Chapter 1

INTRODUCTION

1.1 Preliminaries

For this course the student should be well acquainted with the concepts of differentiation and integration covered in Mathematics I and II. Review these topics as you feel necessary, since they are the basis for solving differential equations problems.

Differential equations are of basic importance in many basic areas of science. It has applications in a number of fields, including physics and engineering. In this course we shall deal exclusively with *Ordinary Differential Eauqtions*, which involve only functions of one variable. Let $y = y(x) = f(x)$ be an adequately differentiable function on some given domain. Its derivative will be denoted as y' *or* dy/dx. Higher order derivatives will be denoted by y'' or d^2y/dx^2, y''' or d^3y/dx^3, etc. Higher order derivatives will be denoted with a superscript in parenthesis. For instance, the fourth derivative will be denoted as $y^{(4)}$ rather than y''''. The partial derivatives of a function $F(x, y) = 0$ are denoted

$$\frac{\partial F}{\partial x} \; or \; F_x \quad \text{and} \quad \frac{\partial F}{\partial y} \; or \; F_y \; .$$

The exponential function $e^{u(x)}$ can be written alternatively as $\exp(u(x))$. This is particularly useful when $u(x)$ has a complex expression.

A function of the form $y = f(x)$ is given in explicit form. For instance, the functions

$$y = \frac{x - \sin x}{x^2 + \ln|x|} \quad \text{and} \quad y = \tan(1+x^2) - e^{-x}$$

are explicit functions. On the other hand, functions given by an equation of the form $F(x, y) = 0$ are implicit functions. Examples of implicit functions are

$$x^2y + 2y^2 + x^3 - 1 = 0 \quad \text{and} \quad \ln\left(\frac{y}{x}\right) + \cos(xy-1) = 0.$$

In some cases, an explicit expression of an implicit function can be found by solving the equation for y. The function

$$x^2 \ln\left|y(x^2-1)\right| - 2x + 5 = 0$$

for instance, admits an explicit form. For this,

$$\ln\left|y(x^2-1)\right| = \frac{2x-5}{x^2} \quad \Rightarrow \quad y(x^2-1) = \exp\left(\frac{2x-5}{x^2}\right) \Rightarrow \; y \; = \frac{1}{x^2-1}\; \exp\left(\frac{2x-5}{x^2}\right)$$

Other times, finding an explicit expression may be too hard or impossible. To calculate the derivative of an implicit function $F(x, y) = 0$, we first calculate the differential

$$dF = F_x\, dx + F_y\, dy = 0,$$

which leads to

$$y' = dy/dx = -F_x/F_y.$$

For example, the implicit derivative of $x^2y + 2y^2 + x^3 - 1 = 0$ is

$$y' = \frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 + 4y}$$

1.2 Definition

In this course, differential equations will involve only functions of one variable, $y = y(x)$ and one or more of its derivatives, $y', y'', \ldots y^{(n)}$. In the quadratic equation $3x^2 - x - 2 = 0$, x is the unknown. A solution, such as $x = 1$, is a value of x that makes the equality true.. In the differential equation

$$y' - y - e^x = 0$$

$y = y(x)$ is a function, and y is the unknown. Solving the equation means finding functions y that make the equality true for all values of x in some given interval (a, b). It's easy to see that $y = xe^x$ is a solution of the differential equation. Indeed, differentiating, $y' = e^x + xe^x$. Substituting y and y' in the equation we get

$$e^x + xe^x - xe^x - e^x = 0$$

which is true. While $y = xe^x$ is a solution, the general solution of the equation is the set of all functions that make the equality true

DEFINITION. An equation involving a function $y = y(x)$ and one or more of its derivatives is called a **differential equation**. The general form of a differential equation is

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

The **order** of a differential equation is the order of the highest derivative involved.

EXAMPLES.

(1) $(1-x^2)\dfrac{dy}{dx}=2y$ *or* $y'-x^2y'=2y$ *or* $(1-x^2)\,dy=2y\,dx$

(2) $x(x+y)\,y' = y\,(x - y)$

(3) $4y''-12y'+9y = 0$

(4) $y^{(3)} + 3y'' - 4y = 0$

Equations (1) and (2) above are first order differential equations. Equation (3) is second order, and equation (4) is third order.

DEFINITION. A ***solution*** of a differential equation is a function $y = y(x)$ in an interval $a < x < b$ that satisfies the equation. The ***general solution*** is the set of all particular solutions of the equation.

EXAMPLE. Show that $y = x^2$ is a solution of the differential equation $xy' = 2y$.

Substituting the function y and its derivative, $y' = 2x$ into the differential equation, we get

$$x(2x) = 2(x^2), \quad \text{or} \quad 2x^2 = 2x^2,$$

which is true for all real values of x.

EXAMPLE. Show that $y = x^2 + \dfrac{1}{x}$ is a solution of $2xy - 2x^2y' + x^3y'' - 6 = 0$

The first and second derivatives of y are $y'=2x-\dfrac{1}{x^2}$, $\quad y''=2+\dfrac{2}{x^3}$

Substituting in y y' and y'' in the equation we get

$$2x\left(x^2+\frac{1}{x}\right)-2x^2\left(2x-\frac{1}{x^2}\right)+x^3\left(2+\frac{2}{x^3}\right)-6=0$$

Performing the products and simplifying,

$$2x^3+2-4x^3+2+2x^3+2-6=0 \quad \text{that is } 0 = 0$$

so the equality holds. In this course we will learn how to solve differential equations.

Differential equations occur in many practical problems in engineering, physics, mechanics, and in sciences in general. This is why it is important to learn how to solve differential equations. Most of the times the solution of a differential equation is not unique. There is a family of functions, depending in one or more constants, all of which solve the given differential equation.

Example. A family of solutions of the differential equation

$$y - 2y' + y'' = x^2 - 3x \tag{3}$$

is given by

$$y = x^2 + x + Ce^x, \tag{4}$$

where C is an arbitrary constant. For this,

$$y' = 2x + 1 + Ce^x \text{ and } y'' = 2 + Ce^x .$$

Substituting in (3) we get

$$x^2 + x + Ce^x - 2(2x + 1 + Ce^x) + 2 + Ce^x = x^2 - 3x$$

$$x^2 + x + Ce^x - 4x - 2 - 2Ce^x + 2 + Ce^x = x^2 - 3x$$

$$x^2 - 3x = x^2 - 3x$$

so (4) is indeed the ***general solution*** of (3). A ***particular solution*** satisfying an additional constraint can be found. Suppose that the particular solution must satisfy the additional condition $y(0) = 1$. The particular solution that satisfies this condition is for $C = 1$. Indeed,

$$y(0) = Ce^0 = C = 1$$

We say in this case that the function $y = x^2 + x + e^x$ is a solution of the ***initial value*** problem of the differential equation (4) with the initial value constraint $y(0) = 1$.

The solution of a differential equation can sometimes be found in explicit form, $y = \mathrm{y}(x) = f(x)$. While explicit solutions are preferable to implicit ones, sometimes explicit solutions may be too hard to find, but an implicit solution of the form, $f(x, y) = 0$, may be accessible. Unless you are requested specifically to find an explicit solution, you can solve the problem by giving an implicit solution of a differential equation.

Often the solution consists of a family of functions involving an arbitrary constant C.

EXAMPLE. The solutions of the equations above are (verify)

(1) $y = C(1+x)/(1 - x)$,

(2) $\ln|xy| - x/y + C = 0$

(3) $y = e^{3x/2}(C_1 + C_2)$

(4) $y = C_1e^x + e^{-2x}(C_2 + C_3)$

Here the solution (2) is given in implicit form, all the others are explicit solutions.

<u>Verification of (2)</u>. The implicit derivative of the proposed solution of equation (2) is

$$y' = -\frac{f_x}{f_y} = -\frac{\dfrac{y}{xy} - \dfrac{1}{y}}{\dfrac{x}{xy} + \dfrac{x}{y^2}},$$

which after simplification (multiply numerator and denominator by xy), becomes

$$y' = \frac{y(x-y)}{x(x+y)}.$$

as we wanted to prove.

Some differential equations that occur in physics and engineering applications may be too difficult to solve. In many cases, a transcendental function may even not exist (a function is transcendental if it can be expressed in terms of a combination of algebraic formulas, trigonometric, logarithmic or exponential expressions). In these cases, approximate solutions can be found by the techniques of Numerical Analysis. The differential equations you will have to solve in this first part of the course are simple and of a very specific form, for which general solutions are straightforward to be computed.

Initial value solutions. The solutions of (1) through (4) are *general solutions*. They consist of a family of functions, one for each particular value of the constants C_i. A particular solution requires finding the value(s) of the constant(s) that satisfy additional condition(s). For instance, the particular solution of (1) that satisfies the initial value condition $y(0) = 2$, requires the specific value $C = 2$ (just substitute x by 0 in the solution of (1) and solve for C).

1.3 Linear Differential Equations

A differential equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x),$$

where the coefficients $a_i(x)$ and $g(x)$ are some given functions, is called a ***linear differential equation of order n.***

Non-linear equations contain nonlinear terms of the form y^2, yy', yy'', $y'y''$, $y^2y'y''$, etc. Equations (1), (3) and (4) above are linear. Equation (2) is not, since it has the nonlinear terms yy' on the left and $-y^2$ on the right-hand side. In Chapter 2 of these notes we cover the general solution of various types of first order differential equations. Chapter 3 deals with second and higher order linear differential equations.

EXAMPLE. The differential equation $3y'' - x^2y' + y/x = \ln x$ is linear, since it is linear on y and its derivatives, even though it is not linear on x. The differential equation $y''y' - 2xy^2 = x + 1$ is not linear.

Linear differential equations are of very important. Their solution is always easier to obtain that certain solutions of non-linear equations.

In Chapter 2 we tackle the solution of first order differential equations which, of course, are the simplest ones to be solved.

1.4 Memory refreshing

In this section we present some basics of the calculus, which are needed to perform well in this course. The list may not be exhaustive. More complete tables of integrals can be found on the cover of any Calculus book.

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int x^{-1}\,dx = \ln|x| + C \qquad \int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C \qquad \int u'(x)e^{u(x)}\,dx = e^{u(x)} + C \qquad \int \ln(ax)\,dx = x\ln(ax) - x + C$$

$$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + C \quad \text{(requires integration by parts)}$$

$$\int x^2e^x\,dx = e^x(x^2 - 2x + 2) + C \quad \text{(requires integrating twice by parts)}$$

$$\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + C \qquad \int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C \qquad \int \sec^2(ax)\,dx = \frac{\tan(ax)}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \quad \text{(the inverse sin function)} \qquad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int x\sin x\,dx = \sin x - x\cos x + C \quad \text{(requires integrations by parts)}$$

$$\int x^n \sin(x^{n+1)})\,dx = -\frac{\cos\left(x^{n+1}\right)}{n+1} + C \qquad \int x^n \cos(x^{n+1)})\,dx = \frac{\sin\left(x^{n+1}\right)}{n+1} + C \qquad \int x^n e^{x^{n+1}}\,dx = \frac{e^{x^{n=1}}}{n+1} + C$$

The last three integrals are solved by using the substitution $u = x^{n+1}$

Exponential and logarithmic functions. By definition, the logarithmic function is the inverse of the exponential function. This means that

$$\ln\left(e^x\right) = e^{\ln x} = x\text{, and with more generality,}\quad x^a = e^{a\ln x}$$

Moreover, for any pair of functions $u(x)$ and v(x), the equation ln |($u(x)$ | = $v(x)$ can be transformed into $u(x) = e^{v(x)}$ = exp ($u(x)$).

Inverse trigonometric functions. By definition, $y = \sin^{-1}(x)$ = arcsin x, $y = \cos^{-1}(x)$ = arcscos x and $y = \tan^{-1}(x)$ = arctan x are the inverse sine, inverse cosine, and inverse tangent functions respectively. Thus,

$$\sin(\sin^{-1}x) = \sin^{-1}(\sin x) = x, \quad \cos(\cos^{-1}x) = \cos^{-1}(\cos x) = x \quad\text{and}\quad \tan(\tan^{-1}x) = \tan^{-1}(\tan x) = x$$

And for any pair of functions $u(x)$ and v(x), the equation $\sin^{-1}(u(x)) = v(x)$ can be transformed into $u(x) = \sin(v(x))$. Similar relationships exist for cos x and tan x.

HOMEWORK

Kreyszig 9th ed. p. 8, problems 5 – 14

Additional Problems

1. Find the implicit derivative of the following functions

(a) $\ln\left|xy^2\right| - \dfrac{x+y}{x-y} + y^2 = 0$ (b) $\cos\left(x^2y^2\right) - 3\sin\left(\dfrac{x}{y}\right) + 7 = 0$ (c) $\sqrt{x^2+y^2} - 2xy - 5 = 0$

2. Verify that the given function is a general solution of the differential equation, and solve the initial value problem that satisfies the additional initial condition.

(a) $y'' + 2y' - 3y = 0$; solution $y(x) = Ce^{-3x}$; initial condition $y(0) = -1$

(b) $x^2y'' + 5xy' + 4y = 0,\ x > 0$; solution $y_1 = C\,x^{-2}$ init. cond. $y_1(1) = 2$; 2nd solution $y_2 = x^{-2}\ln x$;

(c) $y'' + y = \sec x,\ \ 0 < x < \pi/2$; $y = (\cos x)\ln\cos x + x\cos x$

(d) $xy' - 3y = x^3$; solution $y = x^3(C + \ln x)$; initial condition $y(1) = 3$

(e) $y' + y\tan x = \cos x$; solution $y = (x + C)\cos x$; initial condition $y(\pi) = 0$

SOLUTION

2 (a) $y'' + 2y' - 3y = 0$; solution $y(x) = Ce^{-3x}$; initial condition $y(0) = -1$

$y'(x) = -3Ce^{-3x}$, $y''(x) = 9Ce^{-3x}$

Substituting, $9Ce^{-3x} + 2(-3Ce^{-3x}) - 3(Ce^{-3x}) = Ce^{-3x}(9 - 6 - 3) = 0$

Solving the initial value problem means finding the value of C that satisfies the given condition, in this case $y(0) = -1$

Thus $-1 = y\,(0 = Ce^0 = C \Rightarrow C = -1$; and the particular solution is $\boldsymbol{y(x) = -e^{-3x}}$