Chapter 2

COMPLEX INTEGRATION

Some of the properties of complex functions are somewhat similar to the properties of real functions of two variables studied in MATH II. In fact, a function of a complex variables is function of two variables, the real part x and the imaginary part y. Of course, most of the behavior of complex functions is intrinsic to complex analysis.

In this chapter we will learn how to integrate complex functions. In general, integrating a complex functions can be performed by calculating two real valued integrals, one for the real part and one for the imaginary part. So, the student is encouraged to review some of the techniques of integration covered in previous calculus courses.

2.1 Line Integral in the Complex plane

Real functions are integrated along the real axis. Complex functions lay in the complex plane, and they are also integrated along lines or curves in the plane, also called *paths*. Before undertaking the task of integration, we must review the concept of *curve* -covered in MATH II in the two-dimensional real plane- in the complex plane.

2.1.1 Paths. In MAS117 the student was introduced to the concept of parametric equations in the two-dimensional real plane, which yield curves in the plane. Similar features are valid for complex functions, which can also be viewed as functions of two variables, x and y, the real and the complex part of z.

DEFINITION. A ***curve*** or ***path*** in the complex plane is a complex function $z(t)$ of a real variable t defined in some interval. We write

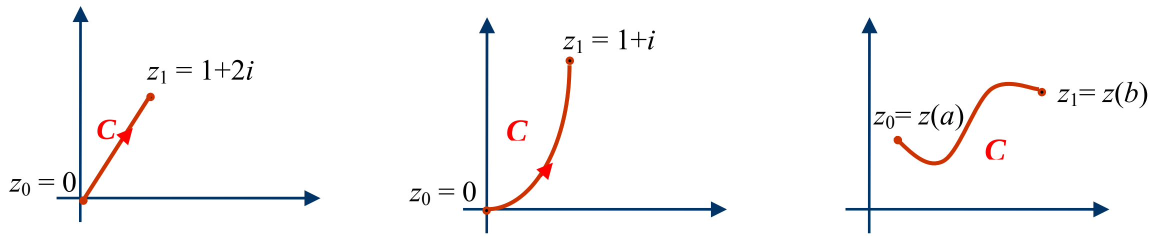
$$C: z(t) = x(t) + iy(t), \qquad a \le t \le b$$

Here $[a, b]$ is the interval in which the path is defined. The curve is said to be ***smooth*** if C has continuous non vanishing derivative at all points of $[a, b]$ that is, $z'(t) = x'(t) + iy(t)$ is always defined and never zero.

Geometrically it means that the curve is continuous or has no gaps and that is has no angular points.

Thus a curve is given by the parametric equation of the real and the imaginary parts, similarly as the parametric equations covered in MATH II. Figure 1 shows examples of paths in the complex plane. Note that the real variable t is an auxiliary variable, and does not show in the graph. The initial point of the path, z_0, corresponds to the value $t = a$, i.e., $z_0 = z(a)$ and the endpoint of the path is $z_n = z(b)$.

EXAMPLE. (a) If the functions $x(t)$ and $y(t)$ are linear then the path is a straight segment.

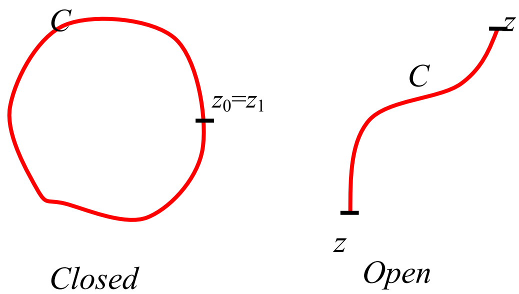


(a) $z(t) = t + 2ti,\ 0 \le t \le 1$ (b) $z(t) = t + i\,t^2,\ 0 \le t \le 1$ (c) $z(t) = x(t) + iy(t),\ a \le t \le b$

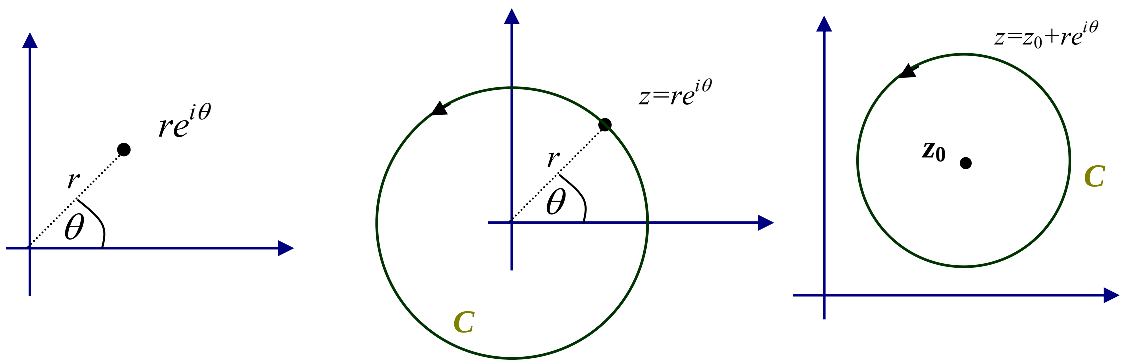
(b) The x coordinate is linear and the y coordinate is quadratic, this produces a parabola.
(c) In general the parametric equations would yield any curve or path.

Include examples of lines, rect, triang, circles, such as $z = z_0 + re^{it},\ a \le t \le b$

DEFINITION. A curve is *closed* if the initial point, $z_0 = z(a)$, and the end point, $z_1 = z(b)$, coincide, that is, if $z_0 = z_1$



EXAMPLE. Perhaps on of the most important closed curves is $z(t) = re^{it},\ 0 \le t \le 2\pi$. This is a circle of center at the origin and radius r, while $z(t) = z_0 + re^{it},\ 0 \le t \le 2\pi$ is a circle of center z_0 and radius r.



From Chapter 1 we know that $re^{i\theta}$ is the complex number whose modulus is 1 and whose argument is θ (here the letter θ is replaced by t, but both indicate the same thing, the argument of angle.) As

θ varies between 0 and 2π, the point $re^{i\theta}$ describes a circle with center at the origin and radius r. In (c), by adding z_0 to the equation we have $z(t) = z_0 + re^{i\theta}$, the center of the circle is moved to z_0. That is, the last equation represents a circle of center at the origin and radius r.

2.1.2 Partition of a path. If we include a set of points in an interval $[a, b\}$, it forms a set of disjoint intervals, called *partition.* The interval $[a, b]$ is thus partitioned in several intervals by inserting points t_1 t_2, … t_{n-1}, t_n such that

$$a = t_0 < t_1 < t_2 \;\ldots.\; < t_{n-1} < t_n = b$$

and there correspondent points in the complex plane will be

$$z_0 = z(a), \quad z_1 = z(t_1), \quad \ldots\ldots, \quad z_{n-1} = z(t_{n-1}), \quad z_n = z(t_n)$$

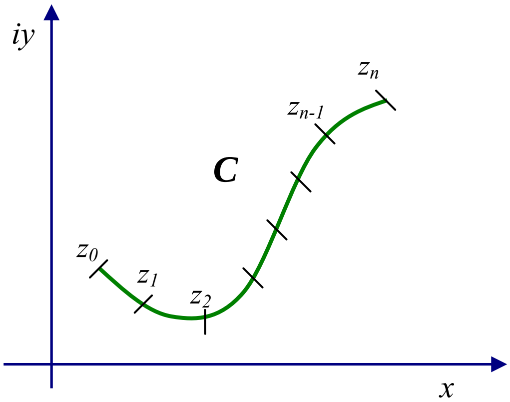


Fig 1. Complex Path

2.1.3 Definition of the complex integral. It is similar to the definition of integral for a function of two real variables studied last semester. Next we will define the line integral

$$\int_C f(z)\,dz$$

where $f(z)$ is the *integrand* and C is the *path of integration*. In other words, to pose the problem of complex integration, it is required a function f to integrate, and a curve C along which we will integrate. So we have the following:

(1) A set of points in the real interval $[a, b]$, $a = t_0 < t_1 < t_2 \;\ldots.\; < t_{n-1} < t_n = b$

(2) Corresponding points in the complex plane $z_0 = z(a)$, $z_1 = z(t_1)$, ……, $z_{n-1} = z(t_{n-1})$, $z_n = z(t_n)$

(3) The complex values of f at these points $f(z_0)$, $f(z_1)$, …… $f(z_{n-1})$, $f(z_n)$,

(4) For each interval we set the differences $\Delta z_i = z_i - z_{i-1}$

(5) We chose a point ξ_i on the curve C between z_{i-1} and z_i.

Then we form the sum

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta z_i$$

Now let n, the number of intervals in the partition, increase to infinity in such a way that the largest interval $t_i - t_{i-1}$ approaches zero. If the limit exists, it is the line integral along C of f.

DEFINITION. Given a complex function $f(z)$ and a smooth path C, the line integral of f along C, denoted by

$$\int_C f(z)\,dz$$

is defined as

$$\int_C f(z)\,dz = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(\xi_i)\Delta z_i\right) \tag{1}$$

provided that the limit exists.

The existence of the line integral is guaranteed if the function *f* and the curve *C* satisfy some pretty general assumption, as in the next theorem. The integral along a curve, therefore, is a complex number.

THEOREM 1. If $f(z)$ is continuous and C is a piecewise smooth path, then the line integral (1) exists

A piecewise path is a continuous path, which is smooth except perhaps on a finite number of points. All the functions we are going to deal with in this curse will be continuous, except eventually on a few points (such as ln z, which is not continuous at $z = 0$ but it is continuous everywhere else.) and all the curves we will consider will be piecewise smooth, so the student do not have to cope with existence problems.

Three basic properties of the complex line integral are

(1) Linearity

$$\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right) dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$$

(2) Sense reversal

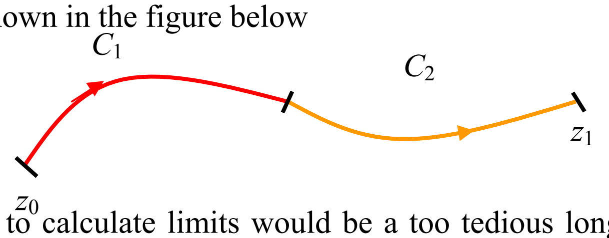
$$\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz$$

where the symbols indicate integrals from z_0 to z_1 and from z_1 to z_0 respectively along a path C.

(3) Partitioning of the path

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

where $C = C_1 + C_2$ as shown in the figure below



Applying the definition to calculate limits would be a too tedious long job to perform. There are methods of solving integrals by calculating the integral of real functions which make the task easier. We will se next several methods. But before we must cover some topics in topology, that is, some definitions of sets in the complex plane.

The definition of integral above refers to a *definite integral*. The definite integral is a number. The *indefinite integral*, on the other hand, is a function. It is also expressed as “antiderivative”.

DEFINITION. Given a function $f(z)$, the *indefinite integral*, also called *antiderivative*, is another function $F(z)$ such that $F'(z) = f(z)$. We also write

$$F(z)=\int f(z)\,dz$$

Observe that the notations of both integrals is almost the same, with the sole difference that in the indefinite integral there is no curve or path of integration. But the concepts are entirely different, since the definite integral is a complex number while the indefinite integral is a function.

EXAMPLE. Find the indefinite integral $\int e^{3z}\,dz$.

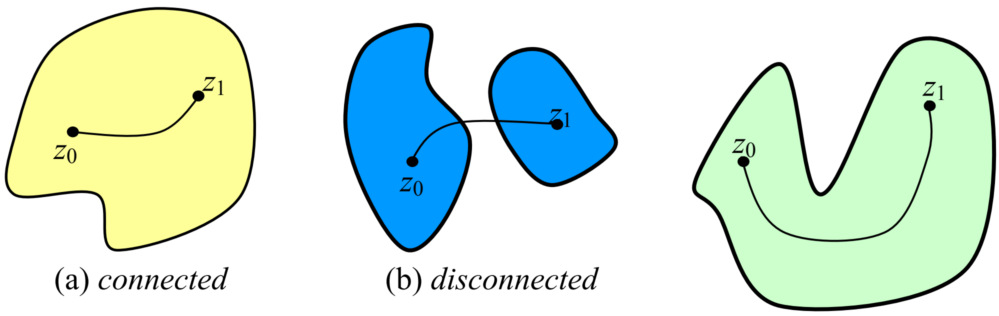
We have
$$F(z)=\int e^{3z}\,dz=\frac{e^{3z}}{3}$$

2.2 Regions of the Complex Plane

A *region* or *domain* of the complex plane is a set of points. Some regions have some important characteristics related with integration.

DEFINITION. A region S of the complex plane is *connected* if any pair of points in S can be joined by a continuous path which is entirely inside S.

The figure below shows examples of connected and disconnected sets

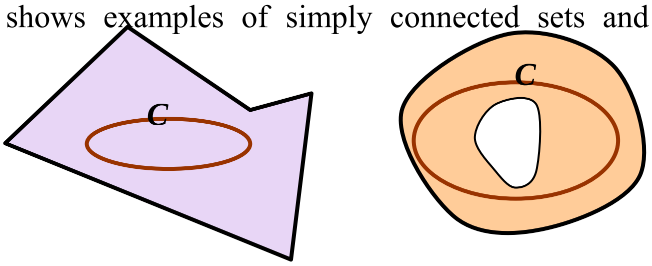


(a) *connected* (b) *disconnected* (c) *connected*

In other words, a connected set comes "in one single piece" (this is not a mathematical term) and a disconnected set comes in "several pieces" (neither this is).

DEFINITION. A region S of the complex plane is *simply connected* if every close curve in S encloses exclusively points of S.

The figure below shows examples of simply connected sets and sets which are not simply connected.



Simply connected *Not simply connected*

In other words, a simply connected set has "no holes"; if it has holes it is not simply connected ("hole" is not a mathematical word.)

2.3 Calculation of the Complex Integral

We will se here several method of calculation of integrals. Some methods are easier than others, but only apply under some particular conditions. Other techniques are somewhat longer but can be used in most of cases.

First Method: Indefinite Integration

This method is very simple and it is preferred to the other methods; alas, it only applies under some particular conditions. So, as long as the assumptions of the following theorem are met, we will use this method. But if the conditions of the theorem do not apply to our function, we will have to resort to the more complicated techniques.

THEOREM 2. Let $f(z)$ be analytic in a simply connected set S. Then there exists an indefinite integral $F(z)$ of $f(z)$ in the set S and for any path inside S joining two points z_0 and z_1 in S we have

$$\int_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad (2)$$

Remarks.

1. Keep in mind that, by the definition of the indefinite integral, $F'(z) = f(z)$, that is, $F(z)$ is the antiderivative of $f(z)$.

2. While $F(z)$ is the indefinite integral, the integral sign in (2) refers to a definite integral.

3. Theorem 2 implies that, if the integrand $f(z)$ is analytic (that is, if it is differentiable) in a simply connected set, then the integral between two points z_0 and z_1 is independent of the path joining them. This is why, in (2), there is no need to consider the path C, since any path would result in the same integral value.

EXAMPLE 1. Calculate $\int_C 4z^3\,dz$ $\quad C: z(t) = 1+t + it^2,\ 0 \le t \le 1$

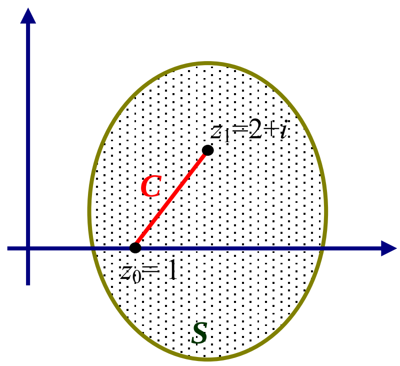
Because the function z^3 is entire, we can apply Theorem 2. Thus, the value of the integral does not depend on the particular path C, it depends only on the initial and the terminal points, $z_0 = z(0) = 1$ and $z_1 = z(1) = 2 + i$. Because the indefinite integral of $f(z) = 3z^3$ is $F(z) = z^4$, we have

$$\int_C 4z^3\,dz = \int_1^{2+i} 4z^3\,dz = z^4\Big|_1^{2+i} = (2+i)^4 - 1$$

This theorem is very practical for evaluating definite integrals, but before u using it, the student must check whether the assumptions (analytic in a simply connected set) are met, since otherwise the theorem is not valid. Next is an example

EXAMPLE 2. Calculate $\int_C \frac{1}{z}\,dz$, $\quad C: z(t) = 1+t + it,\ 0 \le t \le 1$

The function has a singularity at $z = 0$, and it is analytic everywhere else. To apply Theorem 2, we must verify the assumptions of the theorem, that is, find a simple connected domain S containing entirely the path C but excluding the point 0. The figure below shows how



We can draw a set S (green) containing entirely the path C (red) and excludes the point 0. Moreover, S is simply connected (contains no holes). Therefore, we can apply Theorem 2. The indefinite integral of

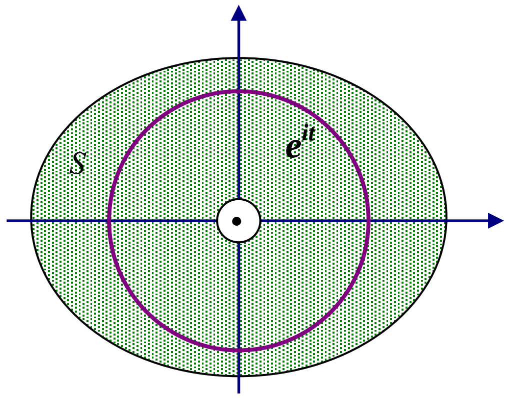
$$f(z) = 1/z \text{ is } F(z) = \ln z.$$

The initial and the terminal points are

$$z_0 = z(0) = 1,\ z_1 = z(1) = 2+i$$

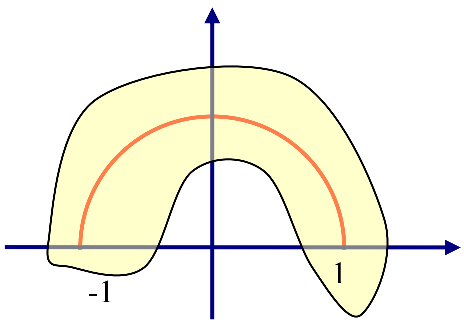
Therefore, by Theorem 2

$$\int_C \frac{1}{z}dz = \int_1^{2+i}\frac{1}{z}dz = \ln z\Big|_1^{2+i} = \ln(2+i) - \ln 1 = \ln(2+i)$$



EXAMPLE 3. Calculate $\oint_C \frac{1}{z}dz \qquad C: z(t) = e^{it},\quad 0 \le t \le 2\pi$

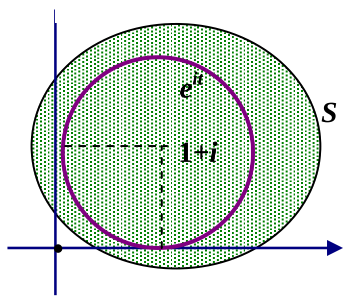
The integrand is the same as in the previous example, but the path is different. As we have seen in the previous section, C is a circle with center at the origin and radius 1. This is a closed path. The variation of the integral sign above only means that the curve is closed. The result is that any region S that contains entirely the path C would also contain the origin, unless we exclude (make a hole) a small are around the origin. But if we do that, then the region S is not simply connected. The result is that the assumption of Theorem 2 are not met, and therefore we cannot use it to calculate the integral in this example. We will be able to calculate this integtral later on, after seeing other, more powerful (and also yielding more difficult calculations



EXAMPLE 3a. Calculate $\int_C \frac{1}{z}dz \qquad C: z(t) = e^{it},\quad 0 \le t \le \pi$

Now we can find a simply connected domain containing the path C and leaving outside the singularity $z = 0$. Therefore, we can now apply Theorem 2.

$$\int_C \frac{1}{z}dz = \ln z\Big|_1^{-1} = \ln(-1) - \ln 1 = \pi i$$



EXAMPLE 4. Calculate $\oint_C \frac{1}{z}dz \quad C: z(t) = 1+i+e^{it},\ 0 \le t \le 2\pi$

This seemingly identical example, in fact allows the use of Theorem 2. Indeed, the path C now is the circle with center $1+i$ and radius 1. It is possible to draw a simply connected region S containing entirely the path C and excluding the origin, so the function is analytic inside S, and the assumptions of the theorem apply. The initial point is $z_0 = z(0) = 2$ and the terminal point is $z_1 = z(2\pi) = 2$ (in a closed curve the initial and terminal points is the same point.) Hence

$$\oint_C \frac{1}{z}dz = \int_{2+i}^{2+i}\frac{1}{z}dz = \ln(2+i) - \ln(2+i) = 0$$

EXAMPLE 5. Calculate $\int_C \mathrm{Re}(z)\,dz$ along C: $z(t) = t + 2ti,\ 0 \le t \le 1$.

For no complex point z the function Re(z) is analytic, therefore we can't apply Theorem 2 to solve it. We will solve this integral after we cover the next theorem.

Second Method: Use of a representation of the path

This method is more general and applies to all continuous functions, whether analytic or not. Therefore, if the assumptions of Theorem 2 failed to be true, we must use this, more general theorem.

THEOREM 3. Let C be a piecewise smooth path $z = z(t),\ a \le t \le b$ and let $f(z)$ be any complex continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt \tag{3}$$

As we will see at once, to calculate an integral applying (3) would require just the calculation of two real integrals. We still need to solve Example 3 above.

EXAMPLE 3 (revisited). Calculate $\oint_C \frac{1}{z}dz,\quad C: z(t) = e^{it},\ 0 \le t \le 2\pi$

C is the circle of center at the origin and radius 1. The only requirement of Theorem 3 is that the function be continuous on C, which is, since the only discontinuity pf $f(z)$ is at the origin, and the circle C does not pass through the origin. We must first calculate the derivative

$$z'(t) = ie^{it} \tag{4}$$

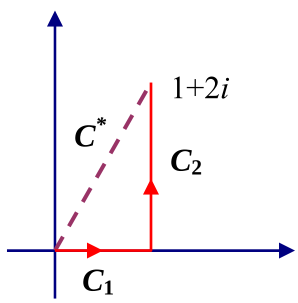
For this, just use the rules of integration covered in MATH I; in this case, just use the chain rule. Second, se must calculate $f[z(t)]$. This is done by substituting the values of $z(t)$ in the function, which yields.

$$f[z(t)] = \frac{1}{e^{it}} = e^{-it} \tag{5}$$

The last step consists of evaluating the integral, which must be split in two integrals, the integral of the real part and the integral of the imaginary part. It follows from (4), (5) and Theorem 3 that

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} e^{-it}ie^{it}\,dt = \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = 2\pi i$$

EXAMPLE 3 (revisited). Calculate $\int_C \mathrm{Re}(z)\,dz$



(a) Along C^*: $z(t) = t + 2ti,\ 0 \le t \le 1$.

(b) Along the path $C = C_1 + C_2$ as shown in the figure

Since the function is non analytic everywhere, we can't apply Theorem 2. So we must apply Theorem 3.

(a) C^* is the segment of line joining the origin and the point $1 + 2i$. The derivative of $z(t)$ is

$$z'(t) = 1 + 2i$$

and $f(z) = \text{Re}(z) = \text{Re}(x+iy) = x$

$$f[z(t)] = t.$$

Thus

$$\int_C \text{Re}(z)\,dz = \int_0^{1+2i} t\,(1+2i)\,dt = (1+2i)\frac{t^2}{2}\bigg|_0^{1+2i} = \frac{1}{2} + i$$

(b) We must calculate two integrals separately, one for each section of the curve. Notice that the path is piecewise smooth, since C_1 and c_2 are both smooth, but there is an angular point at the vertex where the two paths connect together. We have

C_1: $z(t) = t$, $0 \le t \le 1$, $z'(t) = 1$ $f[z(t)] = t$

C_2: $z(t) = 1+2ti$, $0 \le t \le 1$. $z'(t) = 2i$ $f[z(t)] = 1$

Hence,

$$\int_C \text{Re}(z)\,dz = \int_{C_1} \text{Re}(z)\,dz + \int_{C_2} \text{Re}(z)\,dz = \int_0^1 t\ dt + \int_0^1 2i\ dt = \frac{1}{2} + 2i$$

Observe that the initial and the terminal points in both (a) and (b) are the same, but the path of integration is different, and so is the results of the integrals. This means that, if the function is not analytic, then the integral is dependent of the path, and different paths may produce different values of the integral, even if the initial and the terminal points are the same. In Theorem 2 we have seen that the integral is independent of the path, and that regardless of the path, if the initial and the terminal points are the same then the value of the integral remains unchanged. But this is true only if the function is analytic, which is not the case in this example, since Re(z) is not analytic.

The following formula would be used often

Theorem 4.

$$\oint_C (z - z_0)^m\,dz = \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1 \end{cases} \qquad C:\ z(t) = z_0 + re^{it},\ 0 \le t \le 2\pi$$

Here C is a circle with center at z_0 and arbitrary radius r. To prove it, write

$$z'(t) = ire^{it},\ f[z(t)] = r^m e^{imt}$$

and applying Theorem 3

$$\oint_C (z - z_0)^m\,dz = \int_0^{2\pi} \left(r^m e^{imt}\right)\left(ire^{it}\right)dt = ir^{m+1}\int_0^{2\pi} e^{i(m+1)t}\,dt$$

If $m = -1$, the last integral equals

$$i\int_0^{2\pi} 1\,dt = 2\pi$$

If $m \neq -1$, the integral can be calculated as

$$ir^{m+1}\int_0^{2\pi} e^{i(m+1)t}\,dt = \frac{ir^{m+1}e^{i(m+1)t}}{i(m+1)}\bigg|_0^{2\pi} = \frac{ir^{m+1}}{i(m+1)}\left[e^{i(m+1)2\pi} - e^{i(m+1)0}\right] = \frac{ir^{m+1}}{i(m+1)}(1-1) = 0$$

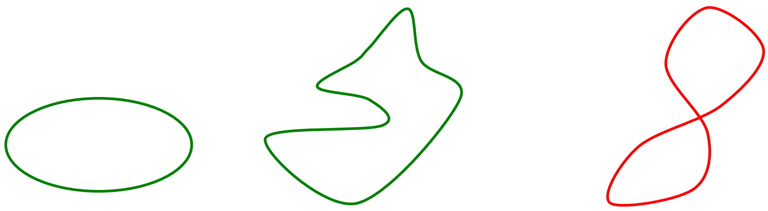
2.4 Cauchy's Integral Theorem

Before we can state the theorem, we must introduce one additional definition about curves. We have seen that a curve can be open or closed. A curve can also be simple or not simple.

2.4.1 Paths and regions of the complex plane.

DEFINITION. A path C: $z(t)$ $a \le t \le b$ is called simple if for any two points t_1 t_2, with $a \le t_1 < t_2 \le$ b we have $z(t_1) \neq z(t_2)$.

This definition means that a simple path does not intercept itself, while a path which is not simple does intercept itself. Below are some graphical examples

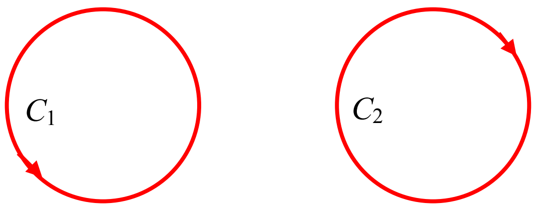


Closed simple *Closed simple* *Closed not simple*

A closed simple path can be oriented clockwise or counterclockwise. For instance

C_1 $z(t) = e^{it}$, $0 \le t \le 2\pi$ is oriented counterclockwise, and

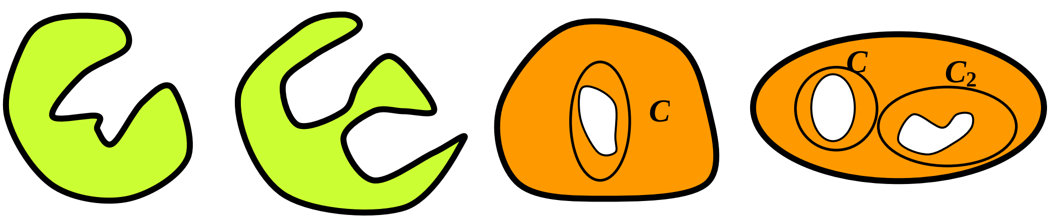
C_2 $z(t) = e^{-it}$, $0 \le t \le 2\pi$ is oriented clockwise.



counterclockwise *clockwise*

The following equality follows $\oint_C f(z)\,dz = -\oint_C f(z)\,dz$

At this point a reminder: *a simply connected region S is a region of the complex plane such that every simple closed path in S enclosed only points of S.* T book uses the word "domain" which in this case is synonym of the word region we use here. It can also be designated as "set".



(a) *Simply connected* (b) *Simply connected* (c) *Doubly connected* (e) *Triply connected*

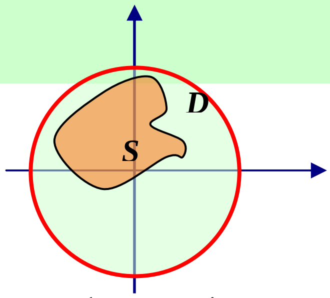
(c) is not simply connected, since there are curves (such as *C*) that are entirely contained in the region but enclose points that are not in the region (the white hole). It is called doubly connected

(d) is not simply connected for the same reason, but in this case there are two separate curves in the same situation (C_1 and C_2), so it is called triply connected. In general a region which is not simply connected is called ***multiply connected*** and the order of multiplicity depends on the number of "holes" inside the region.

(a) and (b) enclose no holes, that is, any closed curve entirely contained inside them contains only points of the region.

DEFINITION. A regions *S* is **bounded** if it is entirely contained inside a disk D_R of arbitrary radius *R*

$$S \subset D_r = \{z \in \mathbf{C}: |z - z_0| \leq R\}$$



2.4.2 Cauchy's Integral Theorem. Now we are in good standing to state the most important theorem perhaps in complex integration

THEOREM 5. (Cauchy's integral theorem). If *f*(*z*) is analytic in a simply connected domain S, then for every simple closed path *C* inside *S* we have

$$\oint_C f(z)\,dz = 0$$

Problems of this kind are pretty simply to solve, since they involve no calculation, only verification of the hypotheses. Be well aware that **all** the assumptions must be satisfied before you can apply the theorem. There are three things the student must verify:

1. That the function is analytic inside a set *S* (that the students must figure out) that contains the curve *C*.

2. That the curve is closed and simple (does not intercept itself)

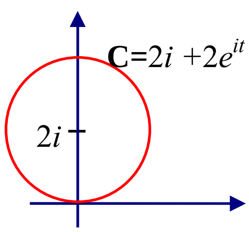
3. That the set *S* is simply connected.

If one of these three conditions fails to be true, then this theorem cannot be applied, and the calculation of the integrals must be carried out by the procedures of the previous sections.

The following chart provides some help in order to verify the assumptions of the theorem. We remind you that a function is entire if it is analytic in the whole complex plane, therefore, entire functions always meet the conditions of the theorem, regardless of the curve *C* given.

$$f(z) = e^z, \quad f(z) = \cos z, \quad f(z) = = \sin z, \quad f(z) = z^a \;\; a \geq 0$$

Composition of entire functions result also in entire functions. For instance $f(z) = e^{z^2}$ is entire, since it is the composite function of the entire functions e^z and z^2.



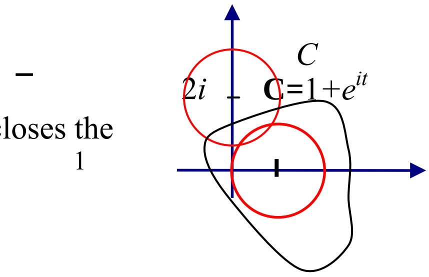
$$f(z) = z^{-a} = \frac{1}{z^a} \quad a > 0$$

is analytic for all complex values except $z = 0$, and more generally

$$f(z) = (z - z_0)^{-a} = \frac{1}{(z - z_0)^a} \quad a > 0$$

is analytic for every complex z except $z = z_0$. Thus, a close simple curve enclosing the point z_0 invalidates the hypotheses of the theorem and it cannot be applied. For instance

$$\oint_C \frac{1}{z - 2i} = 2\pi i \quad C: z(t) = 2i + e^{it} \quad 0 \le t \le 2\pi$$



This integral is not zero (its value is $2\pi i$), because the curve C encloses the point $2i$. The same integral, however but let C_1 be the curve

$$z_1(t) = 1 + e^{it} \quad 0 \le t \le 2\pi$$

$$\oint_{C_1} \frac{1}{z - 2i} = 0 \quad C_1: z_1(t) = 1 + e^{it} \quad 0 \le t \le 2\pi$$

In this case the theorem applies, and the value of the integral is 0, since the curve does not enclose inside the singularity (the point $2i$ for which the function is not analytic).

If the path is not closed, then Cauchy's integral theorem does not apply. For instance

$$\oint_C \frac{1}{z - 2i} \neq 0 \quad C: z(t) = 2i + e^{it} \quad 0 \le t \le \pi$$

because the path is not closed, then the theorem does not apply.

2.4.3 Singularities of functions. In this section we review some of the concepts of analytic functions covered in Chapter 1, namely, we list the most common used functions and their singularities.

1. Polynomial functions are entire (analytic throughout the complex plane.) The functions e^z, $\cos z$, $\sin z$ are also entire functions. Sums and products of entire functions are entire. Quotient of entire functions, $f(z) / g(z)$ are analytic for all values of z for which $g(z) \neq 0$.

2. $f(z) = \ln z$ is analytic throughout the complex plane except $z = 0$; more general, $f(z) = \ln(z - z_0)$ has only one singularity $z = z_0$.

3. $f(z) = \sec z = 1/\cos z$ has singularities at all points for which $\cos z = 0$, i.e., $z = n\pi/2$, n = odd

4. $f(z) = \csc z = 1 / \sin z$ has singularities at all points $z = n\pi$

5. $f(z) = \dfrac{1}{z^2 + a^2}$ (a real number) is not analytic at $z = \pm ia$

6. $f(z) = \dfrac{1}{(z-a)(z-b)}$ (a, b complex numbers) is not analytic at $z = a$ and $z = b$.

7. $f(z)=\dfrac{g(z)}{az^2+bz+c}$, where *g*(*z*) is entire, has singularities at the two roots of the denominator

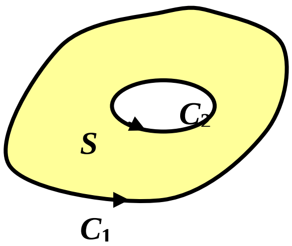
8. tan *z* = sin *z* / cos *z* as singularities at all the points z = $n\pi/2$, *n* = odd

EXAMPLE. Let the function $f(z)=\dfrac{e^z}{z^2+z+1}$. Because e^z is entire, let us find the roots of the denominator $z=\dfrac{-1\pm\sqrt{1-4}}{2}=\dfrac{1\pm i\sqrt{3}}{2}$. Thus the function has two singularities at these two points. And the function $f(z)=\dfrac{\ln z}{z^2+z+1}$ has three singularities, the two roots of the denominator, and also at *z* =0.

2.4.4 Consequences of Cauchy's integral theorem. A direct consequence of Cauchy's integral theorem is the following

THEOREM 5. If *f*(*z*) is analytic in a doubly connected domain *S*, and if C_1 and C_2 are two simple closed curves (oriented counterclockwise) in *S*, then

$$\oint_{C_1} f(z)\,dz=\oint_{C_2} f(z)\,dz$$



2.6 Partial Fractions

Before starting to solve problems on integration, it is worthwhile to review the subject of partial fractions, applied to solving integrals of rational functions. Suppose that we have a function of the form

$$f(z) = \frac{1}{(z-z_1)(z-z_2)}$$

It can be decomposed into partial fractions of the form

$$\frac{1}{(z-z_1)(z-z_2)}=\frac{A}{z-z_1}+\frac{B}{z-z_2} \tag{6}$$

where the coefficients *A* and *B* can be found by solving

$$A(z_1-z_2) = 1, \quad B = -A$$

EXAMPLE 4. Find the partial fraction decomposition of $f(z)=\dfrac{1}{z^2-z-2}$

$$\frac{1}{z^2-z-2}=\frac{1}{(z-2)(z+1)}=\frac{A}{z-2}+\frac{B}{z+1} \quad \text{and} \quad A(2+1)=1 \Rightarrow A=1/3, \quad B=-1/3$$

$$\frac{1}{z^2-z-2}=\frac{1/3}{z-2}-\frac{1/3}{z+1}$$

An expression of the form

$$f(z)=\frac{M\,z+N}{(z-z_1)(z-z_2)}$$

can be written as

$$\frac{M\,z+N}{(z-z_1)(z-z_2)}=\frac{A}{z-z_1}+\frac{B}{z-z_2}$$

where A and B are solution of the system

$$A+B=M,\quad Az_2+Bz_1=-N$$

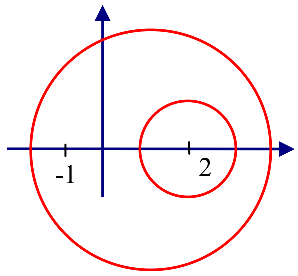
EXAMPLE 5. Find the partial fraction decomposition of $f(z)=\dfrac{2z-1}{z^2-5z+6}$

$$\frac{2z-1}{z^2-5z+6}=\frac{2z-1}{(z-3)(z-2)}=\frac{A}{z-3}+\frac{B}{z-2}\qquad A+B=2,\ \ 2A+3B=1\Rightarrow A=5,\ \ B=-3$$

2.7 Applications

In this section we will show several examples, some of which satisfy the assumptions of Cauchy's integral theorem, and some which do not. It is strongly suggested that the student draw a graph with the curve and the singularities, so as to figure out whether the points of singularities lie inside or outside the curve. In the first case the theorem cannot be applied and in the second case it can be applied.

EXAMPLE 6. Calculate $\oint_C \dfrac{dz}{z^2-z-2}$ where



(a) $C: 2+e^{it}\quad 0\le t\le 2\pi$ (b) $C: 2+4e^{it}\quad 0\le t\le 2\pi$

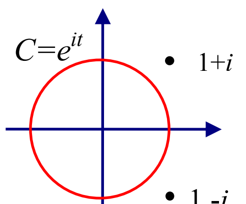
(a) As we have seen in example before, the integrant can be decomposed in partial fraction as

$\oint_C \dfrac{dz}{z^2-z-2}=\dfrac{1}{3}\oint_C \dfrac{dz}{z-2}-\dfrac{1}{3}\oint_C \dfrac{dz}{z+1}$. The first integral, according to Theorem 4, is $2\pi i$. The second integral, according to Cauchy's integral theorem, is 0, since the singularity -1 is outside the path of integration.

(b) In this case the circle contains both singularities, so

$$\oint_C \frac{dz}{z^2-z-2}=\frac{1}{3}\oint_C \frac{dz}{z-2}-\frac{1}{3}\oint_C \frac{dz}{z+1}=\frac{1}{3}2\pi i-\frac{1}{3}2\pi i=0$$

EXAMPLE 7. Calculate $\oint_C \dfrac{3z-2}{(z^2-2z+2)}dz$, where C is



(a) $C=e^{it}$

(a) $z(t)=e^{it},\quad 0\le t\le 2\pi$

(b) $z(t) = 1+i + e^{it}, \quad 0 \le t \le 2\pi$

(c) $z(t) = 1 + 2e^{it}, \quad 0 \le t \le 2\pi$

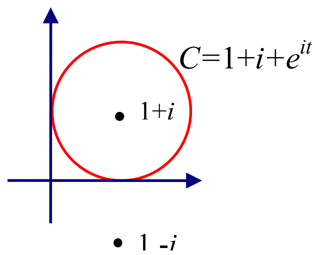
First factor the denominator to find the singularities of the function. For this, we solve the quadratic equation $z^2 - 2z + 2 = 0$, which yields

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

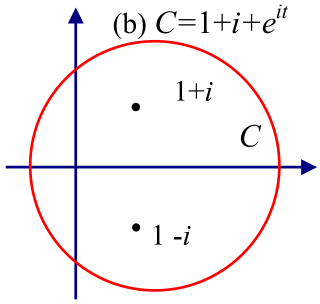
and the singularities happen at these two points ($1 \pm i$). The integrand is

$$\oint_C \frac{3z-2}{(z-1+i)(z-1-i)}dz$$

(a) The function is analytic inside C, therefore by Cauchy's integral theorem the integral is 0.



(b) $C=1+i+e^{it}$



(c) $C=1+2e^{it}$

(b) The curve encloses one singularity at the point $1 + i$, therefore the theorem cannot be applied. We can either use Theorem 3 in section 2.3 or, better, leave this example to be solved by using a result given in the next section.

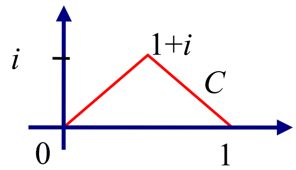
(c) Same as (b), the curve here encloses two singularities, at both $1+i$ and $1-i$. We will come back to these examples in the next section.

EXAMPLE 7. Evaluate $\oint_C \frac{dz}{z-3i}$, $C: |z| = \pi$ oriented counterclockwise

The path C is a circle with center at $z = 0$ and radius π, which encloses the point $3i$, thus $z(t) = \pi e^{it} \quad 0 \le t \le 2\pi$. Applying Theorem 4 in section 2.3 we obtain

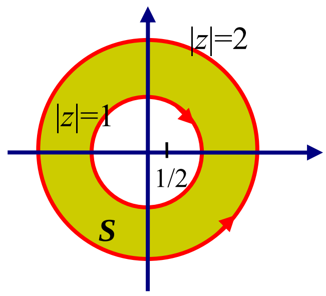
$$\oint_C \frac{dz}{z-3i} = 2\pi i$$

EXAMPLE 8. Evaluate $\oint_C \ln(i-z)\,dz$, where C is the triangle of vertices 0, $1+i$, 2 oriented counterclockwise.



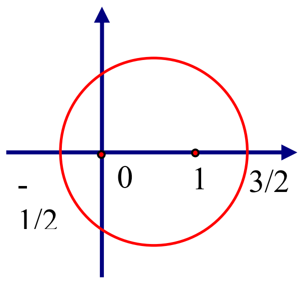
$\oint_C \ln(i-z)\,dz = 0$ because the only singularity of this function is at $z = i$, which is outside the triangle

EXAMPLE 9. Evaluate $\oint_C \frac{\ln z}{2z-1}dz$, where C consists of C_1: $|z| = 1$ clockwise and C_2 $|z| = 2$ counterclockwise.



The function has two singularities, at $z = 0$ and $z = ½$. Thus the function is analytic in the region S formed by the two circles (shaded in green). Applying Theorem 5, the result of the integral is 0.

EXAMPLE 10. Evaluate $\oint_C \frac{2z-1}{z^2-z}dz$, where C is the circle of center at $z = ½$ and radius 1.

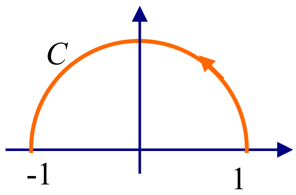


The partial fraction decomposition of the integrand yields

$$\frac{2z-1}{z^2-z}=\frac{1}{z}+\frac{1}{z-1}\Rightarrow\oint_C\frac{2z-1}{z^2-z}dz=\oint_C\frac{1}{z}dz+\oint_C\frac{1}{z-1}dz$$

The two singularities occur at z = 0 and z = 1. Both lie inside C. Applying Theorem 3, each of the two integrals is $2\pi i$, therefore the answer of the problem is $4\pi i$.

EXAMPLE 11. Evaluate $\oint_C \mathrm{Re}(z)\,dz$, where C is the path given in the graph.



This function is not analytic at any point of the complex plane. We must therefore apply Theorem 2. The equation of C is $z(t)$ = ≤, $0 \le t \le \pi$; its derivative $z'(t) = ie^{it}$. Because $e^{it} = \cos t + i \sin t$, along the path, $\mathrm{Re}(z) = \cos t$.

$$\oint_C \mathrm{Re}(z)\,dz = i\int_0^{\pi}\cos t e^{it}\,dt$$

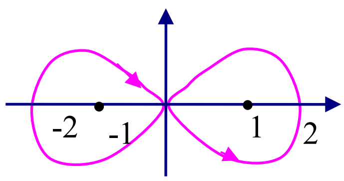
Using the definition of cos z given in section 1.6,

$$\cos t=\frac{1}{2}\left(e^{it}+e^{-it}\right)$$

so the last integral becomes

$$\oint_C \mathrm{Re}(z)\,dz=\frac{i}{2}\int_0^{\pi}\left(e^{it}+e^{-it}\right)e^{it}\,dt=\frac{i}{2}\int_0^{\pi}\left(e^{2it}+1\right)e^{it}\,dt=\left[\frac{1}{4}e^{2it}+\frac{it}{2}\right]_0^{\pi}=\frac{\pi i}{2}$$

EXAMPLE 12. Evaluate $\oint_C \frac{dz}{z^2-1}$, where C is the path given in the graph.



The function has singularities at z = 1 and z = -1. We can split the curve in two parts, the right part, C_1 , oriented clockwise, and the left part C_2 counterclockwise oriented. On the other hand,

$$\frac{1}{z^2-1}=\frac{1/2}{z-1}+\frac{1/2}{z+1}$$

Thus we have

$$\oint_C\frac{dz}{z^2-1}=\frac{1}{2}\oint_{C_1}\frac{dz}{z-1}+\frac{1}{2}\oint_{C_2}\frac{dz}{z+1}$$

Now applying Theorem 4, and taking into consideration that the integral along C_2 must be multiplied by -1 (counterclockwise), we have that

$$\oint_C\frac{dz}{z^2-1}=\frac{1}{2}\oint_{C_1}\frac{dz}{z-1}+\frac{1}{2}\oint_{C_2}\frac{dz}{z+1}=\pi i+\pi i=2\pi i$$

2.8 Cauchy's Integral Formula

A consequence of Cauchy's integral theorem is Cauchy's integral formula.

THEOREM. Let $f(z)$ be analytic in a simply connected region S. Then for any point z_0 in S and any simple closed path C that encloses z_0 we have

$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

where the integral is calculated counterclockwise.

Alternatively, we can write

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

EXAMPLE 13. Evaluate $\oint_C \frac{e^z}{z-2}\,dz$

(a) Along $C = |z| = 1$
(b) Along $C = |z| = 3$

(a) Because the point $z = 2$ is outside C, by Cauchy's integral theorem the integral is 0.
(b) Because the point $z = 2$ is inside C, Cauchy's integral formula yields

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i e^2$$

2.9 Derivatives of Analytic Functions

Further consequences of Cauchy's integral theorem is the following

THEOREM. If $f(z)$ is analytic in a region S then it has derivatives of all order in S and

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{7}$$

for any simple closed curve enclosing z_0.

Formula (7) can be used either to calculate derivatives, or to calculate integrals, by using

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0) \tag{8}$$

EXAMPLE 14. Evaluate $\oint_C \frac{\cos}{(z-\pi i)^2}dz$ along the circle with center at the origin and radius 2π

Because the *C* encloses the poing *z* =π*i,* applying (8) we get

$$\oint_C \frac{\cos}{(z-\pi i)^2}dz = 2\pi i\left(\cos z\right)'\Big|_{\pi i} = 2\pi i \sin \pi i$$

HOMEWORK

Section 13.1 # 15-24

Additional problems. In problems 1 - 11 solve the given integral.

1. $\int_C e^{-3z}\,dz$ C: $z(t) = 2 - t^2 + i(t+1)$ $1 \le t \le 2$

2. $\int_C \sin z\,dz$ C: $z(t) = 1 + t + it$ $0 \le t \le \pi$

3. $\int_C (\mathrm{Re}(z+3))\,dz$ C: $z(t) = t^2 + i\,(t+1)$ $0 \le t \le 1$

4. $\int_C \mathrm{Re}(z^2)\,dz$ C: $z(t) = 2t + i$ $0 \le t \le 1$

5. $\oint_C \left(\tan z - 2z^3 e^{z^2}\right)dz$ $z(t) = 1 - 3i + 2e^{it}$ $0 \le t \le 2\pi$

6. $\oint_c \frac{1}{z-i}\,dz$ C: $z(t) = i + e^{it}$ $0 \le t \le 2\pi$

7. $\oint_c \frac{1}{(z-i)^2}\,dz$ C: $z(t) = i + e^{it}$ $0 \le t \le 2\pi$

8. $\int_C \frac{1}{z-i}\,dz$ C: $z(t) = i + e^{it}$ $0 \le t \le \pi$

9. $\int_C \frac{1}{z-i}$ where C is the segment line between 0 and $1 + i$

10. $\oint_c \frac{1}{2i-z}\,dz$ C: : $z(t) = e^{it}$ $0 \le t \le 2\pi$

11. $\oint_C \frac{2}{z-2+i}\,dz$
(a) C is the rectangle of vertices 0, 3, 3+3i, 3i oriented counterclockwise
(b) C is the rectangle of vertices 0, 3, 3-3i, -3i oriented counterclockwise

12. $\oint_C \frac{e^z}{(z - i\pi/2)}\,dz$

(a) C: $z(t) = i\pi/2 + 3e^{it}$ $0 \le t \le 2\pi$
(b) C: $z(t) = -i\pi/2 + e^{it}$ $0 \le t \le 2\pi$

13. $\oint_C \frac{e^z}{z^2 - 2z + 2} dz \quad C: z(t) = 1 + i + e^{it} \quad 0 \le t \le 2\pi$

14. $\oint_C \frac{\sin z}{(z-i)^2} dz$

15. $\oint_C \frac{\sin z}{(z-i)^3} dz$ (a) $z(t) = i + 2e^{it} \quad 0 \le t \le 2\pi$ (b) $z(t) = i + 2e^{it} \quad 0 \le t \le \pi$

16. $\oint_C \frac{1-3z}{(z-i)^4} dz \quad C: z(t) = 1 + e^{it} \quad 0 \le t \le 2\pi$

SOLUTIONS

3. $\int_C \left(\operatorname{Re}(z+3)\right) dz$ $C: z(t) = t^2 + i\,(t+1) \quad 0 \le t \le 1$

$\operatorname{Re}[z(t)] = \operatorname{Re}[t^2 + i\,(t+1) + 3] = t^2 + 3 \qquad z'(t) = 2t + i$. Hence

$$\int_C \left(\operatorname{Re}(z+3)\right) dz = \int_0^1 (t^2+3)(2t+i)\,dt = \int_0^1 \left[2t^3 + 6t + i(t^2+3)\right] dt = \frac{t^4}{2} + 3t^2 + i\left(\frac{t^3}{3} + 3t\right)\Bigg|_0^1 = \frac{7}{2} + \frac{10}{2} i$$