Economics 686

Mid-Term Examination

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Candidates should attempt each and every question. Please answer questions 1–10 on the examination question paper, by circling the best answer. Questions 11–20 should be answered in the examination answer booklet provided to you. Each answer can earn a maximum of 5 marks.

Throughout the examination, the linear model $y = X\beta + \varepsilon$ will be used, y and ε being $(n \times 1)$ vectors, X being an $(n \times k)$ matrix of rank k and β being a $(k \times 1)$

vector of coefficients. X and y are observed, ε and β are not observable, but β is known to lie in a subset of a k-dimensional subspace of R^n. Sometimes $X\beta = X_1\beta_1 + X_2\beta_2$ in which X_1 is $(n \times k_1)$ and X_2 is $(n \times k_2)$, $k_1 + k_2 = k$. $R[X] = L$, $R[X_1] = L_1$ and $R[X_2] = L_2$; $X\beta = \mu$ with components $\mu_1 \in L_1$ and $\mu_2 \in L_2$. L will be considered a fixed subspace and $y \sim N(\mu; \Sigma\sigma^2)$ with $\mu \in L$ and Σ a pd matrix of order n. Finally, when $\Sigma = I_n$, $s^2 = \left\{\frac{y^\top (I_n - P)y}{(n - k)}\right\}$.

Part I

Multiple Choice Questions

1. The distribution of $A^\top y + b$, where A is a known, fixed, $(n \times m)$ matrix of rank m and b is a fixed vector in R^m, is

a) $N\left(A^\top\mu;\ A^\top\Sigma A\sigma^2\right)$

b) A degenerate normal distribution with mean $A^\top\mu + b$

c) $\chi^2(m; \delta)$ $\qquad \delta = \left\{\frac{(A^\top\mu + b)^\top (A^\top\mu + b)}{\sigma^2}\right\}$

d) $N\left(A^\top\mu + b, A^\top\Sigma A\sigma^2\right)$

e) None of the above

2. If $F = X\left(X^\top\Sigma^{-1}X\right)^{-1}X^\top\Sigma^{-1}$, then F is

a) A projection matrix on L along $[\Sigma^{-1}L]^{\perp}$

b) Symmetric but not idempotent

c) Idempotent but not symmetric

d) A projection matrix on L along $N\left[\Sigma^{-1}X^{\top}\right] = \Sigma^{-1}L^{\perp}$

e) Both a) and c)

3. If Q is a non-singular, $(n \times n)$ matrix such that $Q\Sigma Q^{\top} = I_n$, then $Qy \sim N(Q\mu, I_n\sigma^2)$ and $\left\{\frac{y^{\top}\Sigma^{-1}y}{\sigma^2}\right\}$ is distributed as

a) $\chi^2(n, \delta)$; $\delta = \left\{\frac{\mu^{\top}\Sigma^{-1}\mu}{\sigma^2}\right\}$

b) Some unknown form, because Σ^{-1} is not idempotent

c) A doubly non-central $F(n, 1)$ distribution

d) $\chi^2(k, n)$

e) None of the above

4. Let $\Sigma = I_n$. $F = \left\{\frac{y^{\top}(P-P_1)y(n-k)}{k_2 s^2}\right\}$ is a test for the null hypothesis

a) $\mu = 0$

b) $\beta = 0$

c) $\beta_1 = 0$

d) $\mu_1 = \mu_2$

e) $\beta_2 = 0$

5. $\left\{\frac{y^\top(P-P_1)y}{y^\top(I-P_1)y}\right\}$ will be distributed as

a) $\chi^2(k_2; \delta)$ $\quad \delta = 0$ when $\mu \in L_1$

b) $F(k_2; n - k_1)$ centrally when $\mu \in L_1$

c) $\beta\{(k_2/2); (n - k)/2\}$, centrally when $\mu \in L_1$

d) $t(n - k_1)$, non-central at all times

e) $\beta\{(k_2/2); (n - k_1)/2\}$, centrally when $\mu \in L_1$

6. Let Z be an $(n \times k)$ fixed matrix of rank k such that $\left(Z^\top X\right)^{-1}$ exists. Let $R[Z] = M$. The projection matrix $X\left(Z^\top X\right)^{-1} Z^\top$ may be described as

a) On M along $L^\perp$

b) On L along M

c) On M along L

d) On L along $M^\perp$

e) None of the above

7. In question 6, let Z now be $(n \times m)$, $m \geq k$, of rank m, and consider the orthogonal projection matrix on $\mathcal{M}$, say $Q = Z\left(Z^{\top}Z\right)^{-1}Z^{\top}$. Let $\mathcal{R}[QX] = \mathcal{M}_0$ a k-dimensional subspace of $\mathcal{M}$. The projection matrix on $\mathcal{M}_0$ along $\mathcal{L}^{\perp}$ is

a) $X\left(X^{\top}QX\right)^{-1}X^{\top}Q$

b) $QX\left(X^{\top}QX\right)^{-1}X^{\top}$

c) $ZM_0\left(X^{\top}ZM_0\right)^{-1}X^{\top}$ where M_0 is an $(m \times k)$ matrix such that $\mathcal{M}_0 = \mathcal{R}[ZM_0]$

d) (Q - P)

e) Both b) and c)

8. $\left(\frac{y^{\top}\Sigma^{-1}X\left(X^{\top}\Sigma^{-1}X\right)^{-1}X^{\top}\Sigma^{-1}y}{\sigma^2}\right)$ is distributed as

a) $\chi^2(n; \delta)$

b) $\chi^2(k; \delta)$

c) An unknown distribution because $\Sigma^{-1}X\left(X^{\top}\Sigma^{-1}X\right)^{-1}X^{\top}\Sigma^{-1}$ is not idempotent

d) $\chi^2(n - k; \delta)$

e) None of the above

9. If H_0: $A^\top \beta = 0$; $\beta \in L$ is to be tested, the regression subspace on H_0 is L_0 and $L_0^\perp \cap L$ is

a) $L \cap N\left[A^\top\right]$

b) $R[P - P_0]$ where P_0 is the orthogonal projection matrix on L_0

c) $R[PA]$

d) a) and b)

e) b) and c)

10. If $B^\top$ is a partial isometry matrix of order $(k \times n)$ for the subspace L, then B is defined by

a) $\|B^\top x\| = \|x\|$ for all $x \in L$

b) $B^\top x = 0$ for all $x \in L^\perp$

c) $B^\top B = I_k$

d) $BB^\top = P$

e) a) and b) OR c) and d)

Part II

Other Questions

11. If $\hat{\beta}$ is the OLS estimator of β and $\tilde{\beta}$ is the GLS estimator of β, state the necessary and sufficient condition for $\hat{\beta} = \tilde{\beta}$. Illustrate your answer with a simple example.

12. You are given the following two-equation model

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}$$

where y, X and β are as discussed above, w is $(n \times 1)$; γ is $(k \times 1)$ and

$$\begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \sim N(0; \Omega)$$

where

$$\Omega = \begin{bmatrix} I_n\sigma_\varepsilon^2 & I_n\sigma_{\varepsilon\eta} \\ I_n\sigma_{\eta\varepsilon} & I_n\sigma_\eta^2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon\eta} \\ \sigma_{\eta\varepsilon} & \sigma_\eta^2 \end{bmatrix} \otimes I_n.$$

Demonstrate that the OLS estimator of $\begin{bmatrix} \beta \\ \gamma \end{bmatrix}$ is the same as the corresponding GLS estimator, by writing

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = I_n \otimes X.$$

13. In $y = X\beta + \varepsilon$; $\varepsilon \sim N(0; \Sigma\sigma^2)$, given that $\hat{\beta}$ (OLS) $= \hat{\beta}$ (GLS), $D\left[\hat{\beta}\right] = D\left[\tilde{\beta}\right]$. Find $D\left[\hat{\beta}\right]$ and show that it is equal to $D\left[\tilde{\beta}\right]$, when $\hat{\beta} = \tilde{\beta}$.

14. The model $y = X\beta + \varepsilon;\ \Sigma = I_n$, is divided into two time periods $1, 2, \cdots, q$ and $q+1, q+2, \cdots, n$. Let q and $(n - q)$ be each greater than k. In the first period (I) the coefficients vector is β_I, in the second period (II) the corresponding vector is β_{II}. Briefly explain how to test the null hypothesis $\beta_I = \beta_{II}$.

15. In question 14, how would you check that σ^2 for period I is the same as σ^2 for period II. Why is this important?

16. Suppose now, in question 14, that $(n - q) < k$. Explain how you would modify your answer to question 14 to test H_0: $\beta_I = \beta_{II}$.

17. In the model $y = X\beta + \varepsilon;\ \Sigma = I_n$, it is desired to test that H_0: $B^\top\beta = 0$ where $k = 5$ and B is the (5×3) matrix of rank 3

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Express H_0 in freedom equation form and confirm that, if $\beta = M\alpha$ is this form, then $B^\top M = 0$.

18. In the model $y = X\beta + \varepsilon;\ \Sigma = I_n$ and two estimators of β are proposed $\hat{\beta} = \left(X^\top X\right)^{-1} X^\top y$ and $\tilde{\beta} = \left(X^\top A X\right)^{-1} X^\top A y$ where A is the orthogonal projection on another subspace of dimension $K \geq k$, such that $X^\top A X$ is non-singular. $E A\varepsilon =$

$AE\varepsilon = 0$. Show that

$$\tilde{\beta} - \hat{\beta} = \left(X^\top AX\right)^{-1} X^\top AMy$$

where $M = (I_n - P)$.

19. Given $\left(\tilde{\beta} - \hat{\beta}\right)$ takes that form displayed in question 18, find its dispersion (variance-covariance matrix), treating both X and A as fixed, and hence show that

$$\left(\tilde{\beta} - \hat{\beta}\right)^\top \left[D\left(\tilde{\beta} - \hat{\beta}\right)\right]^{-1}\left(\tilde{\beta} - \hat{\beta}\right) = \frac{y^\top MAX\left(X^\top AMAX\right)^{-1} X^\top AMy}{\sigma^2} = Q.$$

How is Q distributed?

20. Let $AX = W$ in question 19. An estimate $\hat{Q}$ of Q in the same question may be obtained by replacing σ^2 with $\hat{\sigma}^2$. This may be done by calculating the F–statistic for the null hypothesis H_0: $\gamma = 0$ in the regression

$$y = X\beta + W\gamma + r,$$

where, on H_0, $r \sim N(0; I_n\sigma^2)$. Show that

$$\hat{Q} = kF$$

and state the approximate distribution of $\hat{Q}$ when n is 'large'.