Chapter 3

SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

Second order differential equations can be classified in two large groups, linear and non linear. While second-order linear equations are relatively simple to solve, non linear equations may be much more complicated. We will see in this chapter several standard techniques for solving second-order linear equations. The importance of the linear equations arises from the many applications in science.

3.1 Second-Order Linear Differential Equations

A second-order linear differential equation is of the form

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad (1)$$

where $p(x)$, $q(x)$ and $r(x)$ are any functions (not necessarily linear) of x. It is, therefore, linear on y but not necessarily linear on x. Note that the coefficient of y'' is 1. This form given in (1) is called the standard form. If the coefficient of y'' is not 1, say, $a(x)$, then we divide the whole equation by $a(x)$, to convert it in the standard form, operation which would be valid only for all values of x such that $a(x) \neq 0$,

EXAMPLES.

$$y'' - e^{3x} y' + \frac{1}{x^2+1} y = \sin x$$

$$(3x+1)y'' - 5y' + \sqrt{x^3 - 2}\, y = 2xe^{x-1}$$

In the second example, dividing by $(3x+1)$ (valid for $x \neq -1/3$) we transform the equation in the format (1), called the standard form, since the coefficient of y'' is 1. We subdivide again the class of second-order linear equations into two types, homogeneous and nonhomogeneous. Homogenous equations are those for which $r(x) = 0$; they are nonhomogeneous otherwise. If the coefficients $p(x)$ and $q(x)$ are constant, then we have another class of equations called second-order linear with constant coefficients. In this chapter we undertake the solution of differential equation gradually, step by step, form the simpler to the more complicated types of equations.

3.2 Homogeneous Second-Order Linear Differential Equations

A particular type of second-order linear equations are the homogeneous second-order linear equations, that occur when the term $r(x)$ on the right-hand side of (1) is zero.

DEFINITION. The differential equation

$$\boldsymbol{y'' + p(x)\, y' + q(x)\, y = 0} \tag{2}$$

is called **homogeneous second-order linear equation**. If the terms $\boldsymbol{p(x)}$ and $\boldsymbol{q(x)}$, called **the coefficients,** are constant numbers, not dependent of x, we have another important type of equations, called **second-order linear equations with constant coefficients**.

We still have a long way to go before we can start solving differential equations. The following theorem is of vital importance in the solution of homogeneous equations.

THEOREM. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two solutions of the homogenous equation (2), then any linear combination $\boldsymbol{y = c_1 y_1 + c_2 y_2}$ is also a solution of (2).

Proof. Because y_1 and y_2 are solutions of (2), we have

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 = 0$$
$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0$$

Let $y = c_1y_1 + c_2y_2$, so $y' = c_1y'_1 + c_2y'_2$, and $y'' = c_1y''_1 + c_2y''_2$. Multiplying the first equation above by c_1 and the second equation above by c_2, and adding them, we obtain

$$(c_1\, y_1'' - c_2\, y_2'') + p(x)\, (c_1\, y_1' + c_2\, y_2') + q(x)(c_1\, y_1 - c_2\, y_2) = 0$$

substituting, we readily obtain

$$y'' + p(x)\, y' + q(x)\, y = 0$$

and then y is also e solution of the equation (2).

The importance of this theorem is that, if we can find two different solutions of (2), then we can also find infinitely many solutions. As we will see later, in some cases, to find the general solution of a homogenous equation, all we have to do is to find two particular solutions.

EXAMPLE 1. The homogeneous second-order linear equation $y'' - y = 0$ admits the solutions, $y_1 = e^x$ and $y_2 = e^{-x}$. To see this, $y_1'' = e^x$ and thus the equation becomes $e^x - e^x = 0$. The theorem above implies that any linear combination

$$y = c_1\, y_1 + c_1\, y_2 = c_1\, e^x + c_2\, e^{-x} \tag{3}$$

is also a solution of the equation. That is, we found infinitely many solutions of the given equation, one for each arbitrary values of the constants c_1, c_2. As we will see later, (3) is the general solution, which means that any solution to this equation is given in (3) for some particular values of c_1 and c_2.

3.3 Initial Value Problem

Suppose that, from the solution set (3) of the differential equation given in the previous example we want the particular initial value solution with $y(0) = 2$ and $y'(0) = 0$. Because $y'(x) = c_1e^x - c_2\, e^{-x}$, substituting in (3) we get the system

$$c_1 + c_2 = 2, \quad c_1 - c_2 = 0,$$

from which we get the values $c_1 = 1$, $c_2 = 1$. In general, a second-order equation requires two initial conditions, since two constant are associated with the general solution.

In the example above, $y = c_1 e^x + c_2 e^{-x}$ is a solution for any pair of arbitrary values of c_1 and c_2. The question that naturally arises is whether this is the general solution, that is, if all solutions of this differential equation can be written in this form for suitable values of the constants. The answer, as we will see, is yes, because they e^x and e^{-x} are "linearly independent" and therefore they "span" the space of solutions. We need here to clarify these terms; for this, we need to briefly review some concepts of Vector Spaces.

3.4 Vector Spaces, Linear Independence, Basis

The student who already took MAS210 (MATH III) should be familiar with vector spaces. We present here a brief review of the most important concepts, such as linear independence and basis of a vector space, which will be needed in the sequel.

DEFINITION. A ***vector Space V*** is a set of elements **u**, called vectors; a set of scalars (real numbers); and two operations, addition of vectors and scalar multiplication (multiplication of a vector by a scalar). The basic property of a vector space **V** is that any linear combination of vectors of **V** is also a vector of **V**. That is, if $\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_n$ belong to **V** then the vector **v** expressed as a linear combination of the $\mathbf{u}_i$'s , belongs to **V**

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_n\mathbf{u}_n$$

The addition of vectors and the scalar multiplication posses all the standard properties of algebra: commutative, associative, distributive, the existence of 0 and 1, etc. For a complete list of the properties you can review any linear algebra textbook.

Vector space is an abstract concept. We call their elements vectors, but we do not say what they are, we just give their properties. In fact, vectors can be of many different kinds, provided that they posses the properties of vector spaces. Vectors, cam be matrices, n-tuples, functions, ordered pairs of numbers, or triples, etc. The reason for including the study of vector spaces here is that the set of solutions of a homogenous linear differential equation forms a vector space, and that all we have to do to solve one such equation is to find a pair of linearly independent solutions.

DEFINITION. A set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n$ is said to be ***linearly independent*** if the trivial linear combination is the only linear combination equal to zero. In other words,

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots c_n\mathbf{u}_n = 0$$

only if $c_1 = c_2 = \ldots = c_n = 0$

In other words, the linear combination above is not zero unless all the constants c_i are zero. That is, the only way to get the linear combination equal to zero is by multiplying all vectors by 0.

DEFINITION. A ***basis*** of the vector space is a set of linearly independent vectors, $\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u}_n$ that span (generate) the vector space. In other words, any vector $\mathbf{u}$ of the vector space can be represented by a suitable linear combination of the elements of the basis

$$\mathbf{u} = c_1 \mathbf{u_1} + c_2 \mathbf{u_2} + \ldots c_n \mathbf{u}_n$$

and this representation is unique

If a basis for a vector space has n vectors, then n is called the ***dimension*** of the vector space. . It is the minimum number of vectors that generate the space. The set of solutions of a given homogeneous linear equation forms a vector space since, as we have seen above, linear combination of solutions is also a solution. The dimension of the set of solutions of a second-order linear homogeneous equations is exactly two, which means that we can have at most two linearly independent solutions of a given equation. Two solutions y_1, y_2, are linearly independent (*l.i.*), if $c_1 y_1 + c_2 y_2 = 0$ implies $c_1 = c_2 = 0$. (Sometimes it is easier to understand for the student the equivalent statement that if at least one of c_1 and c_2 are different from zero, then $c_1 y_1 + c_2 y_2 \neq 0$.) The solutions of a homogeneous second-order linear equation form a vector space of dimension 2, which means that we need only two linearly independent solutions to span the space.

The space of solution of a linear homogeneous equation of order n is a vector space of dimension n. If $y_1, y_2, \ldots y_n$ is a basis, then they span the vector space, in other words

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

is the general solution. Hence, to fin a general solution we need to find n linearly independent solutions.

For second order equations, the dimension is 2. We need only two linearly independent solutions. In Example 1, if the solutions $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent, then they form a basis, and the general solution of the equation is

$$y = c_1 e^x + c_1 e^{-x}$$

THEOREM. Two solutions y_1, y_2, of the homogeneous linear equation are linearly independent if their quotient is not a constant, that is, if $y_1/ y_2 = u(x)$ is not constant.

Proof. If for some $k \neq 0$ $y_1/ y_2 = k$, then $y_1 - ky_2 = 0$, but this is impossible because they are linearly independent. Thus this constant k does not exist.

In Example 1, $y_1//y_2 = e^x/e^{-x} = e^{2x}$ and therefore, they are linearly independent and $y = c_1 e^x + c_1 e^{-x}$ is the general solution.

THEOREM. If y_1 and y_2 are two linearly independent solutions of the homogeneous second-order linear equation, then they are a basis for the space of solutions.

It follows that if y_1, y_2 are linearly independent, they span the space of solutions, and any other solution y is of the form $y = c_1 y_1 + c_2 y_2$. This is, therefore, the general solution of the equation.

EXAMPLE. Consider again the equation $y'' - y = 0$, with its two solutions $y_1 = e^x$ and $y_2 = e^{-x}$. Because $y_1/y_2 = e^x/e^{-x} = e^{2x}$ is not constant, they are linearly independent and $y = c_1 e^x + c_2 e^{-x}$ is the general solution. All solutions can be written in this form for suitable values of the constants.

EXAMPLE. Find the general solution of the differential equation $y'' + y = 0$.

First we show that $y_1 = \sin x$ and $y_2 = \cos x$ are two solutions. Indeed, $y_1" = -\sin x$ and $y_2" = -\cos x$, plugging in the equation we confirm that they are both solutions. On the other hand, they are linearly independent, since $y_1/y_2 = \tan x$, which is not constant. The last theorem implies that $y = c_1 \sin x + c_2 \cos x$ is the general solution..

Thus to find the general solution of the homogeneous second-order linear equation all we have to do is find two linearly independent solutions and form a linear combination of them.

3.5 Method of Reduction of Order

So far we have reviewed many required concepts, covered definitions and properties of solutions of differential equations, but we have not seen yet how to solve them. The student eager to know how to solve equations, will have to wait still a little further. In this section we learn how to find a second, linearly independent solution, if we know already one solution. But we only see how to find the first solution in the next section.

The method of reduction of order applies to the homogenous equation with non-constant coefficients. Suppose that we know one solution, y_1, of the homogenous equation with non-constant coefficients

$$y'' + p(x)\, y' + q(x)\, y = 0$$

In order to find a second solution linearly independent of y_1, we may apply the method of reduction of order. For this, set $y_2 = u(x)\, y_1(x)$ for some suitable non constant $u(x)$, and find u so that y_2 is also a solution. Clearly, $y_2/y_1 = u(x)$, which, being non constant, results in that y_1 and y_2 are linearly independent.

We can find u by the following formula

$$u' = U = \frac{1}{y_1^2}\exp\left(-\int p(x)\, dx\right) \qquad (4)$$

But $U = u'$, we calculate $u = \int U\, dx$, and from there we obtain $y_2 = u(x)\, y_1(x)$.

Prof of the formula. Differentiating,

$$y'_2 = u'y_1 + uy'_1, \text{ and } y''_2 = u''y_1 + 2u'y'_1 + uy''_1.$$

Substituting these equalities in (2) and grouping terms for u, it follows that

$$u''y_1 + u'(2y'_1 + py_1) + u(y''_1 + py'_1 + qy_1) = 0$$

The last parenthesis in the last equality is 0 because y_1 is a solution of (2) . Hence, after simplification, dividing the equation by y_1, the last equality becomes

$$u'' + u'\frac{2y_1' + py_1}{y_1} = 0$$

Letting $U = u'$ (and therefore $u'' = U'$), we end with the first order differential equation in U

$$U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0 \text{ which is the separable equation } \frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx$$

whose solution is $\ln|U| = -2\ln|y_1| - \int p\, dx$, or, explicitly,

$$u' = U = \frac{1}{y_1^2} \exp\left(-\int p(x)\, dx\right) \tag{4}$$

But $U = u'$, we calculate $u = \int U\, dx$, and from there we obtain y_2.

EXAMPLE. Taking once again the example $y'' - y = 0$. Suppose that we already found $y_1 = e^x$ be a solution. Applying (4) with $p(x) = 0$, we get $u' = U = 1/y_1^2 = e^{-2x}$. Then $u = (-1/2)e^{-2x}$, and a second linearly independent solution is $y_2 = u(x)\, y_1(x) = (-1/2)e^{-2x}e^x = (-1/2)e^{-x}$. Neglecting the constant (-1/2), which can be embedded into the constant c_2, we find $y_2 = e^{-x}$.

EXAMPLE. Knowing that $y_1 = x^{-1}$ is a solution of

$$2x^2y'' + 3xy' - y = 0$$

find a second linearly independent solution y_2.

First write the equation in the format (2).

$$y'' + \frac{3}{2x} y' - \frac{1}{2x^2} y = 0$$

Then, set $y_2 = ux^{-1}$. Using (4), with $p(x) = 3/2x$

$$\int p(x)\, dx = \int \frac{3}{2x} dx = \frac{3}{2}\int \frac{1}{x} dx = \frac{3}{2}\ln|x| \Rightarrow \exp\left(-\int p(x)\, dx\right) = \exp\left(-\frac{3}{2}\ln|x|\right) = x^{-3/2}$$

$$u' = U = \frac{1}{y_1^2} \exp\left(-\int p(x)\, dx\right) \Rightarrow u' = \frac{x^{-3/2}}{x^{-2}} = x^2 x^{-3/2} = x^{1/2}$$

Integrating left and right sides,

$$u = \int u'(x)\, dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2}$$

Hence, the answer of the problem is

$$y_2 = u(x)\, y_1(x) = (2/3)x^{3/2}x^{-1} = (2/3)\, x^{1/2}.$$

If we would like to find the general solution, neglecting the constant 2/3, which can be embedded with the constant c_1, the general solution of the equation

$$y = c_1\, x^{-1} + c_2\, x^{1/2}.$$

Note that the method of reduction of order helps to find a second solution, but a first solution must be found first by some other resources.

3.6 2nd-Order Linear Homogeneous With Constant Coefficients

In the previous section we have seen how to obtain a second solution y_2 linearly independent of y_1, given that one solution, y_1, is known. We have yet to know how to find the first solution. In this section we find a first solution y_1. We start by the easier particular case of constant coefficients, p and q. The equation takes the form

$$y'' + p\, y' + q\, y = 0 \tag{5}$$

It is a well-known fact that one solution has the form $y = e^{kx}$, for some constant k to be determined. To see this, $y' = ke^{kx}$ and $y'' = k^2e^{kx}$. Substituting in (5) and factoring we get

$$e^{kx}(k^2 + pk + q) = 0.$$

Since the exponential term is never zero, then the quadratic expression in parenthesis, called the ***characteristic equation***, must be zero. Solving the quadratic equation we find the suitable value of k. Let $\Delta = p^2 - 4q$ be the discriminant of the quadratic equation above. Three different cases are possible depending on the roots of the characteristic equation.

Case 1. $\Delta > 0$, the equation has two distinct real roots.

Case II. $\Delta = 0$, the equation has one double real root.

Case 3. $\Delta < 0$, the equation has two complex conjugate roots.

Each of these cases is solved in a different way.

Solution of Case 1. The characteristic equation has two different real valued roots, k_1 and k_2. Because $y_1 = e^{k_1x}$ and $y_2 = e^{k_2x}$ are two linearly independent solutions, the general solution is

$$y = c_1 e^{k_1x} + c_2 e^{k_2x} \qquad (6)$$

Example. Find the general solution of $y'' - y' - 2y = 0$.

The characteristic equation is $k^2 - k - 2 = 0$, with roots $k_1 = 2$, $k_2 = -1$. Hence the general solution is $y = c_1 e^{2x} + c_2 e^{-x}$.

Solution of Case 2. The characteristic equation has one double real root $k = -p/2$. In this case, $y_1 = e^{-px/2}$ is one solution. The second solution can be found by the method of reduction of order covered in the previous section. Applying (4)

$$U = e^{px} e^{-\int p\,dx} = 1 \quad \Rightarrow \quad u = \int U\,dx = x$$

so the second solution is $y_2 = x\,e^{-px/2}$ and the general solution is $y = e^{kx}(c_1 + c_2 x)$

Example. The equation $y'' - 4y' + 4y = 0$ has characteristic equation $k^2 - 4k + 4 = 0$, $(k - 2)^2 = 0$, with double root $k = 2$. One solution is $y_1 = e^{2x}$, the other is $y_2 = xe^{2x}$. The general solution therefore is

$$y = e^{2x}(c_1 + c_2 x). \qquad (7)$$

The case of two complex conjugate roots is discussed in the next section.

3.7 Brief Review of Complex Numbers

The student who already took MAS210 (MATH III) should be familiar with complex numbers. We present here just a brief review, needed to solve equation for the case III given in the previous section. The equation $x^2 = -1$ has no solution in the field of real numbers. In order the last equation to have a solution, we must expand the set of real numbers with the inclusion of new numbers, called complex numbers. First it is defined the number $i = \sqrt{-1}$, called imaginary. We have $i^2 = -1$.

Definition. Let a and b be real numbers. A **complex number z** is defined as

$$z = a + bi$$

a is called the ***real part* of *z*** and bi the **imaginary part**. If $b = 0$ then z is real. Two complex numbers, z and $\bar{z}$, are said complex conjugates if $z = a + bi$ and $\bar{z} = a - bi$.

We use complex numbers to solve quadratic equation whose discriminant, Δ is negative.

EXAMPLE. Solve the equation $x^2 - 2x + 8 = 0$.

Using the quadratic formula $x', x'' = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we find that this equation has two complex conjugate roots

$$\frac{2 \pm \sqrt{4-32}}{2} = \frac{2 \pm \sqrt{-28}}{2} = \frac{2 \pm 2\sqrt{-7}}{2} 1 \pm \sqrt{-7}$$

$\sqrt{-7} = \sqrt{(7)(-1)} = \sqrt{7}\sqrt{-1} = \sqrt{7}i$ the roots of the equation are $1 + \sqrt{7}\, i$ and $1 - \sqrt{7}\, i$.

Next we define the exponential function of a complex variable. For real x, the number ix is pure imaginary.

DEFINITION. Given a real x, the complex function $f(x) = e^{ix}$ is defined as

$$e^{ix} = \cos x + i \sin x$$

Because sin x is an odd function, $e^{-ix} = \cos x - i \sin x$. Adding and subtracting, them, we obtain two new complex functions

DEFINITION. The complex functions cos x and sin x are defined as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \; ; \quad \textit{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Let in general z be any complex number, $z = x + yi$. Euler's formula becomes

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \tag{8}$$

3.8 Case Δ < 0

Suppose that the characteristic equation has a pair of conjugate complex roots. If the discriminant is negative. Let $a \pm bi$ be the roots of the characteristic equation. Then the general solution of the differential equation is

$$y = e^{ax} (C_1 \sin bx + C_2 \cos bx)$$

Proof.

$$k_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} = -\frac{p}{2} + \frac{i\sqrt{4q - p^2}}{2} = -\frac{p}{2} + i\sqrt{q - p^2/4}$$

and the second complex conjugate, root is

$$k_2 = \frac{-p-\sqrt{p^2-4q}}{2} = -\frac{p}{2} - \frac{i\sqrt{4q-p^2}}{2} = -\frac{p}{2} - i\sqrt{q-p^2/4}$$

or, in short, $k_1 = -p/2 + i\,w$, and $k_2 = -p/2 - i\,w$, with $w = \sqrt{q-p^2/4}$. Applying (8) we obtain two complex solutions

$$u_1 = e^{k_1x} = e^{-px/2}(\cos wx + i\sin wx) \quad \text{and} \quad u_2 = e^{k_2x} = e^{-px/2}(\cos wx - i\sin wx)$$

Because any linear combination of solutions is another solution, we add these two solutions and then subtract them, and get two real valued solutions, which form a basis on the space of real valued solutions

$$y_1 = (u_1 + u_2)/2 = e^{-px/2}\cos wx, \quad \text{and} \quad y_2 = (u_1 - u_2)/2i = e^{-px/2}\sin wx \tag{9}$$

The general solution is

$$\boldsymbol{y = e^{-px/2}(C_1\cos wx + C_2 \sin wx)}$$

EXAMPLE. We have seen at the beginning of this chapter that the equation $y'' + y = 0$ admits the two *l.i.* independent solutions $y_1 = \sin x$ and $y_2 = \cos x$. For this equation, the characteristic equation is k^2- + 1 = 0, with two complex conjugate roots i and $-i$, and $a = 0$ *and* $b = 1$, so $w = 1$, from which, applying (9), the two solutions y_1 and y_2 above readily follow.

EXAMPLE. Find the general solution of $y'' - 4y' + 5 = 0$.

The characteristic equation $k^2 - 4k + 5 = 0$ hast two complex roots

$$k_1, k_2 = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm \sqrt{-1} = 2 \pm i$$

From (9) the general solution readily follows,

$$y = e^{2x}(c_1 \sin x + c_2 \cos x)$$

3.9 Euler-Cauchy Equation

This is a particular case of a second-order homogeneous equation with non-constant coefficients. It is of the form

$$x^2y'' + axy' + by = 0 \tag{10}$$

We try the solution $y = x^m$ and search for the value of m that solves the equation (10). We have $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$. Substituting into the equation and factoring yields

$$x^m[m^2 + (a-1)m + b] = 0 \tag{11}$$

This gives the auxiliary equation $m^2 + (a-1)m + b = 0$. Solving the quadratic equation, we find the value of m that solves the equation. As before, three cases can be identified, depending on whether this equation has two distinct real roots, a double real root, and two complex conjugates roots. The general solutions are as follows

Case 1. The auxiliary equation has two distinct real roots m_1, m_2. The general solution is

$$y = c_1x^{m_1} + c_2x^{m_2}$$

EXAMPLE. Find the general solution of the Euler-Cauchy equation $2x^2y'' - 5xy' - 4y = 0$.

Dividing by 2 we get the standard form of the Euler-Cauchy equation.

$$x^2y''-\frac{5}{2}xy'-2y=0$$

Here a = -5/2 and b = -2. The auxiliary equation (11) is

$$m^2-(7/2)\,m-2=0 \;\Rightarrow\; 2m^2-7m-4=0 \;\Rightarrow\; (2m+1)(m-4)=0,$$

The roots are m_1 = -1/2, m_2 = 4. The general solution, therefore, is

$$y=c_1x^{-1/2}+c_2x^4 \;\Rightarrow\; y=\frac{c_1}{\sqrt{x}}+c_2x^4, \qquad x>0.$$

EXAMPLE. Solve the equation $x^2y''+7xy'+8y=0$.

The auxiliary equation is $m^2+6m+8=0$, with the roots m_1 = -4 and m_2 = -2. The general solution is

$$y=c_1x^{-5}+c_2x^{-3}.$$

<u>Case 2</u>. One double real root. In this case the discriminant of the auxiliary equation is zero, and the double root becomes $m=(1-a)/2$. Substituting m by this value, one solution is

$$y_1=x^{(1-a)/2} \tag{12}$$

The second solution is found by the method of reduction of order. First, we must divide the equation (10) by x^2 to have it in the standard form

$$y''+\frac{a}{x}y'+\frac{b}{x^2}y=0$$

From (3) it follows that

$$U=x^{a-1}\exp\left(-\int\frac{a}{x}dx\right)=e^{-a\ln|x|}=x^{a-1}x^{-a}=\frac{1}{x}$$

so $u=\int U\,dx=\int\frac{1}{x}dx=\ln|x|$, from which we obtain the second solution

$$y_2=x^{(1-a)/2}\ln|x| \tag{13}$$

Combining (12) and (13) we have the general solution

$$y=(c_1+c_2\ln x)\,x^{(1-a)/2}$$

EXAMPLE. Find the general solution of $x^2y''-3xy'+4y=0$.

The auxiliary equation is $m^2-4m+4=0$, with the double root $m=2$. The general solution is

$$y=(c_1+c_2\ln x)x^2. \tag{14}$$

We skip Case 3, two complex conjugate roots, for not having many applications.

3.10 Nonhomogeneous, Constant Coefficients Equations

The non-homogeneous equation with constant coefficients is given by

$$y''+py'+qy=r(x) \tag{15}$$

where p and q are constant numbers and $r(x)$ is a function. The corresponding homogeneous equation to the nonhomogenous equation (15) is formed by making $r(x)=0$, that is

$$y''+py'+qy=0 \tag{16}$$

where p and q are constant numbers and $r(x)$ is a function.

Let Y_1 and Y_2 be two solutions of the nonhomogeneous equation (15). Then, their difference $Y = Y_1 - Y_2$ is a solution of (16), because substituting Y in (15) we get

$$Y'' + pY' + qY = (\, Y_1'' + pY_1' + qY_1 \,) - (\, Y_2'' + pY_2' + qY_2 \,) = r(x) - r(x) = 0$$

This proves the following

THEOREM. The difference of any two solutions of the nonhomegemous equation (15) is a solution of the homogeneous equation (16).

Now, let Y be the general solution of (15) and let Y_1 be a particular solution of (15). Then their difference,

$$y = Y - Y_1 \tag{17}$$

is a solution of (16), and applying the results on homogeneous equations in the previous sections, y is of the form

$$y = c_1 y_1 + c_2 y_2 \tag{18}$$

where y_1 and y_2 are two *l.i.* solutions of (16). Combining (17) and (18), we have proved the following

THEOREM. If Y_1 is a particular solution of the nonhomogeneous equation (15), and y_1 and y_2 are two linearly independent solutions of the homogeneous equation (16), then the general solution of (15) is

$$Y = Y_1 + c_1 y_1 + c_2 y_2 \tag{19}$$

Hence, to find the general solution of the nonhomogeneous equation with constant coefficients (15), we must first solve the homogeneous equation and then find a particular solution of the nonhomogeneous equation (15). The method of ***undetermined coefficients*** is used to find a particular solution of (15). There is no standard procedure for this, and we must guess the form of the particular solution of (15). Guessing the right form of the solution may not always be a simple task, but for the problems we will have to solve in this class, the form of the particular solution should be obvious after we solve a few examples.

EXAMPLE. Find the general solution of $y'' - 3y' - 4y = 3e^{2x}$.

First we solve the homogeneous equation, whose characteristic equation is $k^2 - 3k - 4 = 0$, with two distinct real roots $k_1 = 4$, $k_2 = -1$. By (6), the general solution of the homogeneous equation is

$$y = c_1 e^{4x} + c_2 e^{-x}.$$

We must yet find a particular solution of the nonhomogeneous equation. We guess the particular solution $Y_1 = Ae^{2x}$, where A the undetermined coefficient, must be found, by substituting Y_1 in the equation. Because $Y_1' = 2Ae^{2x}$ and $Y_1'' = 4Ae^{2x}$, we have

$$3e^{2x} = e^{2x}(4A - 3(2A) - 4A) = -6Ae^{2x}$$

from where we get $A = -1/2$. Thus the particular solution of the nonhomoegenous equation is

$$Y_1 = -(1/2)\, e^{2x}$$

and the general solution of the nonhomogeneous equation is

$$Y = -(1/2)\, e^{2x} + c_1 e^{4x} + c_2 e^{-x}.$$

3.11 Particular Solution of the Nonhomogeneous Equation

In general, the problem of finding a particular solution is not straightforward, and most of the times not an easy one. Trial and error, experience and patience are necessary for finding particular solutions. There are, however, a few typical cases in which the particular solution of (15) can easily be found. These cases are listed below. In the exam, the student may be asked to find a particular solution, which will be a problem among the functions in the list.

Term in $r(x)$	Particular solution Y_1
$k\,e^{ax}$	Ae^{ax}
$kx\,e^{ax}$	$e^{ax}(Ax+B)$
$k\,x$	$Ax+B$
$k\,x^2$	Ax^2+Bx+C
$k\,x^3$	Ax^3+Bx^2+Cx+D
$k_1\,x+k_2$	$Ax+B$
$k_1\,x^2+k_2x+k_3$	Ax^2+Bx+C
$k\,x^n$	$A_n\,x^n+A_{n-1}\,x^{n-1}+\ldots+A_1\,x+A_0$
$k_1\,x^n+k_2\,x^{n-1}+\ldots+k_n$	$A_n\,x^n+A_{n-1}\,x^{n-1}+\ldots+A_1\,x+A_0$
$k\sin ax$	$A\sin ax+B\cos ax$
$k\cos ax$	$A\sin ax+B\cos ax$
$k_1\cos ax+k_2\sin ax$	$A\sin ax+B\cos ax$
$k\,e^{ax}\sin bx$	$e^{ax}(A\sin bx+B\cos bx)$
$k\,e^{ax} cos\,bx$	$e^{ax}(A\sin bx+B\cos bx)$

Where the constants $A, B, C, D, A_1 \ldots A_n$ must be calculated by substituting the proposed Y_1 into the equation by the method of undetermined coefficients.

EXAMPLE. Solve the nonhomogeneous equation $y'' + 4y = 8x^2$.

The table above suggest trying the particular solution $y_p = Ax^2 + Bx + C$. So, $y'' = 2A$. Substituting in the equation we get $2A + 4Ax^2 + 4Bx + 4C = 8x^2$. Hence, $4A = 8$, $B = 0$, and $2A + 4B = 0$. Thus $A = 2$, $B = 0$, $C = -1$, which leads to the particular solution $y_p = 2x^2 - 1$.

3.12 Modification Rules

There are some cases in which the suggested solutions on the table above do not work. In this case, the following modifications must done.

(a) If the function given on the second column of the table above is a solution of the homogeneous equation, we can't use it as our particular solution y_p, in this case, multiply the function on the table by x (or by x^2 in the case of the double root of the characteristic equation).

(b) If $r(x)$ is a sum of functions of the type given in the first column, then make y_p the sum of the corresponding functions given in the second column.

EXAMPLE. Solve the nonhomogeneous equation $y'' - 2y' - 3y = e^{3x}$.

The characteristic equation $k^2 - 2k - 3 = 0$ has the roots $k_1 = 3$ and $k_2 = -1$. The general solution of the homogeneous equation is $y = c_1e^{3x} + c_2e^{-x}$. The table above suggests the particular solution $y_p = e^{3x}$. However, because this is a solution of the homogeneous equation, cannot be a solution of the nonhomogeneous equation. Modification (a) indicates to try $y_p = Axe^{3x}$. Then we have $y_p' = Ae^{3x}(3x + 1)$, $y_p'' = Ae^{3x}(9x + 6)$. Substituting into the equation, after simplification, yields $4Ae^{3x} = e^{3x}$, so $A = ¼$. The particular solution, therefore, is $y_p = xe^{3x}/4$.

EXAMPLE. Solve the nonhomogeneous equation $y'' - 2y' + y = e^x$.

The characteristic equation $k^2 - 2k + 1 = 0$ has the double root $k = 1$. The general solution of the homogeneous equation is $y = c_1e^x + c_2xe^x$. Modification (a) indicates to try $y_p = Axe^x$. Then we have $y_p' = Ae^x(x^2 + 2x)$, $y_p'' = Ae^x(x^2 + 4x + 2)$. Substituting into the equation, after simplification, yields $2Ae^x = e^x$, so $A = 1/2$. The particular solution, therefore, is $y_p = x^2e^x/2$.

3.13 Variation of Parameters

This method applies to the general equation given in (1):

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

where now, $p(x)$ and $q(x)$ are not necessarily constant functions. The particular solution for this equation is

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx \qquad (20)$$

where y_1, y_2 form a basis for the solution of the homogeneous equations and W, called the *Wronskian* is given by

$$W = y_1 y'_2 - y_2 y'_1$$

EXAMPLE. Solve the equation $y'' - \sin x\, y' + (1+\cos x)\, y = -\sin^2 x$.

Two linearly independent solutions of the corresponding homogeneous equation are equation are $y_1 = \sin x$ and $y_2 = \cos x - 1$. The Wronskian is $W = \cos x - 1$. Applying (20) a particular solution, after simplification, is

$$y_p = \sin x \int \sin^2 x\,dx + (1-\cos x)\int \sin x(1+\cos x)\,dx$$

where the integrals are standard functions with well-known antiderivatives. We omit the details since this case will not be included in the exams for being in general too laborious.

HOMEWORK

Kreyszig, p.52, #1-14, 17-22; p.59, #1-13, 21-32; p.72, #1-15; p.83, #1-20; p. 101, #1-17

Additional problems

1. Find the general solution of the following equations

(a) $y'' - 2y' - 10y = 0$ (b) $y'' - 2y' - 8y = 0$ (c) $y'' - 6y' - 9y = 0$

(d) $y'' + y' + y = 0$ (e) $y'' + y' - y = 0$ (f) $2y'' + 3y' - 2y = 0$

(g) $y'' - 4y' + 13y = 0$

2. Find the initial value problem

(a) $y'' - 5y' + 4y = 0$; $y(0) = 1$; $y'(0) = -1$ (b) $4y'' - 4y' + y = 0$; $y(0) = 2$; $y'(0) = 0$

(c) $y'' - 4y' + 13y = 0$; $y(0) = 1$; $y'(\pi/2) = -1$

3. Find a particular solution to the following equations

(a) $y'' - 5y' + 4y = e^{2x}$ (b) $y'' + y' + y = xe^{x}$ (c) $y'' - 3y' + 2y = 3x^2$

(d) $y'' - 3y' + y = \cos x$ (e) $y'' - 5y' + 6y = x^2 - 2x + 1$ (f) $y'' - y' - 2y = 2\cos x - \sin x$

(g) $y'' - 5y' + 6y = e^{3x}$ (h) $y'' - y' - 12y = x + e^{2x}$

4. Find the general solution of the following equations

(a) $y'' + 2y' - 3y = e^{2x}$ (b) $y'' - 2y' + y = e^{x}$ (c) $y'' - 4y = e^{2x}$

(d) $x^2 y'' + 5xy' - 5y = 0$ (e) $x^2 y'' - xy' + y = 0$

5. Given the equation $y'' + (\sin x)\, y' + (1-\cos x)\, y = 0$

(a) show that $y_1 = \sin x$ is one solution

(b) use the method of reduction of order to find a second, *l.i.* solution y_2.

SOLUTION TO SELECTED PROBLEMS

Problem 1

(a) $y'' - 2y' - 10y = 0$ Characteristic equation is $k^2 - 2k - 10 = 0$. The roots are

$\frac{2 \pm \sqrt{4+40}}{2} = \frac{2 \pm \sqrt{44}}{2} = \frac{2 \pm 2\sqrt{11}}{2} = 1 \pm \sqrt{11}$ The general solution is

$$y = C_1 \exp\left[\left(1+\sqrt{11}\right)x\right] + C_2 \exp\left[\left(1-\sqrt{11}\right)x\right]$$

(b) $y'' - 2y' - 8y = 0$ Characteristic equation is $k^2 - 2k - 8 = 0$. The roots are $(k - 4)(k + 2) = 0$

The roots are 4 and -2. The general solution is

$$y = C_1 e^{4x} + C_2 e^{-2x}$$

(c) $y'' - 6y' - 9y = 0$. Characteristic equation is $k^2 - 6k - 9 = 0$. The roots are $(k - 3)(k - 3) = 0$. Double root is 3. The general solution is

$$y = e^{3x} (C_1 + C_2 x)$$

(d) $y'' + y' + y = 0$. Characteristic equation is $k^2 + k + 1 = 0$. The roots are

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The general solution is

$$y = e^{x/2} \left[C_1 \sin(\sqrt{3}\, x/2) + C_2 \cos(\sqrt{3}\, x/2)\right]$$

(g) $y'' - 4y' + 13y = 0$

Characteristic equation is $k^2 - 4k + 13 = 0$

$\frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\sqrt{-1}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$. Therefore the general solution is

$$y = e^{2x} (C_1 \cos 3x + C_2 \sin 3x)$$

Problem 2

(a) $y'' - 5y' + 4y = 0$; $y(0) = 1$; $y'(0) = -1$. Characteristic equation $k^2 - 5k + 4 = 0$. The roots are $(k - 4)(k - 1) = 0$ thus 4 and 1. General solution

$$y = C_1 e^{4x} + C_2 e^{x}.$$

Initial value: $y' = 4C_1 e^{4x} + C_2 e^{x}$

$y(0) = 1 \Rightarrow C_1 + C_2 = 1$
$y'(0) = -1 \Rightarrow 4C_1 + C_2 = -1$

Solving the system, subtract the equations

$-3C_1 = 2 \Rightarrow C_1$ -2/3 and $C_2 = 5/3$. The solution of the initial value problem is

$$y = (-2/3)e^{4x} + (5/3)e^{x}$$

Problem 3

(a) $y'' - 5y' + 4y = e^{2x}$

We guess $y = Ae^{2x}$. $y' = 2Ae^{2x}$, $y'' = 4Ae^{2x}$. Hence, plugging into the equation

Ae^{2x} (4 + (-5)(2) + 4) $= e^{2x}$. Solving the equation we get $-2A = 1 \Rightarrow A = -1/2$. The particular solution is

$$Y = -(1/2)\, e^{2x}.$$

(c) $y'' - 3y' + 2y = 3x^2$

We guess the solution $Y = Ax^2 + Bx + C$. $Y' = 2Ax + B$; $Y'' = A$. Plugging in the equation

$A - 3(2Ax + B) + 2 (Ax^2 + Bx + C) = 3x^2$.

Regrouping terms

$$2Ax^2 + (-6A + 2B)\, x + (A - 3B + 2C) = 3\, x^2.$$

We must solve the system

$$2A = 3; \quad -6A + 2B = 0; \quad A - 3B + 2C = 0$$

From which we obtain

$A = 3/2$; $B = 9/2$; $C = 6$. The particular solution is

$$Y = (3/2)\, x^2 + (9/2)\, x + 6$$

(d) $y'' - 3y' + y = \cos x$

We try the solution $Y = A \cos x + B \sin x$. $Y' = -A \sin x + B \cos x$; $Y'' = -A \cos x - B \sin x$

Plugging into the equation $-A \cos x - B \sin x - 3(-A \cos x - B \sin x) + A \cos x + B \sin x = \cos x$

Factoring

$$\cos x\, (-A + 3A + A) + \sin x\, (-B + 3B + B) = \cos x$$

$$3A \cos x + 3B \sin x = \cos x$$

Thus, $3A = 1$, $3B = 0$. The particular solution is

$$Y = (1/3 \cos x)$$

(e) $y'' - 5y' + 6y = x^2 - 2x + 1$

We try $Y = Ax^2 + Bx + C$, $Y' = 2Ax + B$; $Y'' = 2A$. Plugging in the equation

$2A - 10Ax - 5B + 6Ax^2 + 6Bx + 6C = x^2 - 2x + 1$, grouping terms

$6Ax^2 + (-10A + 6B)\, x + 2A - 5B + 6C = x^2 - 2x + 1$

$6A = 1$; $-10A + 6B = -2$; $2A - 5B + 6C = 1 \Rightarrow A = 1/6$; $B = -1/18$; $C = 17/108$

(f) $y'' - y' - 2y = 2 \cos x - \sin x$

We try $Y = A \sin x + B \cos x$; $Y' = A \cos x - B \sin x$; $Y'' = -A \sin x - B \cos x$. Plugging in the equation

$-A \sin x - B \cos x - A \cos x + B \sin x - 2A \sin x - 2B \cos x = 2 \cos x - \sin x$

$(- B - A - 2B) \cos x + (-A + B - 2A) \sin x = 2 \cos x - \sin x$

$-A - 3B = 2$; $-3A + B = -1$, $B = -7/10$; $A = 1/10$

The particular solution is $Y = -(1/10) \sin x - (7/10) \cos x$

(g) $y'' - 5y' + 6y = e^{3x}$.

When looking for a particular solution of the form $Y = Ae^{3x}$, this will be true **only** if e^{3x} **is not** a solution of the homogeneous equation. In this case, the characteristic equation

$k^2 - 5k + 6 = (k - 3)(k - 2)$. Hence, e^{3x} is a solution of the homogeneous equation. We **cannot** use $Y = Ae^{3x}$. Therefore, we need to modify Y. Using the procedures of section 3.12, we try

$Y = Axe^{3x}$. $Y' = 3Axe^{3x} + Ae^{3x} = Ae^{3x}(3x + 1)$; $Y'' = 3Ae^{3x}(3x + 1) + 3Ae^{3x} = Ae^{3x}(9x + 6)$

Plugging in the equation

$$Ae^{3x}(9x + 6) - 5[Ae^{3x}(3x + 1)] + 6(Axe^{3x}) = e^{3x}$$

Regrouping terms

$$Ae^{3x}\left[(9x - 15x + 6x) + 6 - 5\right] = e^{3x} \Rightarrow Ae^{3x} = e^{3x} \Rightarrow A = 1$$

The particular solution is $Y = xe^{3x}$.

(h) $y'' - y' - 12y = x + e^{2x}$

We try $Y = Ax + B + Ce^{2x}$; $Y' = A + 2Ce^{2x}$; $Y'' = 4Ce^{2x}$;

Plugging in the equation, $4Ce^{2x} - A - 2Ce^{2x} - 12Ax - 12B - 12Ce^{2x} = x + e^{2x}$

$$-12Ax + (-A - 12B) + e^{2x}(4C - 2C - 12C) = x + e^{2x}$$

$$-12A = 1; \quad -A - 12B = 0; \quad -10C = 1 \Rightarrow A = -1/12; \; B = 1/144; \; C = -1/10$$

Problem 4.

(d) $x^2 y'' + 5xy' - 5y = 0$

This is an Euler-Cauchy equation. We use the formula (12), and form the auxiliary equation

$m^2 + (a - 1)m + b$, with $a = 5$ and $b = -5 \Rightarrow m^2 + 4m - 5 = 0$. $(m + 5)(m - 1) = 0$, $m = 1, -5$.

General solution is $y = c_1 x + c_2 x^{-5}$

(e) $x^2 y'' - xy' + y = 0.$

This is an Euler-Cauchy equation. We use the formula (12), and form the auxiliary equation

$m^2 + (a - 1)\,m + b$, with $a = -1$ and $b = 1 \Rightarrow m^2 - 2m + 1 = 0$; $(m - 1)^2 = 0$. Double root.

The general solution is $y = c_1 x + c_2\, x \ln|x|$

Problem 5.

5. Given the equation $y'' + (\sin x)\, y' + (1 - \cos x)\, y = 0$
   (a) show that $y_1 = \sin x$ is one solution
   (b) use the method of reduction of order to find a second, *l.i.* solution y_2.

Use formula (4)

$$u' = U = \frac{1}{y_1^2} \exp\left(-\int p(x)\, dx\right)$$

$$p(x) = \sin x \;\Rightarrow\; \int \sin x\, dx = -\cos x \;\Rightarrow\; \exp\left[-\int \sin x\, dx\right] = e^{\cos x} \;\Rightarrow\; U = \frac{e^{\cos x}}{\sin^2 x}$$

$u = \int U\, dx$, and from there we obtain $y_2 = u(x)\, y_1(x)$.