Chapter 4

LAPLACE TRANSFORMS

In Chapter 3 we studied methods of solution of linear differential equations with constant coefficients. The Laplace Transform technique is an alternative method of finding solution of differential equations, which in some situations is preferable to the methods covered before. The Laplace transform has also numerous applications in many areas of physics.

Given a function *f*(*t*), its Laplace transform is another function, $\mathfrak{I}(s)$, also denoted $\mathcal{L}[f](s)]$, of independent variable *s*. We shall use both notations depending on the convenience of using one or another. Viewed in this way, the Laplace Transform is an operator over the set of functions, which, for each function *f*, yields another function $\mathfrak{I}$, in mathematical notation: .

$$\mathcal{L}:\ f \longrightarrow \mathfrak{I}$$

In this chapter we shift our previous notation of *x* as independent variable by *t*, to be consistent with the book's notation. So we will write *f*(*t*) and *y*(*t*) and *dy/dt*. The change of letter for the independent variable from the original function to its Laplace transform is merely for simplification of notation and understanding. So, if a function has independent variable *t*, it refers to the original function, and if the independent variable is denoted by *s*, the function is the Laplace transform.

4.1 Definition of the Laplace Transform

Before giving the formal definition of the Laplace transform, we recall an important concept of improper integrals, which was covered in MATH I. An improper integral is defined as the limit of a definite integral when one or both limits of integration diverge to infinity:

$$\int_0^\infty f(t)\,dt = \lim_{b\to\infty}\int_0^b f(t)\,dt$$

If the limit exists, we say that the improper integral converges. Otherwise, we say that it diverges, which means that the improper integral for that function is not defined.

DEFINITION. Given a function $f(t)$ defined for all values of $t \geq 0$, the ***Laplace transform*** of f is the function $\Im(s)$, also denoted $\mathcal{L}[f](s)]$ defined as follows:

$$\Im(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{1}$$

for all values of s for which the improper integral converges.

EXAMPLE. Let $f(t) \equiv 1$ for all $t \geq 0$. Calculate its Laplace transform

$$\Im(s) = \int_0^\infty 1 \cdot e^{-st}\,dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = \lim_{b\to\infty} -\frac{1}{s}e^{-st}\bigg|_0^\infty = \lim_{b\to\infty}\left(-\frac{1}{s}e^{-bs} + \frac{1}{s}\right) = \frac{1}{s}$$

EXAMPLE. Evaluate the Laplace transform of $f(t) = e^{at}$.

$$\Im(s) = \int_0^\infty e^{at}\,e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}e^{st}\,dt = -\frac{e^{-(s-a)t}}{s-a}\bigg|_0^\infty$$

If $s > a$, then $e^{-(s-a)t} \to 0$ as $t \to \infty$, but if $s \leq a$ the improper integral diverges or doesn't exist. So we have $\Im(s) = \dfrac{1}{s-1}$ $\;s > a$ and is not defined for $s \leq a$.

EXAMPLE. Evaluate the Laplace transform of $f(t) = t,\; t \geq 0$

It's done by integration by parts

$$\Im(s) = \int_0^\infty t\,e^{-st}\,dt = -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\bigg|_0^\infty = \frac{1}{s} - \frac{1}{s^2} = \frac{s-1}{s^2}$$

In fact, for the usual problems we have to solve, we will never have to calculate Laplace transforms, since they are tabulated. A table of the most frequent Laplace transforms is provided at the end of this chapter. This table will also be provided during exams.

4.2 Existence and Uniqueness of the Laplace Transform

The Laplace transform of a function f exists under pretty general conditions f, namely

DEFINITION. A function f is said to be of ***exponential order*** if there exist nonnegative constants M, c, and T, such that

$$|f(t)| \leq Me^{ct} \quad \text{for all } t > T. \tag{2}$$

In other words, a function is of exponential order if its growth is bounded by the growth of an exponential function. The following theorem insures the existence of the Laplace transform under quite general conditions.

THEOREM. If $f(t)$ is defined and piecewise continuous on every finite interval or the positive real line $t \geq 0$ and is of exponential order as in (2), then the Laplace transform of f exists for all $s > c$.

THEOREM. The Laplace transform of a function f, is unique on the domain where it converges.

4.3 Basic Properties of the Laplace Transform

The properties covered here are useful for quickly finding of Laplace transforms. Next theorem shows the *linearity* property of the Laplace transform.

THEOREM. For constants a and b, we have

$$\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

EXAMPLE. Evaluate the Laplace transform of $f(t) = 3 - 5t$. Using the examples above and the theorem, we have

$$\mathcal{L}[3\text{-}5t] = 3\mathcal{L}[1]\text{-}\ 5\mathcal{L}[t] = -\frac{2s-5}{s^2}$$

The Lplace transform of the derivatives of a function are also easy to find. The next two theorems relate the Laplace transform of the first and second derivatives to the Laplace transform of the function itself.

First derivative theorem

THEOREM. If f is continuous and of exponential order (2), and f' is piecewise continuous on an interval $[0, A]$, then

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$$

for $s > c$.

Second derivative theorem

THEOREM. If f and f' are is continuous and of exponential order (2), and f'' is piecewise continuous on an interval $[0, A]$, then

$$\mathcal{L}[f''] = s^2\mathcal{L}[f] - s\,f(0) - f'(0).$$

There is a general theorem for the *n*-th derivative of a function, which extends the last two theorems above, but in this course we will not work with differential equations of order larger than two. The next theorem relates to Laplace transform of the integral of a function to the Laplace transform of the function itself.

THEOREM. If $f(t)$ is piecewise continuous and of exponential order (2), then

$$\mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\mathcal{L}\,[f]$$

for all $s > c$.

4.4 Inverse Laplace Transform

Suppose that $\mathfrak{I}(s)$ is the Laplace transform of certain function $f(t)$, then we say that f is the inverse transform of $\mathfrak{I}$.

DEFINITION. If $\mathfrak{I}(s)$ is the Laplace transform of a function f, that is, if there exists a function f such that $\mathfrak{I}(s) = \mathcal{L}[f(t)]$, then $f(t)$ is called the inverse Laplace transform of $\mathfrak{I}(s)$, and write

$$f(t) = \mathcal{L}^{-1}[f(s)].$$

Existence and uniqueness of the inverse Laplace transform are the next problems to consider. We will not study existence conditions; that is, the assumptions that guarantee that, given a function $g(s)$, there is another function $f(t)$ such that g is the Laplace transform of f. At the end of this chapter there is a table of functions and their transforms. We can use this table to find the inverse Laplace transform of the functions tabulated there. On the other hand, the inverse Laplace transform of a function, if it exists, is unique. This is insured by the following theorem(s) (both theorems are equivalent, so one of them is redundant, but we stated both here just for clearness of the exposition)

THEOREM. If $f(t) \neq g(t)$ for all values of t on some interval, then $\mathcal{L}[f] \neq \mathcal{L}\,[\mathrm{g}]$.

An equivalent, and somewhat more rigorous statement is

THEOREM. If $\mathcal{L}[f](s) = \mathcal{L}\,[g](s)$ for $s \in [c, \infty)$, then $f(t) = g(t)$ for all t wherever both functions are continuous.

In other words, if the Laplace transforms of two functions are equal, then the functions can only disagree on their points of discontinuity. The uniqueness of the Laplace transform and the uniqueness of the inverse Laplace transform indicate that there is a one-to-one correspondence between functions and their Laplace transforms.

EXAMPLE. Using a previous example above, if *f*(s) = $\dfrac{1}{s-1}$, $s > a$, then

$$\mathcal{L}^{-1}[f(t)] = e^{at}.$$

Notice that we did not calculate the inverse transform of *f*(*s*), we just used a previous example. This means that we do not know how to evaluate inverse transforms. We can just find the inverse transforms of some well-known functions. The calculation of the inverse transform requires the use of complex-function integration, and it is postponed until the chapter on Fourier transforms that we will cover later.

THEOREM (shifting). Suppose that, for some $k > 0$, $\mathfrak{I}(s)$, $s > k$, is the Laplace transform of *f*(*t*), then

$$\mathcal{L}[e^{at}f(t)] = \mathfrak{I}(s-a),$$

or, equivalently

$$e^{at} f(t) = \mathcal{L}^{-1}[\mathfrak{I}(s-a)].$$

4.5 Review of Partial Fractions; Finding Inverse Transforms

Laplace transforms can be used, among other purposes, to solve differential equations. But this entails to finding inverse transforms of rational functions, which in turn requires the use of partial fractions. But before we can proceed to the partial fractions technique, we must first review another item: polynomial factorization. Any polynomial can be factored in such a way that each factor is either a first degree binomial or a second degree trinomial with two complex conjugate roots. That is, a polynomial *P*(*t*) can be written as

$$P(x)=A(x-a_1)^{k_1}(x-a_2)^{k_2}...(x-a_p)^{k_p}(x^2+b_1x+c_1)^{q_1}(x^2+b_2x+c_2)^{q_1}...(x^2+b_mx+c_m)^{q_m}$$

and this representation is unique.

A rational function is the quotient of two polynomials. Let *P*(*t*) and *Q*(*t*) be two polynomials of any degree, then the function

$$R(t)=\frac{P(t)}{Q(t)}$$

is called rational function. Suppose that the degree of *P* is less than the degree of *Q*, then *R* can be written as the sum of simple terms of the form

$$R(t)=\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+...+\frac{A_k}{x+a_k}+\frac{B_1x+C_1}{x^2+b_1x+c_1}+\frac{B_2x+C_2}{x^2+b_2x+c_2}+....+\frac{B_px+C_p}{x^2+b_px+c_p} \tag{3}$$

where the denominators of the quadratic polynomials have no real roots (two complex roots). Simple example of partial fraction decomposition are as follows

$$\frac{x^2+mx+n}{(x-a)(x^2+px+q)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+px+q}$$

where the second degree trinomial has no real roots. Another is

$$\frac{x^2+mx+n}{(x^2+p_1x+q_1)(x^2+p_2x+q_2)}=\frac{A_1x+B_1}{(x^2+p_1x+q_1)}+\frac{A_2x+B_2}{(x^2+p_2x+q_2)}$$

In all the cases the coefficients *A, B, C,* etc. are found by taking common denominator on the right side and equating the coefficients of *x* of the same degree.

EXAMPLE. Prove the following identities

(a) $\frac{x+19}{x^2+3x-10}=\frac{3}{x-2}-\frac{2}{x+5}$

(b) $\frac{3x^2-8x+5}{x^3-3x^2-3x-4}=\frac{2x-1}{x^2+x+1}+\frac{1}{x-4}$

The verification is left to the reader, since it only entails simple but tedious algebraic manipulations. For the sake of this course, however, we will use only simple expressions. We need to decompose a rational expression in such a way that the partial fractions fit one of the functions for which there exist well known inverse transforms, which are tabulated at the end of this chapter.

Suppose that we want to decompose the following rational expression as a sum of partial fractions, so as to enable us to use one the tabulated functions.

$$\frac{ms+n}{s^2+ps+q}$$

We can always assume that $m = 1$, since otherwise, we write

$$\frac{ms+n}{s^2+ps+q}=m\frac{s+n/m}{s^2+ps+q}$$

There are three different cases, according to the roots of the denominator. The denominator has

1. Two different real roots
2. One real double root
3. No real roots

Problem 1. The denominator has two different real roots, *a* and *b* (that for the sake of this course, and to avoid unnecessary cumbersome calculations) they will always be rational roots. Write

$$\frac{s+n}{s^2+ps+q}=\frac{s+n}{(s-a)(s-b)}=\frac{s}{(s-a)(s-b)}+n\frac{1}{(s-a)(s-b)}$$

Each one of the two terms on the right-hand side has a tabulated inverse transform shown on lines 11 and 12 of the table at the end of this chapter.

Problem 2. The denominator has one double root, *a*. That is

$$\frac{s+n}{s^2+ps+q}=\frac{s+n}{(s-a)^2}=\frac{s-a+a+n}{(s-a)^2}=\frac{1}{(s-a)}+(a+n)\frac{1}{(s-a)^2}$$

and again, both terms on the right are tabulated on lines 7 and 8.

Problem 3. No real root. Completing the square of the denominator, we have

$$s^2+ps+q=\left(s-\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$$

But the second parenthesis is positive, since the discriminant, $\Delta = p^2 - 4q < 0$. Thus we can write

$$s^2+ps+q=\left(s-\frac{p}{2}\right)^2+w^2\,,\ \text{with}\ \ w^2=\left(q-\frac{p^2}{4}\right)$$

Finally

$$\frac{s+n}{s^2+ps+q}=\frac{s-p/2+p/2+n}{(s-p/2)^2+w^2}=\frac{s-p/2}{(s-p/2)^2+w^2}\ +(p/2+n)\frac{1}{(s-p/2)^2+w^2}$$

and inverse transform exist for both terms on the right, namely, formulae 17 and 18.

Example. Find the inverse Laplace transform of

$$\frac{s-1}{s^2-s-2}$$

Factoring the denominator

$$\frac{s-1}{s^2-s-2}=\frac{s-1}{(s-2)(s+1)}=\frac{s}{(s-2)(s+1)}-\frac{1}{(s-2)(s+1)}$$

using formulae 11 and 12 of the table of inverse transforms, with a = 2 and b = -1,

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2-s-2}\right]=\frac{e^{at}(a-1)-e^{bt}(b-1)}{a-b}=\frac{e^{2t}+2e^{-t}}{3}$$

Example. Find the inverse Laplace transform of

$$\frac{s-2}{s^2-2s+2}$$

The denominator has no real roots. Write

$$\frac{s-2}{s^2-2s+2}=\frac{s-2}{(s-1)^2+1}=\frac{s-1-1}{(s-1)^2+1}=\frac{s-1}{(s-1)^2+1}-\frac{1}{(s-1)^2+1}$$

and 17 and 18 provide us with the answer:

$$f(t) = e^t(\cos t - \sin t)$$

EXAMPLE. Find the inverse Laplace transform of

$$\frac{s^2+1}{s^3-2s^2-8s}$$

Factoring the denominator,

$$\frac{s^2+1}{s^3-2s^2-8s} = \frac{s^2+1}{s(s-4)(s+2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+2}$$

Taking common denominator on the right

$$\frac{A(s-4)(s+2)+Bs(s+2)+Cs(s-4)}{s(s-4)(s+2)}$$

Equating the numerators

$$A(s-4)(s+2)+Bs(s+2)+Cs(s-4) = s^2+1$$

Here there is a shortcut:

For $s = 4$, $B(4)(4+2) = 17 \Rightarrow B = 17/24$
For $s = -2$, $C(-2)(-2-4) = 5 \Rightarrow C = 5/12$
For $s = 0$, $A(-4)(2) = 1 \Rightarrow A = -1/8$

Thus

$$\frac{s^2+1}{s^3-2s^2-8s} = -\frac{1/8}{s} + \frac{17/24}{s-4} + \frac{5/12}{s+2}$$

Lines 1 and 7 of the table of transforms yields

$$\mathrm{L}^{-1}\left[\frac{s^2+1}{s^3-2s^2-8s}\right] = -\frac{1}{8} + \frac{17}{24}e^{4t} + \frac{5}{12}e^{-2t}$$

4.6 Application to Solving Differential Equations

Let us start with an initial value problem of the second-order linear equation with constant coefficients

$$y'' + ay' + by = r(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{4}$$

Let $Y(s)$ be the Laplace transform of $y(t)$ and $R(s)$ the Laplace transform of $r(t)$. By the theorems above about the derivatives of the Laplace transform, we have

$$Y = \mathrm{L}[y],$$

$$\mathrm{L}[y'] = s\,\mathrm{L}[y] - y(0), \tag{5}$$

$$\mathrm{L}[y''] = s^2\mathrm{L}[y] - s\,y(0) - y'(0).$$

Because of the uniqueness of the Laplace transform, we can apply the transform to both sides of the equation in (4)

$$\mathcal{L}\big(y'' + ay' + by\big) = \mathcal{L}\big(r(t)\big). \tag{6}$$

Substituting in (4) the values given in (5) (keep in mind that $Y = Y(s)$ is a function of s while $y = y(t)$ is a function of t), we obtain

$$\big(s^2 Y - s\,y(0) - y'(0)\big) + a\big(sY - y'(0)\big) + bY = R(s)$$

This last is called ***subsidiary equation***. Factoring

$$(s^2 + as + b)\,Y - (s+a)\,y(0) - y'(0) = R(s)$$

Solving for Y

$$Y(s) = \frac{(s+a)Y(0) + y'(0) + R(s)}{s^2 + as + b} \tag{7}$$

Because $y(t) = \mathcal{L}^{-1}\big(Y(s)\big)$, applying the inverse Laplace transform to both sides of (7) we get

$$y(t) = \mathcal{L}^{-1}\big(Y(s)\big) = \mathcal{L}^{-1}\left[\frac{(s+a)Y(0) + y'(0) + R(s)}{s^2 + as + b}\right]$$

Finally, the inverse Laplace transform of the right-hand side is evaluated by the methods seen in the previous section.

EXAMPLE. Solve the initial value problem

$$y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

This is the simplest case: the equation is homogeneous, Applying (7), with $a = -1$ and $b = -6$, we get

$$Y(s) = \frac{2(s-1)-1}{s^2 - s - 6} = \frac{2s-3}{(s-3)(s+2)} = 2\frac{s}{(s-3)(s+2)} - 3\frac{1}{(s-3)(s+2)}$$

and from tables 11 and 12, we get $y(t)$

$$y(t) = \frac{2}{3-(-2)}\left(3e^{3t} + 2e^{-2t}\right) - \frac{3}{3-(-2)}\left(e^{3t} - e^{-2t}\right) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

This equation, of course, can be solved also by employing the methods covered in chapter 3. But Chapter 3 methods can't be applied in the case of discontinuities, and Laplace transforms methods are more suitable in this case.

EXAMPLE. Solve the initial value problem

$$y'' + 4y = \sin 3t; \quad y(0) = y'(0) = 0$$

This equation arises in the motion of a mass-and-spring system with external force. This equation is not homogeneous, so we must first find $R(s)$, the Laplace transform of $\sin 3t$. Line 18 of the table of transforms shows that

$$R(s) = \frac{3}{s^2+9}$$

Hence, (7) yields

$$Y(s) = \frac{R(s)}{s^2+4} = \frac{3}{\left(s^2+4\right)\left(s^2+9\right)}$$

According to formula (3) and subsequent examples, $Y(s)$ may be split in two terms, each of the parenthesis are second degree polynomials with no real roots.

$$\frac{3}{\left(s^2+4\right)\left(s^2+9\right)} = \frac{As+B}{\left(s^2+4\right)} + \frac{Cs+D}{\left(s^2+9\right)}$$

The we take common denominator on the right side, and equate the coefficients of x of similar degree. We get a system of equations where the constants can be found.

$$\frac{Ax+B}{\left(s^2+4\right)} + \frac{Cx+D}{\left(s^2+9\right)} = \frac{(As+B)(s^2+9)+(Cs+D)(s^2+4)}{\left(s^2+4\right)\left(s^2+9\right)}$$

That is to say that

$$(As+B)(s^2+9)+(Cs+D)(s^2+4)=3$$

computing the products

$$s^3(A+C) + s^2(\mathrm{B+D}) + s(9A+4C) + (9B + 4C) = 3$$

and we got the system

$$A + C = 0, \quad B + D = 0, \quad 9A + 4C = 0, \quad 9B + 4D = 3$$

The first and third equations yield $A = B = 0$. The second and fourth equations yield $B = 3/5$, $D = -3/5$. We arrive finally to the form

$$Y(s) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9}$$

and from line 13 of the table

$$y(t)=\frac{3}{5}\frac{1}{2}\sin 2t - \frac{3}{5}\frac{1}{3}\sin 3t = \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t$$

EXAMPLE. Solve the initial value problem

$$y'' + 6y' + 34y = 30 \sin 2t$$

From line 30 of the table of transforms we get $R(s)$

$$R(s)=60\frac{1}{s^2+4}$$

and from formula (7)

$$Y(s) = \frac{60}{(s^2+6s+34)(s^2+4)} = \frac{As+B}{s^2+6s+34} + \frac{Cs+D}{s^2+4}$$

Taking common denominator and equating the numerators

$$\begin{aligned}(As+B)(s^2+4)+((Cs+D)(s^2+6s+34)\\ = s^3(A+C)+s^2(B+D+6C)+s(4A+34C+D)+(4B+34D) = 60\end{aligned}$$

From which we get the system

$$A + C = 0, \quad B + D + 6C = 0, \quad 4A + 34C + D = 0, \quad 4B + 34D = 60$$

Whose solution is $A = B = 10/29$, $C = -10/29$, $D = 50/29$. Plugging these values into the equation of $Y(s)$ above

$$Y(s) = \left[\frac{10}{29}\frac{s+1}{s^2+6s+34}\right] - \left[\frac{10}{29}\frac{s}{s^2+4}\right] + \left[\frac{50}{29}\frac{1}{s^2+4}\right]$$

We can calculate the inverse transform of each bracket separately

First bracket: $\dfrac{10}{29}\dfrac{s+1}{s^2+6s+34} = \dfrac{10}{29}\dfrac{s+3}{(s+3)^2+25} + \dfrac{10}{29}\dfrac{-2}{(s+3)^2+25}$

whose inverse transforms is: $\dfrac{10}{29}e^{-3t}\cos 5t - \dfrac{4}{29}e^{-3t}\sin 5t$

Second bracket: $-\dfrac{10}{29}\cos 2t$

Third bracket: $\dfrac{1}{29}\sin \quad 5\,t$

The final solution of the initial value problem is

$$y(t) = \frac{5}{29}(-2\cos 2t + 5\sin 2t) + \frac{2}{29}e^{-3t}(5\cos 5t - 2\sin 5t)$$