Economics 686

Solutions to Mid-Term Examination

Fall, 1996

by

Gordon R. Fisher

Concordia University

School of Graduate Studies

Departemnt of Economics

Candidates should attempt each and every question. Please answer questions 1–10 on the examination question paper, by circling the best answer. Questions 11–20 should be answered in the examination answer booklet provided to you. Each answer can earn a maximum of 5 marks.

Throughout the examination, the linear model $y = X\beta + \varepsilon$ will be used, y and ε being $(n \times 1)$ vectors, X being an $(n \times k)$ matrix of rank k and β being a $(k \times 1)$

vector of coefficients. X and y are observed, ε and β are not observable, but β is known to lie in a subset of a k-dimensional subspace of R^n. Sometimes $X\beta = X_1\beta_1 + X_2\beta_2$ in which X_1 is $(n \times k_1)$ and X_2 is $(n \times k_2)$, $k_1 + k_2 = k$. $R[X] = L$, $R[X_1] = L_1$ and $R[X_2] = L_2$; $X\beta = \mu$ with components $\mu_1 \in L_1$ and $\mu_2 \in L_2$. L will be considered a fixed subspace and $y \sim N(\mu; \Sigma\sigma^2)$ with $\mu \in L$ and Σ a pd matrix of order n. Finally, when $\Sigma = I_n$, $s^2 = \left\{\frac{y^\top (I_n - P) y}{(n - k)}\right\}$.

Part I

Multiple Choice Questions

1. The distribution of $A^\top y + b$, where A is a known, fixed, $(n \times m)$ matrix of rank m and b is a fixed vector in R^m, is

e) None of the above

2. If $F = X\left(X^\top \Sigma^{-1} X\right)^{-1} X^\top \Sigma^{-1}$, then F is

a) A projection matrix on L along $[\Sigma^{-1}L]^\perp$

c) Idempotent but not symmetric

e) Both a) and c)

3. If Q is a non-singular, $(n \times n)$ matrix such that $Q\Sigma Q' = I_n$, then $Qy \sim N(Q\mu; I_n\sigma^2)$ and $\left\{\frac{y'\Sigma^{-1}y}{\sigma^2}\right\}$ is distributed as

a) $\chi^2(n; \delta)$; $\quad \delta = \left\{\frac{\mu'\Sigma^{-1}\mu}{\sigma^2}\right\}$

4. Let $\Sigma = I_n$. $F = \left\{\frac{y'(P-P_1)y}{k_2 s^2}\right\}$ is a test for the null hypothesis

e) $\beta_2 = 0$

5. $\left\{\frac{y'(P-P_1)y}{y'(I-P_1)y}\right\}$ will be distributed as

c) $\beta\{(k_2/2); (n-k)/2\}$, centrally when $\mu \in L_1$

6. Let Z be an $(n \times k)$ fixed matrix of rank k such that $\left(Z'X\right)^{-1}$ exists. Let $R[Z] = M$. The projection matrix $X\left(Z'X\right)^{-1}Z'$ may be described as

d) On L along $M^{\perp}$

7. In question 6, let Z now be $(n \times m)$, $m \geq k$, of rank m, and consider the orthogonal projection matrix on M, say $Q = Z\left(Z'Z\right)^{-1}Z'$. Let $R[QX] = M_0$ a k-dimensional subspace of M. The projection matrix on M_0 along $L^{\perp}$ is

b) $QX\left(X'QX\right)^{-1}X'$

c) $ZM_0\left(X'ZM_0\right)^{-1}X'$ where M_0 is an $(m \times k)$ matrix such that $M_0 = R[ZM_0]$

e) Both b) and c)

8. $\left(\frac{y^\top \Sigma^{-1} X (X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1} y}{\sigma^2}\right)$ is distributed as

b) $\chi^2(k; \delta)$

9. If H_0: $A^\top \beta = 0$; $\beta \in L$ is to be tested, the regression subspace on H_0 is L_0 and $L_0^\perp \cap L$ is

b) $R[P - P_0]$ where P_0 is the orthogonal projection matrix on L_0

c) $R[PA]$

e) b) and c)

10. If $B^\top$ is a partial isometry matrix of order $(k \times n)$ for the subspace L, then B is defined by

a) $\|B^\top x\| = \|x\|$ for all $x \in L$

b) $B^\top x = 0$ for all $x \in L^\perp$

c) $B^\top B = I_k$

d) $BB^\top = P$

e) <u>a) and b) OR c) and d)</u>

Part II

Other Questions

11. $\hat{\beta} = \tilde{\beta}$ iff L is invariant under Σ. E.g. Let $\Sigma = I_n(1 - \rho) + ee^{\top}\rho;\ 0 < \rho < 1$, where e is the equiangular vector in R^n and the first column of X. Then $Pe = e$ and

$$
\begin{aligned}
P\Sigma P &= P\left(I_n(1-\rho) + ee^{\top}\rho\right)P \\
&= \left(P(1-\rho) + ee^{\top}\rho\right)P \\
&= \left(I_n(1-\rho) + ee^{\top}\rho\right)P \\
&= \Sigma P.
\end{aligned}
$$

If $P\Sigma P = \Sigma P$ for some P on L, then L is invariant under Σ.

12. For $\Omega = \Sigma \otimes I_n$,

$$
\Sigma = \begin{bmatrix} \sigma_{\varepsilon}^2 & \sigma_{\varepsilon\eta} \\ \sigma_{\eta\varepsilon} & \sigma_{\eta}^2 \end{bmatrix}
$$

and the regression subspace $R[I_2 \otimes X] = L_*$, let P_* be the orthogonal projection on L_* then $P_* = (I_2 \otimes P)$. Let $\begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \theta$. Then $\hat{\theta}(OLS) = \tilde{\theta}(GLS)$ iff L_* is invariant under Ω. This is the case because

$$
\begin{aligned}
P_*\Omega P_* &= (I_2 \otimes P)(\Sigma \otimes I_n)(I_2 \otimes P) \\
&= \Sigma \otimes P
\end{aligned}
$$

$$= (\Sigma \otimes I_n)(I_2 \otimes P)$$

$$= \Sigma \otimes P.$$

13. For x and z in R^k

$$D\left[\left(x, \hat\beta\right), \left(z, \hat\beta\right)\right] = D\left(\left(x, \left(x^\top x\right)^{-1} x^\top y\right), \left(z, \left(x^\top x\right)^{-1} x^\top y\right)\right)$$

$$= \left(x, \left(x^\top x\right)^{-1} x^\top \Sigma x \left(x^\top x\right)^{-1} z\right) \sigma^2$$

$$D\left[\left(x, \tilde\beta\right), \left(z, \tilde\beta\right)\right] = D\left\{\left(x, \left(x^\top \Sigma^{-1} x\right)^{-1} x^\top \Sigma^{-1} y\right), \left(z, \left(x^\top \Sigma^{-1} x\right)^{-1} x^\top \Sigma^{-1} y\right)\right\}$$

$$= \left(x, \left(x^\top \Sigma^{-1} x\right)^{-1} x^\top \Sigma^{-1} \Sigma \Sigma^{-1} x \left(x^\top \Sigma^{-1} x\right)^{-1} z\right) \sigma^2$$

$$= \left(x, \left(x^\top \Sigma^{-1} x\right)^{-1} z\right) \sigma^2.$$

Thus $D\hat\beta = \left(x^\top x\right)^{-1} x^\top \Sigma x \left(x^\top x\right)^{-1} \sigma^2$ and $D\tilde\beta = \left(x^\top \Sigma^{-1} x\right)^{-1} \sigma^2$. If $\hat\beta = \tilde\beta$ then it is possible to write $\Sigma x = xM$ for some ns M $(k \times k)$; thus $\Sigma^{-1} x = xM^{-1}$ and

$$\frac{1}{\sigma^2} D\hat\beta = \left(x^\top x\right)^{-1} x^\top x M \left(x^\top x\right)^{-1}$$

$$= M \left(x^\top x\right)^{-1}$$

$$= \left(x^\top x M^{-1}\right)^{-1}$$

$$= \left(x^\top \Sigma^{-1} x\right)^{-1}.$$

Thus $D\hat\beta = D\tilde\beta$.

14. $\begin{bmatrix} y_I \\ y_{II} \end{bmatrix} = \begin{bmatrix} X_I & 0 \\ 0 & X_{II} \end{bmatrix} \begin{bmatrix} \beta_I \\ \beta_{II} \end{bmatrix} + \begin{bmatrix} \varepsilon_I \\ \varepsilon_{II} \end{bmatrix}$ with $n - q > k$ is H. This model has

regression subspace $L_* = R\begin{bmatrix} X_I & 0 \\ 0 & X_{II} \end{bmatrix}$, with op given by $\begin{bmatrix} P_I & 0 \\ 0 & P_{II} \end{bmatrix} = P_*$. When $H_0 : \beta_I = \beta_{II} = \beta$, $\begin{bmatrix} y_I \\ y_{II} \end{bmatrix} = \begin{bmatrix} X_I \\ X_{II} \end{bmatrix}\beta + \begin{bmatrix} \epsilon_I \\ \epsilon_{II} \end{bmatrix}$. Of course, $\begin{bmatrix} X_I \\ X_{II} \end{bmatrix} = X$ the orthogonal projection onto L is P. Thus

$$F = \frac{y^\top (P_* - P) y}{y^\top (I_n - P_*) y} \cdot \frac{n-2k}{k} \sim F(k, n-2k)$$

non-central on H, central on H_0. Hence:

1. Fit each period separately and determine

$$S_I^2 = \hat{\epsilon}_I^\top \hat{\epsilon}_I \qquad S_{II}^2 = \hat{\epsilon}_{II}^\top \hat{\epsilon}_{II}$$

2. Fit the model on H_0, yielding $S^2 = \tilde{\epsilon}^\top \tilde{\epsilon}$, where $\epsilon = \begin{bmatrix} \epsilon_I \\ \epsilon_{II} \end{bmatrix}$ and $\tilde{\epsilon}$ are the residuals on H_0.

3. Form

$$F = \frac{S^2 - S_I^2 - S_{II}^2}{S_I^2 + S_{II}^2} \cdot \frac{n-2k}{k}$$

4. If $F > F(k, n-2k)_{0.95}$ e.g. then H_0 is rejected, if not, do not reject H_0. Alternatively, state a p-value and make an appropriate conclusion.

15. Checking that $\sigma_I^2 = \sigma_{II}^2$ is vital because H_0 and H are formulated on the

assumption that $\sigma_I^2 = \sigma_{II}^2$. The required test is (from Question 14)

$$\left[\frac{\frac{1}{n_1-k}S_I^2}{\frac{1}{n_2-k}S_{II}^2}\right] \sim F(n_1-k; n_2-k):$$

The denominator here is S_{II}^2 and the numerator is S_{II}^2. The tables for F usually given only probabilities in the upper tail. If therefore $S_I^2 > S_{II}^2$, the test above is satisfactory. Otherwise use $\left(\frac{S_{II}^2}{S_I^2}\right) \sim F(n_2-k; n_1-k)$.

16. When $k > (n-q)$, the second-period regression can be made to fit perfectly. Recognizing this

$$P_* = \begin{bmatrix} P_I & 0 \\ 0 & I_{n-q} \end{bmatrix}$$

which is still an op. Then $y^\top(P_* - P)y$ may be written $\left(\hat\beta_I - \tilde\beta\right)^\top X_1^\top X_1 \left(\hat\beta_I - \tilde\beta\right) + \left(y_{II} - X_2\tilde\beta\right)^\top \left(y_{II} - X_2\tilde\beta\right)$ and $y^\top(I_n - P_*)y = y^{\top}_I(I_q - P_I)y_I$. Thus

$$\begin{aligned} F &= \frac{y^\top(P_* - P)y}{y_I^\top(I_q - P_I)y}\cdot\frac{q-k}{n-q} \\ &= \frac{\left(\hat\beta_I - \tilde\beta\right)^\top X_1^\top X_1 \left(\hat\beta_I - \tilde\beta\right) + \left(y_{II} - X_2\tilde\beta\right)^\top \left(y_{II} - X_2\tilde\beta\right)}{y_I^\top(I_q - P_I)y_I}\cdot\frac{q-k}{n-q}: \end{aligned}$$

If $B^\top\beta = 0$ is as given, then

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}\beta = 0$$

implies

$$\beta_1 - \beta_2 = 0$$

$$2\beta_2 + \beta_5 = 0$$

$$2\beta_3 - \beta_4 = 0.$$

Since $k = 5$ and $r = 3$, the number of independent coefficients is 2. Let these be α_1 and α_1. Set $\alpha_1 = \beta_1 \Rightarrow \beta_2 = \alpha_1$ and $\beta_5 = -2\alpha_1$. Setting $\beta_3 = \alpha_2$; $\beta_4 = 2\alpha_2$. Hence

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = M\alpha.$$

Then $B^\top M$

$$B^\top M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

as is required.

18. $\hat{\beta} = \left(X^\top X\right)^{-1} X^\top y \qquad \tilde{\beta} = \left(X^\top AX\right)^{-1} X^\top Ay \qquad EA\varepsilon = 0.$

$$\begin{aligned}
\tilde{\beta} - \hat{\beta} &= \left(X^\top AX\right)^{-1} X^\top Ay - \left(X^\top X\right)^{-1} X^\top y \\
&= \left(X^\top AX\right)^{-1} \left(X^\top A - X^\top AX \left(X^\top X\right)^{-1} X^\top\right) y \\
&= \left(X^\top AX\right)^{-1} \left[X^\top A \left(I_n - P\right)\right] y \\
&= \left(X^\top AX\right)^{-1} X^\top AMy.
\end{aligned}$$

19.

$$\begin{aligned}
D\left(\tilde{\beta} - \hat{\beta}\right) &= \left(X^\top AX\right)^{-1} X^\top AMD[y]MAX\left(X^\top AX\right)^{-1} \\
&= \left(X^\top AX\right)^{-1} X^\top AMAX\left(X^\top AX\right)^{-1} \sigma^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\tilde{\beta} - \hat{\beta}\right)^\top \left[D\left(\tilde{\beta} - \hat{\beta}\right)\right]^{-1} \left(\tilde{\beta} - \hat{\beta}\right) \\
&= \frac{y^\top MAX\left(X^\top AX\right)^{-1} \left(\left(X^\top AX\right)^{-1} X^\top AMAX\left(X^\top AX\right)^{-1}\right)^{-1} \left(X^\top AX\right)^{-1} X^\top AMy}{\sigma^2} \\
&= \frac{y^\top MAX\left(X^\top AMAX\right)^{-1} X^\top AMy}{\sigma^2} \\
&= Q
\end{aligned}$$

$Q \sim \chi^2(k; \delta)$ $\delta = \frac{\beta^\top MAX\left(X^\top AMAX\right)^{-1} X^\top AM\beta}{\sigma}$. Hence if $\beta \in L$; $\delta = 0$.

20. Let $AX = W$. Then

$$q = \frac{y^\top MW\left(W^\top MW\right)^{-1} W^\top My}{\sigma^2}.$$

consider the following regression

$$y = X\beta + W\gamma = r.$$

If this were a regression with $r \sim N(0; \sigma^2 m I_n)$ the standard F-test for $H_0 : \theta = 0$ is

$$F = \frac{y^\top MW \left(W^\top MW \right)^{-1} W^\top My}{y^\top (I_n - P_*) y} . \frac{n - 2k}{k} \sim F(k; n - 2k)$$

P_* being the op onto $R[X : W]$. It follows that

$$kF = \hat{Q}$$

when $\frac{y^\top (I_n - P_*) y}{n - 2k} = \hat{\sigma}^2$. $\hat{Q} \overset{d}{\to} \chi^2(k; \delta)$ $\delta = 0$ when $\mu \in L$.