Chapter 2

FIRST-ORDER DIFFERENTIAL EQUATIONS

This chapter deals with methods of solution of first-order differential equations for several well-known types of equations. Each of these types has a specific procedure for solving it. It is the task of the student to first recognize the type of the given equation and then use the corresponding procedure for solving it.

2.1 Solving by Integration

Equations of the form $y^{(n)} = f(x)$ are solved by integrating $f(x)$ n times with respect to x, using the techniques of integration as you learned in previous math classes. The general solution of $y' = f(x)$ is

$$y = \int f(x)\,dx + C$$

EXAMPLE. Solve $\dfrac{y'}{2x} - \sin x^2 = 0$.

We have $y' = 2x \sin x^2$, so $y = \int 2x \sin x^2\,dx + C \Rightarrow y = -\cos x^2 + C$

EXAMPLE. Find the general solution of $e^x y' - x + 1 = 0$.

By elementary algebraic manipulations this equation can be written $y' = e^{-x}(x-1)$ Hence,

$$y = \int e^{-x}(x-1)\,dx = \int xe^{-x}\,dx - \int e^{-x}\,dx$$

Integrating by parts the first integral, we have

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

so the solution is

$$y = -xe^{-x} - e^{-x} + e^{-x} + C = -xe^{-x} + C$$

The general solution of $y'' = f(x)$ is obtained by integrating twice with respect to x

$$y' = \int f(x)\,dx + C_1 \quad \Rightarrow \quad y = \int \left(\int f(x)\,dx + C_1 \right) dx = \int\int f(x)\,dx + C_1 x + C_2$$

and so on. In general, the solution of $y^{(n)} = f(x)$ can be found by integrating $f(x)$ n times

$$y = \int dx \int dx \dots \int f(x)\,dx + C_1 + C_2 x + C_3 \frac{x^2}{2} + \dots + C_n \frac{x^{n-1}}{(n-1)!}$$

EXAMPLE. Find the general solution of $y'' = \sin x - 6x + 1$

Finding the solution requires integrating the function twice with respect to x

$$y' = \int (\sin x - 6x + 1)\,dx = -\cos x - 3x^2 + x + C_1$$

$$y = \int (-\cos x - 3x^2 + x + C_1)\,dx = -\sin x - x^3 + \frac{x^2}{2} + C_1 x + C_2$$

2.2 Integrating Factors

Sometimes the problem of finding the solution of a differential equation can be simplified by multiplying the whole equation by a suitable function $\mu(x)$, called ***integrating factor.*** Consider for instance the equation

$$y' - 2y = e^x$$

This is a first order linear equation. The solution of this problem can be greatly simplified by multiplying the whole equation by the integrating factor $\mu(x) = e^{-2x}$. Indeed

$$e^{-2x} y' - 2\, e^{-2x} y = e^{-x}$$

The left-hand side is the derivative of the product of the functions $(e^{-2x})(y)$, thus we have

$$\frac{d}{dx}\left(e^{-2x} y\right) = e^{-x} \quad \Rightarrow \quad e^{-2x} y = \int e^{-x}\,dx \Rightarrow \; e^{-2x} y = -e^{-x} + C \quad \Rightarrow \; y = -e^{x} + C\,e^{2x}$$

The student can easily verify that $y = -e^x + C\,e^{2x}$ is indeed the general solution of the equation.

Integrating factors, however, may be sometimes tricky to find. As we will see below, the solution of the general first-order linear equation can be easily obtained by multiplying the equation by some well-known integrating factor. We will see many examples of use of integrating factors later. The problems we are going to cover in this course require pretty standard integrating factors that can be obtained by simple integration, and do not present a big complexity. In the next section we will study a type of equation that requires the finding of an integrating factor.

2.3 First-Order Linear Equations

A first-order linear differential equations is of the form

$$\mathbf{y' + a(x)\,y = b(x)}$$

The general solution of such equation, if it exists, is always straightforward to calculate. We start by the simplest case, when the coefficients $a(x)$ and $b(x)$ are constant terms, not depending on x.

2.3.1 First-order Linear Equation With Constant Coefficients. Are a particular case of the general first-order linear equation, in which $a(x)$ and $b(x)$ are constant numbers (independent of x). The general form is

$$y' = py + q \tag{1}$$

THEOREM. The general solution of (1) is

$$y = -q/p + C\,e^{px} \tag{2}$$

Proof. By standard algebraic manipulations, (1) can be written as

$$y' = p(y + q/p) \;\Rightarrow\; \frac{y'}{y+q/p} = p\,.$$

Because $\frac{d}{dx}\ln u = \frac{u'}{u}$, we have

$$\frac{y'}{y+q/p} = p \;\Rightarrow\; \frac{d}{dx}\left(\ln\left|y+q/p\right|\right) = p\,.$$

Integrating both sides of the equation above yields

$$\ln\left|y+q/p\right| = px + k\,,$$

where k is the constant of integration. Applying the definition of logarithm

$$y + q/p = e^{px+k} = e^{k}\,e^{px} = C\,e^{px},$$

which is (4), and this proves the theorem.

EXAMPLE. Solve the initial value problem $\frac{y'}{3} + \frac{y-1}{2} = 0$, $y(0) = 0$.

Write $$y' = -\frac{3}{2}y + \frac{3}{2}$$

Applying the theorem above with $p = -3/2$ and $q = 3/2$ we get the general solution

$$y = 1 + Ce^{-3x/2}\;.$$

The initial value $y(0) = 0$ makes $0 = 1 + C$ therefore, $C = -1$, so the final answer is

$$y = 1 - e^{-3x/2}$$

2.3.2 General First-Order Linear Equations. Next we study the general case of the first-order linear equation

$$y' + a(x)\,y = b(x)\;, \tag{1}$$

where now $a(x)$ and $b(x)$ are functions of x. The solution is found by multiplying the entire equation by a suitable *integrating factor*, μ(x).

THEOREM. The solution of (1) is

$$y = \frac{\int \mu(x) b(x) dx + C}{\mu(x)} \tag{2}$$

where the integrating factor, $\mu(x)$ is calculated as

$$\mu(x) = \exp \int a(x) dx \tag{3}$$

Proof. Taking logarithms in both sides of (3),

$$\ln \mu(x) = \int a(x) dx$$

Then differentiating on both sides,

$$\frac{d}{dx} \ln \mu(x) = \frac{\mu'(x)}{\mu(x)} = a(x)$$

Therefore, $\mu'(x) = \mu(x)\, a(x)$. Multiplying (1) by $\mu(x)$, gives

$$\mu(x) y' + \mu(x)\ a(x)\ y\ = \mu(x) b(x) \tag{4}$$

Substituting $\mu(x)\, a(x)$ by $\mu'(x)$ yields

$$\mu(x) y' + \mu'(x)\ y\ = \mu(x) b(x)$$

The left-hand side is the derivative of a product, that is

$$\frac{d}{dx}\left(\mu(x) y\right) = \mu(x) b(x)$$

Integrating both sides with respect to x

$$\mu(x) y = \int \mu(x) b(x)\, dx$$

and this is (4) as we wanted to prove.

EXAMPLE. Find the solution of the initial value problem

$$y' - y/2 = e^{-x}, \quad y(0) = -1$$

It follows from (2) that $\int a(x) dx = \int \left(-\frac{1}{2}\right) dx = -\frac{x}{2}$, so $\mu(x) = e^{-x/2}$. Thus the general solution is

$$y = \frac{\int e^{-x/2} e^{-x} dx + C}{e^{-x/2}} = e^{x/2}\left(\int e^{-3x/2} dx + C\right) = e^{x/2}\left(-\frac{2}{3} e^{-3x/2} + C\right) = -\frac{2}{3} e^{-x} + C e^{x/2}.$$

The particular solution $y(0) = -1$ gives $-2/3 + C = -1$ holds with $C = -1/3$.

EXAMPLE. Find the general solution of $y' + 2xy = x$.

Here $a(x) = 2x$ and $b(x) = x$. Thus, $\mu(x) = \exp\left(\int 2x\, dx\right) = e^{x^2}$

Hence, the general solution is $y = \frac{\int x e^{x^2}}{e^{x^2}} = \frac{\frac{1}{2} e^{x^2} + C}{e^{x^2}} = e^{-x^2}\left(e^{x^2} + C\right) = \frac{1}{2} + C e^{-x^2}$

2.4 Separable Differential Equations

An equation of the form

$$g(y)\, y' = f(x)$$

is called ***separable***. Note that separable equations are not linear. Since $y' = dy/dx$, the last equation can be written alternatively as

$$g(y)\, dy = f(x)\, dx$$

The solution is obtained by integrating both sides with respect to *x*,

$$\int g(y)\, dy = \int f(x)\, dx + C$$

Most of the times, the solution will appear in implicit form. You may be required to find an explicit form of the solution.

EXAMPLE. Find the general solution of $2\sqrt{x}\, y' = \sqrt{1-y^2}$

Because

$$2\sqrt{x}\frac{dy}{dx} = \sqrt{1-y^2}\,,$$

the equation is separable. Simple algebraic manipulations lead to

$$\frac{1}{\sqrt{1-y^2}}dy = \frac{1}{2\sqrt{x}}dx\,.$$

Integrating both sides,

$$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{1}{2\sqrt{x}}dx + C\,, \text{ that is. } \sin^{-1} y = \sqrt{x} + C$$

The left-hand side of the equation is the inverse trigonometric function and not to $1/\sin y$. There is an explicit form for this solution; by the definition of the inverse function, since $\sin(\sin^{-1} y) = y$, taking sin in both sides of the last equality,

$$y = \sin\left(\sqrt{x} + C\right).$$

2.5 Exact Equations

A differential equation of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{7}$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{8}$$

EXAMPLE. Verify that the differential equation $(2x + 3y)\, dx + (3x + 2y)\, dy = 0$ is exact.

Solution. $M(x, y) = 2x + 3y$ and $N(x, y) = 3x + 2y$. It follows that

$$\frac{\partial M}{\partial y} = 3 \quad and \quad \frac{\partial N}{\partial x} = 3$$

Solving Exact Equations.

For (7) and (8) to hold together, there must exist a function $F(x, y)$ such that its differential is

$$dF(x, y) = F_x dx + F_y dy = M(x, y)\,dx + N(x, y)\,dy\,. \tag{9}$$

Indeed, if

$$F_x = M(x,y) \text{ and } F_y = N(x,y), \tag{10}$$

since $F_{xy} = F_{yx}$, we have

$$\frac{\partial M}{\partial y} = F_{xy} \quad \text{and} \quad \frac{\partial N}{\partial x} = F_{xy}$$

so (9) readily follows. If we can find the expression of $F(x, y)$, then the general solution of (6) will be $F(x, y) = C$. To find that function F, note that (10) implies that

$$\int M(x, y)\,dx = \int F_x(x, y)dx = F(x, y) + C_1(y) \tag{10a}$$

and

$$\int N(x, y)\,dy = \int F_y(x, y)dy = F(x, y) + C_2(x)$$

The constant C_1 must be constant with respect to the variable of integration, x, but not necessarily constant with respect to y, which means that it may depend on y; this is why we write $C_1(y)$. Same consideration for $C_2 = C_2(x)$. Next step consists of differentiating in (10a) with respect to y

$$F_y(x, y) + C'_1(y) = N(x, y)$$

The variable x disappear (if it doesn't revise your calculations) and we find $C_1(y)$

Solution of an exact equation. Let's solve the exact differential equation given in the example above.

$$(2x + 3y)\,dx + (3x + 2y)\,dy = 0$$

to illustrate the solution.

First Step. Integrate $M(x, y)$ with respect to x

$$F(x, y) = \int (2x + 3y)dx = x^2 + 3xy + C_1(y)\ , \tag{11}$$

Second Step. To find $C_1(y)$, differentiate $F(x, y)$ with respect to y.

$$F_y(x, y) = \frac{d}{dy}\left(x^2 + 3xy + C_1(y)\right) = 3x + C'_1(y) = N(x, y)$$

But

$$F_y(x,y) = N(x,y), \text{ that is } F_y = 3x + C_1'(y) = 3x + 2y$$

We conclude that

$$C_1'(y) = 2y.$$

Note that $C_1'(y)$ cannot depend on x; if your calculations lead to an expression of $C_1'(y)$ that depends on x, revise your calculations, you made a mistake.

Third Step. Integrate both sides of the last equality with respect to y,

$$C_1(y) = \int C_1'(y)\,dy = \int 2y\,dy = y^2 + C \qquad (12)$$

where the last constant, C is a real constant depending neither on x nor on y. From (11) and (12), the solution pf our problem, therefore, is,

$$F(x, y) = x^2 + 3xy + y^2 + C = 0$$

Alternatively, we can use a symmetric argument by interchanging the roles of x and y and M and N. That is, integrating first N with respect to y, getting a constant of integration $C_1(x)$ and then following a symmetrical procedure. We leave this as an exercise for the student to verify that it yields same answer. While both procedures provide same solution, the integrals may not have equivalent degree of difficulty, and one of the methods may yield easier integrals to calculate than the other.

2.6 Substitution Methods

Sometimes a differential equation is neither separable nor exact, but by means of a suitable substitution $u = u(x,y)$, it can be rendered into one of these types.

EXAMPLE. Transform the differential equation $y' = \frac{y}{x} - \frac{x^2}{3y^2}$ into a separable equation by applying the substitution $u = y/x$ and then solve it.

This equation is neither separable nor exact. Employing the substitution $u = y/x$ it becomes separable. Indeed, $y = u\,x$ and $y' = u'x + u$. The equation is thus transformed into

$$u'x + u = u - \frac{1}{3u^2} \Rightarrow 3u^2\,u'x = -1 \Rightarrow 3u^2 du = -\frac{1}{x}dx$$

which is separable. Integrating,

$$u^3 = -\ln|x| + C$$

Returning to the original values of x and y, we finally get the general solution

$$\frac{y^3}{x^3} = -\ln|x| + C \Rightarrow y = -x\left(\ln|x| + C\right)^{1/3}$$

EXAMPLE. Find the general solution of $xy' = -3y - 2\sqrt{xy}$ by employing the substitution $u = \sqrt{xy}$

We have $y = \frac{u^2}{x}$ so $y' = \frac{2uu'x - u^2}{x^2}$. Substituting y and y' in the equation and simplifying

$$x\frac{2uu'x - u^2}{x^2} = -3\frac{u^2}{x} - 2u \Rightarrow \frac{2uu'x - u^2}{x} = -3\frac{u^2}{x} - 2u$$

Multiplying the whole equation by x and dividing by u

$$2u'x - u = -3u - 2x \Rightarrow 2u'x = -2u - 2x \Rightarrow u'x = -u - x \Rightarrow (u + x)\,dx + x\,du = 0$$

which is exact. To solve the exact equation using the method explained above, first integrate $M(x, u) = u + x$ with respect to x, so

$$F(x, u) = xu + x^2/2 + C(u).$$

Then differentiate with respect to u,

$$F_u = x + C'(u) = N(x,u) = x \Rightarrow C'(u) = 0$$

Finally integrate $C'(u) = 0$, which gives $C(u) = C$. The solution of the exact equation is, therefore,

$$xu + x^2/2 = C$$

Substituting u by its expression in x and y

$$x\sqrt{xy} + \frac{x^2}{2} = C \Rightarrow \sqrt{xy} = \frac{C}{x} - \frac{x}{2}$$

So, the general solution of the differential equation is

$$y = \frac{(C/x - x/2)^2}{x}$$

2.7 Integrating Factors Revisited

Sometimes the solution of a differential equation can be greatly simplified by multiplying the whole equation by a function $\mu(x)$ called **integrating factor**. We have seen how to use an integrating factor in order to solve the first-order linear equation. Integrating factors can also be used to transform a non-exact equation into an exact one.

EXAMPLE. The equation $3xy + y^2 + (x^2 + xy)\, y' = 0$ is not exact. Show that the integrating factor $\mu(x) = x$ transforms the equation into an exact equation and then solve it.

This equation can be written alternatively as

$$(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0,$$

which is neither separable nor exact. Multiplying the equation by the integrating factor $\mu(x) = x$,

$$\mu(x)\,(3xy + y^2)\, dx + \mu(x)\,(x^2 + xy)\, dy = 0 \quad \Rightarrow \quad (3x^2y + x\,y^2)dx + (x^3 + x^2y)\, dy = 0$$

which is exact. Indeed $M_y = 3x^2 + 2xy = N_x$

Integrating $M(x,y)$ with tespect to y,

$$F(x,y) = \int \left(3x^2y + xy^2\right)dx = x^3y + \frac{x^2}{2}y^2 + C(y) \tag{13}$$

differentiating on y,

$$F_y(x,y) = N(x,y) = x^3 + x^2y + C'(y) = x^3 + x^2y \Rightarrow C'(y) = 0$$

Therefore $C(y) = C$. Substituting in (13) we obtain the general solution

$$x^3 y + \frac{1}{2}x^2 y^2 = C$$

How did we find in this example the right integrating factor that converts the equation into an exact one? Integrating factors may be hard to find. In section 2.3.2 we have seen a standard procedure to find an integrating factor to solve first-order differential equations. There is also a standard procedure for finding integrating factors to convert a non exact equation into an exact equation such as the case of the previous example. In other cases, if an integrating factor is needed, it will be provided to the student.

THEOREM. Suppose that the differential equatin

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is not exact and that the expression

$$Q(x) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}$$

Depends only on x, then the equation can be converted to an exact equation by multiplying by the integrating factor

$$\mu(x) = \exp\left(\int Q(x)\, dx\right)$$

In the case of the previous example we have

$$Q(x) = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$$

It follows that

$$\mu(x) = \exp\left(\int x^{-1}\, dx\right) = \exp(\ln x) = x$$

2.8 Homogeneous Equations

A first-order differential equation is homogenous if it can be written in the form

$$y' = F(y/x) \quad \text{or} \quad y' = F(x/y) \tag{15}$$

EXAMPLE. The differential equation $2xyy' = 4x^2 + 3y^2$ is homogenous. Indeed, it can be written as

$$y' = \frac{4x^2+3y^2}{2xy} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right) = \frac{2}{(y/x)} + \frac{3}{2}(y/x)$$

$$y' = \frac{2}{(y/x)} + \frac{3}{2}(y/x)$$

Solving first-order homogenous differential equations. Use the substitution $u = y/x$, $y = u\,x$,

$$y' = u + x\,u', \tag{16}$$

in which case the equation is transformed into a separable equation. Indeed, substituting (15) in (14) results in

$$xu' + u = F(u), \text{ or } \frac{du}{F(u)-u} = \frac{dx}{x}$$

which is separable.

Example. Solve the homogeneous differential equation $2xyy' = 4x^2 + 3y^2$

Transforming it $y' = \frac{2}{(y/x)} + \frac{3}{2}(y/x)$. Using the substitution $u = y/x$, $y = u\,x$, $y' = u + x\,u'$,

$$u + xu' = \frac{2}{u} + \frac{3}{2}u \;\Rightarrow\; xu' = \frac{2}{u} + \frac{u}{2} \;\Rightarrow\; xu' = \frac{u^2+4}{2u}$$

This equation is always separable, since it is of the form

$$x\frac{du}{dx} = \frac{u^2+4}{2u} \;\Rightarrow\; \frac{2u}{u^2+4}du = \frac{dx}{x}$$

Integrating $\int \frac{2u}{u^2+4}du = \int \frac{1}{x}dx$ So the general solution is $\ln(u^2+4) = \ln|x| + C_1$

We can find an explicit solution by applying the exponential function to both sides,

$$u^2 + 4 = \exp(\ln|x| + C_1) = e^{C_1}|x| = C|x|$$

$u = (C|x| - 4)^{1/2}$ returning the substitution back $y/x = (C|x| - 4)^{1/2}$

Hence, the explicit solution is $y = \pm\sqrt{Cx^3 - 4x^2}$

2.9 Bernouilli Equations

A first-order differential equation of the form

$$y' + a(x)\,y = b(x)\,y^n \qquad (16)$$

is called a *Bernouilli Equation*. If $n = 0$ or $n = 1$ then the equation is linear. If $n > 1$.

Solving Bernouilli Equations. The substitution $u = y^{1-n}$ transforms equation (16) into the linear equation

$$\boldsymbol{u' + (1-n)\,a(x)\,u = (1-n)\,b(x)} \qquad (17)$$

Indeed, $y = u^{1/(1-n)}$ and $y' = \frac{1}{1-n}u'u^{n/(1-n)}$. Substituting in (16)

$$\frac{1}{1-n}u'u^{n/(1-n)} + a(x)u^{1/(1-n)} = b(x)u^{n/(1-n)}$$

Multiplying the whole equation by $(n-1)\,u^{-n/(1-n)}$ we readily obtain (17)

EXAMPLE. Solve the Bernouilli equation $xy' + 6y = 3xy^{4/3}$

First convert it into the standard form (16) yields $y' + \frac{6}{x} y = 3y^{4/3}$. Here $n = 4/3$, $a(x) = 6/x$, $b(x) = 3x$. The substitution $u = y^{1-4/3} = y^{-1/3}$ ($y = u^{-3}$ $y' = -3u^{-4}u'$) transforms it into the form (17), that is

$$u' - \frac{2}{x}u = -1 \;\Rightarrow\; u' - \frac{2}{x}u = -1$$

which is linear, so we can use the method to solve linear equations. The integrating factor is

$$\mu(x) = e^{\int (-2/x)\,dx} = e^{-2\ln x} = x^{-2}$$

and the general solution is

$$u = \frac{\int x^{-2}(-1)\,dx + C}{x^{-2}} = \frac{x^{-1} + C}{x^{-2}} = x + Cx^2$$

Rewriting the solution in terms of y and x we finally obtain the general solution

$$y^{-1/3} = x + Cx^2 \;\Rightarrow\; y = \left(x + Cx^2\right)^{-3}$$

Chapter 2 Summary

Types of equations, form of the equation, and solution		
Type	**Form of Equation**	**Solution**
Integrals	$y^{(n)} = f(x)$	$y = \int dx \ldots \int f(x)dx + C_1 + C_2 x + C_3 x^2 + \ldots + C_n x^{n-1}$
Linear 1st order const coefficients	$y' = py + q$	$y = -q/p + C\,e^{px}$.
General linear	$y' + a(x)y = b(x)$	$y = \dfrac{\int \mu(x) b(x)\,dx + c}{\mu(x)}, \quad \mu(x) = \exp \int a(x)\,dx$
Separable	$g(y)\,y' = f(x)$ or $f(x)\,dx = g(y)\,dy$	$\int g(y)\,dy = \int f(x)\,dx + C$
Exact	$M(x, y)\,dx + N(x, y)\,dy = 0$ with $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$	**Integrate *M* on *x* and get F(*x*,y) + C(y); differentiate on *y* and find the solution on section 2.5 page 10**
Homogeneous	$y' = F\left(\dfrac{y}{x}\right)$ or $y' = F\left(\dfrac{x}{y}\right)$	**substitution *u = y/x***
Bernouilii	$y' + a(x)\,y = b(x)\,y^n$	**substitution $u = y^{1-n}$ *transforms into*** $u' + (1-n)\,a(x)\,u = (1-n)\,b(x)$
Other substitutions	Various problems	**Homogeneous and Bernouilli equations have mechanical substitutions. For other equations, if substitution is required, it will be suggested**
Integrating factors	Various problems	**Integrating factor will be provided**

HOMEWORK

Kreyszig, p. 18, #2-19; p. 25, #1-20, p. 32, #3-17

Additional problems

1. Find the general solution of the following differential equations and verify your answer

(a) $3y' - 1 = y$ (b) $x^2 y'' - 1 = x^2 \sin x$

(c) $3y' = y + e^{2x}$ (d) $y' + e^x y = e^{-x}$

(e) $y' = \dfrac{x - e^{-x}}{y + e^y}$ (f) $(2y\sin x + x^2 + y)y' + y^2 \cos x + 2xy = 0$

(g) $y' = \dfrac{2x}{1+2y}$ (h) $\dfrac{y'}{y} + 3 = \dfrac{2}{y}$

(i) $xy' = y + \sqrt{x^2 - y^2}$ (j) $xy' = \sqrt{1 - y^2}$

(g) $\dfrac{dy}{dx} + \dfrac{y}{x} = xy^2$ (h) $3x^2 y + 2x - 3y)\,dx + (x^3 - 3x + y^2)\,dy = 0$

(i) $y' = \dfrac{y - 4x}{x - y}$ (j) $y'' - e^x = 6x + 1$

2. Use the proposed integrating factor to transform the given differential equations into an exact equation and solve it

(a) $x^2 y^3 + x(1 + y^2)y' = 0;\quad \mu(x) = 1/xy^3$

(b) $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0;\quad \mu(x) = ye^x$

3. Use the proposed substitution to solve the following differential equations

(a) $y' = (x + y + 3)^2;\quad u = x + y + 3$ (b) $2xe^{2y}\dfrac{dy}{dx} = 3x^4 + e^{2y};\quad u = e^{2y}$

4. Find an explicit solution of the following differential equations and verify your answer

(a) $y' = \dfrac{x(x^2+1)}{4y^3}$ (b) $xy' = \sqrt{1 - y^2}$ (c) (c) $y' - y = 2x^2 e^{2x}$

(d) $\dfrac{y''}{x} + 1 = e^{-x} + \dfrac{2}{x}$ (e) $\sin(2x)\,dx + \cos(3y)\,dy = 0$

5. Solve the following initial value problems

(a) $y' + 2xy = 2xe^{-x^2};\quad y(0) = 1$ (b) $y' + 3x^2 y = 3x^2;\quad y(0) = 0$

(c) $xy' + y = x\sin x;\quad y(\pi) = -1$

SOLUTION

1 (a) $3y' - 1 = y$

Linear constant coefficients, $y' = y/3 + 1/3 \Rightarrow p = 1/3,\ q = 1/3 \Rightarrow$ **$y = -1 + Ce^{x/3}$**

(b) $x^2 y'' - 1 = x^2 \sin x \Rightarrow y'' = \dfrac{1}{x^2} + \sin x$ Hence, direct integration.

$$y' = \int \left(x^{-2} + \sin x\right)dx = -x^{-1} - \cos x + C_1 \quad \Rightarrow \quad y = \int \left(x^{-1} - \cos x + C_1\right)dx = \ln|x| - \sin x + C_1 x + C_2$$

(c) $3y' = y + e^{2x}$

First-order linear non constant coefficients . Write it in the Standard format

$y' - y/3 = (1/3)e^{2x}$ $a(x) = -1/3,\ b(x) = (1/3)e^{2x}$

$$\mu(x) = \exp\left(\int (-1/3)dx\right) = e^{-x/3}$$

$$y = \frac{\int e^{-x/3}(1/3)e^{2x}dx + C}{e^{-x/3}} = (1/3)e^{x/3}\left(\int e^{5x/3}dx + C\right) = (1/3)e^{x/3}\left[(3/5)e^{5x/3} + C\right] = (1/5)e^{2x} + Ce^{x/3}$$

(d) $y' + e^x y = e^{-x}$

Linear first-order non constant coefficients

(e) $y' = \dfrac{x - e^{-x}}{y + e^{y}}$

Separable, $\Rightarrow (y + e^y)\,dy = (x - e^{-x})\,dx$ Thus $\int \left(y + e^y\right)dy = \int \left(x - e^{-x}\right)dx + C$

We get $y^2/2 + e^y = x^2/2 + e^{-x} + C.$

(f) $(2y\sin x + x^2 + y)y' + y^2\cos x + 2xy = 0$

Exact, indeed, write $\left(y^2\cos x + 2xy\right)dx + \left(2y\sin x + x^2 + y\right)dy = 0$

$M_y = 2y\cos x + 2x;\ N_x = 2y\cos x + 2x.$

$$\int \left(y^2\cos x + 2xy\right)dx = y^2\sin x + x^2 y + C(y) = F(x, y)$$

Then $F_y = 2y\sin x + x^2 + C'(y) = 2y\sin x + x^2 + y \Rightarrow C'(y) = y \Rightarrow C(y) = y^2/2$

Hence the solution is **$y^2 \sin x + x^2 y + y^2/2 = C$**