

**EXAMPLE.** Find the Fourier series of

$$f(x) = \begin{cases} 0 & if \quad -1 < x \le 0 \\ k & if \quad 0 < x \le 1 \end{cases} \qquad f(x+2) = f(x)$$

$2L = 2 \quad \Rightarrow \quad L = 1$. Thus



$$a_0 = \frac{k}{2}\int_0^1 dx = \frac{k}{2}, \quad a_n = k\int_0^1 \cos n\pi x\, dx = \frac{k}{n\pi}\sin n\pi x\Big|_0^1 = 0, \quad n = 1, 2\cdots$$

$$b_n = k\int_0^1 \sin n\pi x\, dx = \frac{-k}{n\pi}\cos n\pi x\Big|_0^1 \Rightarrow b_n = \begin{cases} \dfrac{2k}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}.$$

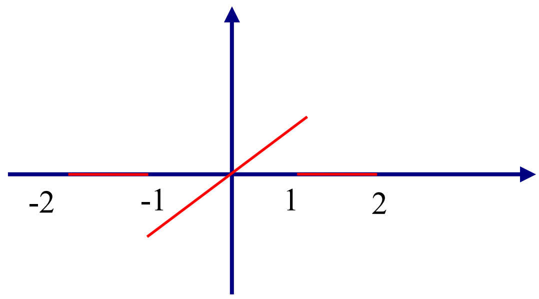
Hence

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\cdots\right)$$

**EXAMPLE**. Find the Fourier series of

$$f(x) = \begin{cases} 0 & if \quad -2 < x \le -1 \;\; or \;\; 1 < x \le 2 \\ x & if \quad -1 < x \le 1 \end{cases} \qquad f(x+4) = f(x)$$

$2L = 4$, so $L = 2$. Since the function is odd, $a_n = 0$, and, integrating by parts,




$$b_n = \frac{1}{2}\int_{-1}^{1} x \sin\frac{n\pi x}{2}\, dx = \frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2}\Bigg|_{-1}^{1} = \frac{-\cos n\pi}{n\pi}$$

and we have

$$f(x) = \frac{1}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \cdots\right)$$

**EXAMPLE**. Find the Fourier series of

$$f(x) = \begin{cases} \cos wx & if \quad \dfrac{-\pi}{2w} < x < \dfrac{\pi}{2w} \\ 0 & if \quad \dfrac{-\pi}{w} < x < \dfrac{-\pi}{2w}, \;\; or \;\; \dfrac{\pi}{2w} < x < \dfrac{\pi}{w} \end{cases} \qquad f(x + 2\pi / w) = f(x)$$

$L = 2\pi/w$. Since the function is even, $b_n = 0$, and, $a_0 = \dfrac{w}{\pi}\displaystyle\int_0^{\pi/2w} \cos wx\, dx = \dfrac{1}{\pi}$

Using the formula $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ we can calculate $a_n$

$$a_n = \frac{w}{2\pi}\int_{-\pi/2w}^{\pi/2w} \cos wx \, \cos nwx \, dx = \frac{w}{2\pi}\int_0^{\pi/2w}\left[\cos(n+1)wx + \cos(n-1)wx\right]dx$$

$$= \frac{w}{2\pi}\left[\frac{\sin(n+1)wx}{(n+1)w} + \frac{\sin(n-1)wx}{(n-1)w}\right]_0^{\pi/2w} = \frac{1}{2\pi}\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)}\right]$$

If $n$ is odd, $a_n = 0$. If $n$ is even, we have $a_n = \dfrac{(-1)^{n/2}}{\pi}\left[\dfrac{1}{n-1} + \dfrac{1}{n+1}\right] = \dfrac{(-1)^{n/2}2}{\pi(n+1)(n-1)}$

Therefore, $f(x) = \dfrac{1}{\pi} + \dfrac{2}{\pi}\left(\dfrac{\cos 2wx}{1\cdot 3} - \dfrac{\cos 4wx}{3\cdot 5} + \dfrac{\cos 6wx}{5\cdot 7} - \cdots\right)$

# 1.7 Integral Formulas

1. $\int \sin nx \, dx = \frac{-\cos nx}{n}$ $\int \cos nx \, dx = \frac{\sin nx}{n}$

2. $\int \sin \frac{n\pi x}{L} dx = -\frac{L}{n\pi} \cos \frac{n\pi x}{L}$ $\int \cos \frac{n\pi x}{L} dx = \frac{L}{n\pi} \sin \frac{n\pi x}{L}$

3. $\int x \cos nx \, dx = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2}$ $\int x \sin nx \, dx = -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}$

4. $\int \sin kx \, dx = \frac{-\cos knx}{kn}$ $\int \cos kx \, dx = \frac{\sin knx}{kn}$

5. $\int x \sin \frac{n\pi x}{L} dx = -\frac{L}{n\pi} x \cos \frac{n\pi x}{L} + \frac{L^2}{(n\pi)^2} \sin \frac{n\pi x}{L}$

6. $\int x \cos \frac{n\pi x}{L} dx = \frac{L}{n\pi} x \sin \frac{n\pi x}{L} + \frac{L^2}{(n\pi)^2} \cos \frac{n\pi x}{L}$

## Other formulas

$\sin x = -\sin(-x)$ $\cos x = \cos(-x)$

$$\sin n\pi = 0 \qquad \cos n\pi = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$$

# HOMEWORK

Textbook Section 10.3, problems 1-12

Additional problems

Find the Fourier series representation of the following functions

(1) $f(x) = x + 2$ *if* $-2 < x \le 0$, $f(x) = 2 - x$, *if* $0 < x \le 2$ , $f(x+4) = f(x)$

(2) $f(x) = 0$, $(-1 < x < 0)$ ; $f(x) = x$, $(0 < x < 1)$ $f(x+2) = f(x)$

(3) $f(x) = x$, $(0 < x < 1)$ ; $f(x) = 1 - x$, $(1 < x < 2)$ $f(x+2) = f(x)$

(4) $f(x) = x + \sin wx$ *if* $\frac{-\pi}{2w} < x < \frac{\pi}{2w}$, $f(x) = 0$ *if* $\frac{-\pi}{w} < x < \frac{-\pi}{2w}$ *or* $\frac{\pi}{2} < x < \pi$, $f\left(x + \frac{2\pi}{w}\right) = f(x)$

(Hint: split the function $f(x)$ in two, $f_1(x) = x$ and $f_2(x) = \sin wx$, and use the formula $2\sin a \sin b = \sin(a+b) + \sin(a-b)$ and the fact that both functions are odd)

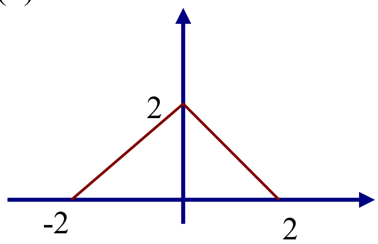
(5) $f(x) = x^2$ *if* $-1 < x < 1$, $f(x) = 0$ *if* $-2 < x < -1$ *or* $1 < x < 2$, $f(x+4) = f(x)$

(6) $f(x) = \begin{cases} 0 \ \ if \ \ -2 < x < -1 \ \ or \ 1 < x < 2 \\ 1 - x \quad if \quad -1 < x < 0 \\ 1 + x \quad if \quad 0 < x < 1 \end{cases}$ Hint: the function $g(x) = f(x) - 1$ is odd

# SOLUTIONS

(1) Find the Fourier series representation of the following functions

$f(x) = x + 2 \ if \ -2 < x \le 0, \quad f(x) = 2 - x, \ if \ 0 < x \le 2, \quad f(x+4) = f(x)$




Because the function is even, $b_n = 0$. $L = 2$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{2}(2-x)\,dx = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \int_{0}^{2} f(x)\cos\frac{n\pi x}{2}dx = \int_{0}^{2}(2-x)\cos\frac{n\pi x}{2}dx$$

$$a_n = 2\int_{0}^{2}\cos\frac{n\pi x}{2}dx - \int_{0}^{2} x\cos\frac{n\pi x}{2}dx = \left[\frac{4}{n\pi}\sin\frac{n\pi x}{2} - \left(\frac{2}{n\pi}x\sin\frac{n\pi x}{2} + \frac{2^2}{(n\pi)^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{4}{n\pi}\sin n\pi - \left(\frac{4}{n\pi}\sin n\pi + \frac{4}{(n\pi)^2}\cos n\pi - \frac{4}{(n\pi)^2}\right) = -\frac{4}{(n\pi)^2}(\cos n\pi - 1)$$

$$a_n = \begin{cases} 0 & n \quad even \\ 8/(n\pi)^2 & n \quad odd \end{cases}$$

$$f(x) = 1 + \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos(3\pi x/2) + \frac{1}{5^2}\cos(5\pi x/2) + \dots\dots\right)$$

(2) $f(x) = 0$, $(-1 < x < 0)$; $f(x) = x$, $(0 < x < 1)$ $\quad f(x+2) = f(x)$