Chapter 1

COMPLEX NUMBERS

While the study of complex numbers may seem unreal and unnecessary to the student, the fact is that complex analysis has vast applications in many areas of physics and engineering. We will undertake in this course a "practical" approach, learning basically the properties and the calculation tools necessary for the understanding and the solving of applied problems. We will cover basically chapters 12 and 13 of the textbook.

1.1 Definition

The equation

$$x^2 = -1$$

does not have solution in the field of real numbers, because for any real x, the number x^2 is positive. In other words, $\sqrt{-1}$ does not exist for real numbers. A new number is then defined, i, such

$$i^2 = -1$$

which is equivalent to

$$\sqrt{-1} = i\,.$$

This number is called ***imaginary***. Because multiplication and addition of numbers must be allowed, given a real number b, the product of b and i, is bi. For instance, $3i$ or $-(1/2)i$. And the sum of a and bi, will be $a + bi$

DEFINITION. A complex number z is of the form

$$z = a + bi$$

where a and b are real numbers and i is the imaginary number. The firs term, a, is called the **real part**, $\mathrm{Re}(z) = a$, and the second term, bi is called the **imaginary part**, $\mathrm{Im}(z) = b$. An alternative way (shorter) to denote a complex number is by $z = (a, b)$.

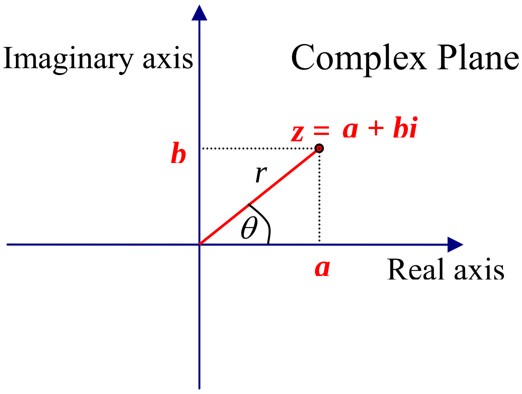
Sum $(a+bi)+(c+di)=(a+c)+(b+d)i$

Product $(a+bi)(c+di)=ac+bci+adi+bdi^2=(ac-bd)+(bc+ad)i$

Quotient $\dfrac{a+bi}{c+di}=\dfrac{(a+bi)(c-di)}{(c+di)(c-di)}=\dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

Powers: $i^2=-1;\ i^3=-i;\ i^4=1;\ i^5=i\ \ldots\ldots$

1.1.2 Geometrical Representation. Complex numbers can be represented in the complex plane.



The **modulus** $|z|$ of z is the length of the segment, and is $r=|z|=\sqrt{a^2+b^2}$. The angle θ is called the **argument** of z, and it is calculated as $\tan\theta=b/a \Rightarrow \theta=\arctan(b/a)$

Complex conjugate. Given a complex number $z=a+bi$, its complex conjugate is $\bar{z}=a-bi$. We have

$$\bar{z}z=(a+bi)(a-bi)=a^2+b^2=|z|^2$$

1.2 Polar Form

The angle θ that the vector forms with the real axis is called the argument of z, so arg (z) = θ. By simple trigonometric identities, we can se that

$$a=r\cos\theta \quad \text{and} \quad b=r\sin\theta. \tag{1}$$

The polar form of the complex number therefore is

$$z=a+bi=r\cos\theta+i\,r\sin\theta, \quad \text{or}$$

$$z=r(\cos\theta+i\sin\theta)$$

which is called the **polar form** of the complex number. Remark that

$$z=r(\cos\theta+i\sin\theta)=r\big(\cos(\theta+2\pi)+i\sin(\theta+2\pi)\big)$$

We always prefer to use the argument in the interval $0\le\theta<2\pi$.

Product and quotient in polar form. In some cases, working with the polar form yields faster and easier calculations. Formula (1) is used to convert the polar form to regular form. Conversely, converting from regular to polar yields

$$|z| = r = \sqrt{a^2+b^2}, \quad \tan\theta = \frac{b}{a}, \text{ or } \quad \theta = \arctan\left(\frac{b}{a}\right) \tag{2}$$

In particular, multiplication and division is better perform in polar form. Again, by using simple trigonometric identities, we can show that, if

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

so

$$|z_1z_2| = |z_1||z_2| \quad \text{and} \quad \arg(z_1z_2) = \arg(z_1) + \arg(z_2)$$

And the division is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

Continuing this process, we come with the formula of powers

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Roots. From where we can deduct the roots of a complex number. But before, observe that the square root of a positive real number has two solution, for example,

$$\sqrt{4} = 2 \text{ and } \sqrt{4} = -2$$

because $(2)^2 = (-2)^2 = 4$. Similarly, the nth root of a complex number has n solutions. Keep in mind that $w = \sqrt[n]{z}$ is, by definition, another complex number w such that $w^n = z$. Let

$$z = r(\cos\theta + i\sin\theta)$$

and let $w = R(\cos\phi + i\sin\phi)$. Then $w^n = R^n(\cos n\phi + i\sin n\phi)$. Therefore

$$R^n = r \Rightarrow R = \sqrt[n]{r} \quad \text{and} \quad n\phi = \theta + 2k\pi$$

Thus nth the root of z is

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), \quad k = 0, 1, 2, \ldots n-1 \tag{3}$$

Thus the nth root of a number has n values

EXAMPLE. Let $z_1 = 1 + 2i$ and $z_2 = -1 + i$.

(a) Find $|z_1|$, $\bar{z}_1$ ➔ $|z_1| = \sqrt{5}$, $\bar{z}_1 = 1 - 2i$

(b) Calculate $z_1 + z_2$. ➔ $z_1 + z_2 = 3i$

(c) Calculate $z_1 z_2$. ➔ $z_1 z_2 = (1)(-1) - (2)(1) + i((1)(1) + (2)(-1)) = -3 - i$.

EXAMPLE. Let $z_1 = 2(\cos\pi/3 + i\sin\pi/3)$ and $z_2 = \cos\pi/2 + i\sin\pi/2$.

(a) Find $z_1\, z_2$ ➔ $z_1\, z_2 = 2(\cos 5\pi/6 + i \sin 5\pi/6)$.

(b) Find $|z_1|$, $\bar{z}_1$ ➔ $|z_1| = 2$, $\bar{z}_1 = z_1 = 2(\cos \pi/3 - i \sin \pi/3)$

(c) Find z_1^3 ➔ $z_1^3 = 8(\cos \pi + i \sin \pi)$

(d) Find $\sqrt[4]{z_1}$ ➔

$$\begin{cases} \sqrt[4]{z_1} = \sqrt[4]{2}\left(\cos\dfrac{\pi/3+2k\pi}{4} + i\sin\dfrac{\pi/3+2k\pi}{4}\right), \; k=0,\,1,\,2,\,3 \\ k=0 \Rightarrow \sqrt[4]{z_1}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right) \\ k=1 \Rightarrow \sqrt[4]{z_1}\left(\cos\dfrac{7\pi}{12} + i\sin\dfrac{7\pi}{12}\right) \\ k=2 \Rightarrow \sqrt[4]{z_1}\left(\cos\dfrac{13\pi}{12} + i\sin\dfrac{13\pi}{12}\right) \\ k=3 \Rightarrow \sqrt[4]{z_1}\left(\cos\dfrac{19\pi}{12} + i\sin\dfrac{19\pi}{12}\right) \end{cases}$$

1.3 Complex Functions, Derivative

We generalize here the concept of function to the case in which the variables are complex numbers

DEFINITION. A complex function $w = f(z)$ is a correspondence that for each value of the complex variable z in the domain S assigns a unique complex number w.

EXAMPLE. $w = f(z) = 3z^2 - \sqrt{z^2+1}$

Because z is a complex variable, $z = x + iy$. And because w is also a complex number, $w = u + iv$. Therefore

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

where $u(x,y)$ and $v(x,y)$ are real functions of two independent variables, x, y; $u(x,y)$ is the real part and $v(x,y)$ the imaginary part.

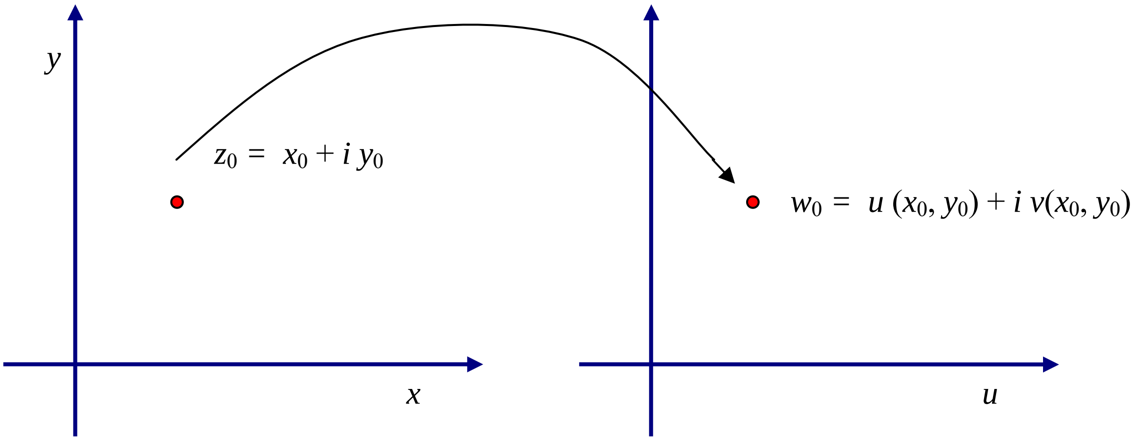
EXAMPLE. $w = f(z) = z^2 + 3z$. Finf the real part and the imaginary part of f.

$z = x + iy$, therefore, $z^2 + 3z = (x^2 - y^2 + 2xyi) + 3x + 3yi = x^2 - y^2 + 3x + (2xy + 3y)\, i$

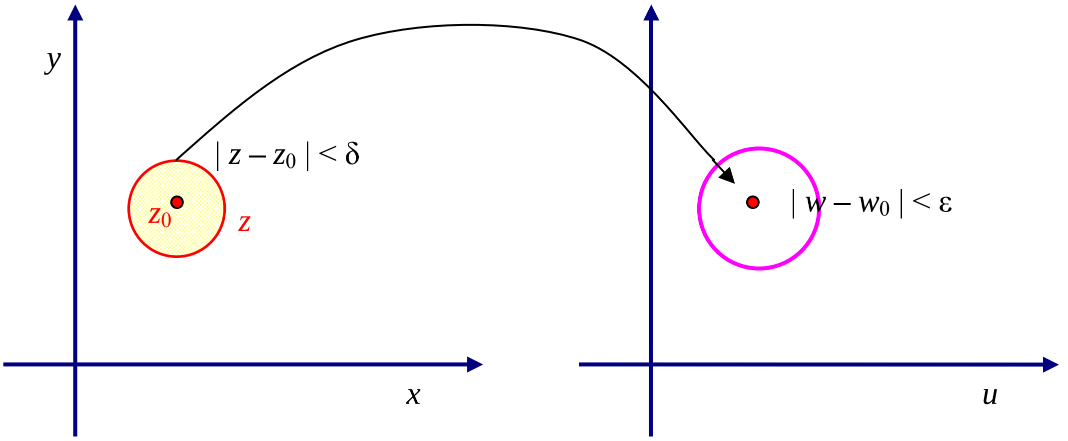
Thus $Re(f) = u(x,y) = x^2 - y^2 + 3x$ and $\mathrm{Im}(f) = v(x,y) = 2xy + 3y$. If $z = 1-2i$, then $w = f(1-2i) = -10i$

Complex functions cannot be represented graphically as real functions of one variable, since it requires 4 dimensions, 2 for the variable z and 2 for the variable w. We will eventually graph the domain of $f(z)$ if needed.

1.3.2 Limits and derivatives. The limit of a complex function is defined in a similar way as the limit for real functions, but because the independent variable now has two dimensions, x and y, we must approach in the complex plane. Given a point z_0 in the domain of the function, its correspondent, w_0 is another point in the complex plane of the dependent variable



A neighborhood around the point z_0 is a region of the form $|z - z_0| < \delta$ and a neighborhood around the point w_0 is a region of the form $|w - w_0| < \varepsilon$



Definition. A complex function $f(z)$ defined in a neighborhood of a point z_0 is said to have **limit** l as z approaches z_0, and is written

$$\lim_{z \to z_0} f(z) = l$$

if for any $\delta > 0$ arbitrary there exists $\varepsilon > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Also the definition of continuity and derivative of a complex function is definied in analogous form as for real functions.

Definition. A complex function is continuous at a point z_0 if it is defined in a neighborhood of the point and satisfies

$$\lim_{z\to z_0} f(z) = f(z_0)$$

The function is said to be continuous in a region S of the complex plane if it is continuous at any point *z* inside S.

DEFINITION. The derivative of a complex function *f* (*z*) at a point z_0 is defined by

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided that the limit exist, in which case we say that the function is differentiable *at* z_0 . A function is said to be analytic in a region S of the complex plane if it is differentiable for any *z* inside S.

EXAMPLE. The function $f(z) = z^2$ is analytic throughout the complex plane and $f'(z) = 2z$. To see this, calculate the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{z^2 - z_0{}^2}{z - z_0} = \lim_{z\to z_0} \frac{(z+z_0)(z-z_0)}{z - z_0} = 2z_0$$

Applying the definition to calculate derivatives would be a tecious and cambersome task. Fortnuately, most of the rules of differentiation for real functions extend to complex functions.

1.3.3 Rules of differentiation. The rules of differentiation of complex functions are basically the same as for real functions. Namely

$$(f + g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Let $f(x) = z^n$. then $f'(z) = nz^{n-1}$

We will see in the next sections the calculation of the derivative of exponential and trigonometric functions

1.4 Cauchy-Riemann Equations

The Cauchy-Riemann equations are one of the pillars of complex analysis. An analytic function must satisfy very stringent conditions.

THEOREM (Cauchy-Riemann equations). Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$¸ and differentiable at z itself. Then the partial derivatives of u and v exist at that point and satisfy

$$\boldsymbol{u_x = v_y} \qquad\qquad \boldsymbol{u_y = -v_x}$$

Hence, if f is analytic in a region S it satisfies the C-R equations at every point of S

EXAMPLE. We have seen in the previous example that the function $f(z) = z^2$ is analytic in the complex plane. Let us calculate the real and imaginary parts of f.

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Thus $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We have

$$u_x = 2x \text{ and } v_y = 2x \qquad u_y = -2y \text{ and } -v_x = -2y$$

The Cauchy-Riemann equation are very handy to find whether a function is analytic or not.

EXAMPLE. Show that $f(z) = \dfrac{1}{z}$ is analytic for all $z \neq 0$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

so

$$u(x,y) = \frac{x}{x^2+y^2} \text{ and } v(x,y) = \frac{-y}{x^2+y^2}$$

By simple differentiation we obtain

$$u_x = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2} = v_y \text{ and } u_y = \frac{2xy}{\left(x^2+y^2\right)^2} = -v_x$$

which proves the assertion. The details on partial differentiation are left to the student.

EXAMPLE. Show that $f(z) = \operatorname{Re}(z)$ is not analytic at any point z of the complex plane.

If $z = x + iy$, then $\operatorname{Re}(z) = x$. It follows that $u(x, y) = x$ and $v(x, y) = 0$. Therefore,

$$u_x = 1 \text{ and } v_y = 0$$

Thus the Cauchy-Riemann equations are not satisfied, and the function Re(z) is not analytic for any value of z

1.4.2 Laplace Equation. Laplace's equation has many applications in physics, ranging from electrostatics, gravitation, fluids, and heat.

THEOREM (Laplace equation) Let $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain S then the real and imaginary parts of f satisfy

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad and \qquad \nabla^2 v = v_{xx} + v_{yy} = 0$$

For every point in S.

EXAMPLE. Take again the example of the function $f(z) = z^2$

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

We have

$$u_x = 2x \qquad u_y = -2y \qquad v_x = 2y \qquad v_y = 2x$$

So

$$u_{xx} = 2 \;\; u_{yy} = -2, \rightarrow \;\; u_{xx} + u_{yy} = 0$$

Similarly,

$$v_{xx} = 0 \;\; u_{yy} = 0, \;\; \rightarrow \;\; v_{xx} + v_{yy} = 0$$

1.5 Exponential Function

We will study now some particular functions, which are well known by the student in the courses of real analysis. In this section we will study the exponential function, and in the afterward we will cover the logarithmic function and the trigonometric functions. In the complex field, these functions generalize the concept of real function, that is, along the real axis they coincide with the previous definitions.

DEFINITION. The exponential complex function $f(z) = e^z$, also written exp(z) is defined by

$$\boxed{e^z = e^{x+iy} = \exp(x + iy) = e^x(\cos y + i \sin y)} \qquad (4)$$

EXAMPLE. Let $z = 1 + 2y$. Find e^z.

$$e^z = e(\cos 2 + \mathrm{i} \sin 2)$$

With this definition, when z is real ($y = 0$), we obtain the function e^x, the same as for a function of a real variable. It has also several other important properties of the exponential function

(a) The function is **entire** (which means analytic in the whole real plane)

(b) The derivative of $f(z) = e^z$ is $f'(z) = e^z$, as it happens with the real function.

(c) $e^{z_1+z_2} = e^{z_1}e^{z_2}$. In particular, if $z_1 = x$ and $z_2 = iy$, we get $e^{x+iy} = e^x e^{iy}$ and if $z = iy$ we obtain the important Euler formula

$$e^{iy} = \cos y + i \sin y$$

and because in its polar form yi = cos y + i sin y, we obtain an alternative way of writing a complex number. If $z = x + iy = r(\cos\theta + i\sin\theta)$, then z can be written as

$$z = re^{i\theta}.$$

Thus,

$$e^{2\pi i} = 1, \quad e^{\pi i} = -1, \quad e^{\pi i/2} = i, \quad e^{3\pi i/2} = -i$$

We also have that

$$e^{z} \neq 0 \text{ for all complex } z$$

From (3) we readily conclude that

$$e^{z+2\pi i} = e^{z} \quad \text{for all complex } z$$

which means that the exponential function is periodic

1.6 Trigonometric Functions

As before, we are going to extend the concept of trigonometric function given for real functions to the complex field. By applying Euler formula we can write

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x + -i \sin x$$

adding and subtracting this equations obtain

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

So, the generalization to the complex plane provides the following

DEFINITION. The trigonometric complex functions $f(z)$ = cos z and $f(z)$ = sin z are defined by

$$\boxed{\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)} \qquad (5)$$

Once defined the two basic trigonometric functions sine and cosine, we can use them to define all the rest of the trigonometric functions known in the real field, that is, tan z, cot z, sec z, csc z, and we do in exactly the same way they were defined in MATH I:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

The derivatives of the trigonometric functions are the same as for real functions:

$$(\sin z)'=\cos z,\ \ (\cos z)'=\sin z;\ (\tan z)'=\sec^2 z=\frac{1}{\cos^2 z}$$

The functions sin *z* and cos *z* are entire.

1.5.2 Hyperbolic Functions. They are defined by the formulae

$$\cosh z=\frac{1}{2}\left(e^z+e^{-z}\right),\qquad \sinh z=\frac{1}{2}\left(e^z-e^{-z}\right)$$

and they are called *hyperbolic cosine* and *hyperbolic sine* respectively.

1.7 Logarithmic Function

This is the last of the functions to define. The natural logarithm is defined as the inverse function of the exponential function, that is,

$$\ln z = w \text{ such that } z = e^w .$$

Because, as we have seen in the section of the exponential function, $e^w \neq 0$ for all complex w, the logarithm is not defined for $z = 0$. Let $w = u + iv$ and $z = re^{i\theta}$. We have

$$e^w = e^{u+iv} = re^{i\theta} \rightarrow e^u e^{iv} = re^{i\theta}$$

From which we conclude that

$$e^u = r \text{ or } u = \ln r \quad \text{and} \quad e^{iv} = e^{i\theta}$$

This provides the with the following definition of the logarithm

DEFINITION. The natural logarithmic function $f(z) = \ln z$, with $z = r\,e^{i\theta}$, is defined by

$$\boxed{\ln z = \ln r + i\theta} \tag{6}$$

EXAMPLE. Calculate ln *z* where $z = 2(\cos \pi/3 + i \sin \pi/3)$

$$\ln z = \ln 2 + i\,\pi/3.$$

Now, because for any real $z = r\,e^{i\theta}$ we have also $z = r\,e^{i\theta+2k\pi}$, in other words, since the argument of a complex number is not unique, the logarithmic function is a multiple-valued function. To avoid this dilemma, we define the ***principal value*** of the natural logarithm, for which $0 \le \theta \le 2\pi$.

The derivative of ln *z* is the same as for real functions

$$(\ln z)'=\frac{1}{z}$$

The function ln *z* is analytic for all $z \neq 0$

HOMEWORK

1. Given $z_1 = 2 + 3i$ and $z_2 = 1 + i$ and $z_3 = -3 + 2i$, calculate
(a) $z_3(z_1 + 2z_2)$ (b) $z_1 \bar{z}_2$ (c) $1/z_1$ (d) z_1/z_2 (e) $\operatorname{Re}(z_1/\bar{z}_1)$ (f) $(z_1)^{16}$

2. Evaluate $\sqrt[4]{z}$ and $\sqrt[5]{z}$ for
(a) $z = 2 - 2i$ (b) $z = 3\sqrt{3} + 3i$ (c) $z = 4\sqrt{4} + 4i$

3. Given $z = \sqrt{3} + i$, find
(a) the polar form of z
(b) write z^3 in polar form and in regular form^{-}
(c) $\sqrt[3]{z}$

4. Let $z = -\sqrt{2} + i\sqrt{2}$. Calculate z^5.

5. Differentiate
(a) $f(z) = \exp(z^2+1)$, (b) $f(z) = \ln(z^3 - 2z)$

6. Use Cauchy-Riemann equations to figure out whether the given functions are analytic
(a) $f(z) = z^2 - 2z + 1$ (b) $f(z) = \bar{z}/z$ (c)) $f(z) = \operatorname{Im}(z)$ (d)) $f(z) = \operatorname{Re}(z) + 2\operatorname{Im}(z)$
(e) $f(z) = \operatorname{Re}(z) + \operatorname{Im}(z)$

7. Calculate exactly $\ln z$ where
(a) $z = (3 - 3i)$ (b) $\sqrt{3} - i$ (c) $z = 2e^{i\pi/3}$ (d) $z = 2(\cos 5\pi/7 + i \sin 5\pi/7)$

8. Find the exact values of $\sin z$ and $\cos z$, for
(a) $z = 0$ (b) $z = \pi$ (c) $z = \pi/2$ (d) $z = \pi i$, (e) $z = \pi i/4$ (f) $z = \pi i/3$ (g) $z = \pi i/2$

SOLUTIONS

Problem 3.

(a) $r^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4, \Rightarrow r = 2 \quad \theta = \arctan\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$ $z = 2\,(\cos \pi/6 + i \sin \pi/6)$

(b) $z^3 = 8\,(\cos \pi/2 + i \sin \pi/2) = 8\,(0 + i) = 8i$.

(c) $\sqrt[3]{z} = \sqrt[3]{2}\left(\cos\dfrac{\pi/6 + 2k\pi}{3} + i\sin\dfrac{\pi/6 + 2k\pi}{3}\right)$ $k = 0, 1, 2$

$$\sqrt[3]{z} = \sqrt[3]{2}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right),\ \sqrt[3]{2}\left(\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right), \sqrt[3]{2}\left(\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right)$$

Problem 4. Let $z = -\sqrt{2} + i\sqrt{2}$. Calculate z^5.

$r^2 = \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 = 4 \Rightarrow r = 2 \qquad \theta = \arctan\dfrac{-\sqrt{2}}{\sqrt{2}} = \arctan(-1) = \dfrac{3\pi}{4}$. Therefore

$z = 2(\cos 3\pi/4 + i \sin 3\pi/4)$ and $z^5 = 32(\cos 15\pi/4 + i \sin 15\pi/4)$

But $15\pi/4 = 2\pi + 7\pi/4 \Rightarrow$ and $z^5 = 32(\cos 7\pi/4 + i \sin 7\pi/4) = 32\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) = 16\sqrt{2} - i16\sqrt{2}$

Problem 5. Differentiate

(a) $f(z) = \exp(z^2+1)$, (b) $f(z) = \ln(z^3 - 2z)$

(a) $f'(z) = 2z\exp(z^2+1)$ (b)

Problem 6. Use Cauchy-Riemann equations to figure out whether the given functions are analytic

(a) $f(z) = z^2 - 2z + 1$ (b) $f(z) = \bar{z}/z$ (c)) $f(z) = \mathrm{Im}(z)$ (d)) $f(z) = \mathrm{Re}(z) + 2\mathrm{Im}(z)$

(e) $f(z) = \mathrm{Re}(z) + \mathrm{Im}(z)$

(a) $z = x + iy$, $f(z) = f(x+iy) = u(x+iy) + i\,v(x+iy)$
$z^2 - 2z + 1 = x^2 - y^2 + 2xyi - 2(x + iy) + 1 = x^2 - y^2 - 2x + 1 + 2i(xy - y)$
$u_x = 2x - 2$ $\quad u_y = -2y$
$v_y = 2x - 2;$ $\quad -v_x = -2y$.

(b) $\bar{z}/z$

$$\frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$$

$$u_x = \frac{2x(x^2 + y^2) - (x^2 - y^2)2x}{(x^2 + y^2)^2} = \frac{4xy^2}{(x^2 + y^2)^2} \quad v_y = -\frac{2x(x^2 + y^2) - 2xy(2y)}{(x^2 + y^2)^2} = -\frac{2x^3 - 2xy^2}{(x^2 + y^2)^2}$$

Therefore, not analytic

Problem 7. Calculate exactly $\ln z$ where

(a) $z = (3 - 3i)$ (b) $\sqrt{3} - i$ (c) $z = 2e^{i\pi/3}$ (d) $z = 2\left(\cos 5\pi/7 + i\sin 5\pi/7\right)$

(a) $r = \sqrt{3^2 + 3^2} = 3\sqrt{3} \quad \theta = \arctan(3/-3) = \arctan -1 = 7\pi/4$

(a) $\ln(3-3i)=\ln\left(3\sqrt{2}\right)+i7\pi/4$

Problem 8. Find the exact values of sin z and cos z, for
(a) $z = 0$ (b) $z = \pi$ (c) $z = \pi/2$ (d) $z = \pi i$, (e) $z = \pi i/4$ (f) $z = \pi i/3$ (g) $z = \pi i/2$