Some Results on the Normal and the Lognormal Distributions

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This note states and proves various results on expectations of functions of normally and lognormally distributed random variables that are frequently applied in the pricing of financial derivatives.

Theorem 1 *If $X \sim N(\mu, \sigma^2)$ and $\gamma \in \mathbb{R}$, then*

$$\mathrm{E}\left[e^{-\gamma X}\right] = \exp\left\{\Leftrightarrow\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right\}.$$

Proof: By definition, we have

$$\mathrm{E}\left[e^{-\gamma X}\right] = \int_{-\infty}^{+\infty} e^{-\gamma x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

Manipulating the exponent, we get

$$\begin{aligned}
\mathrm{E}\left[e^{-\gamma X}\right] &= e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\gamma\mu\sigma^2 + \gamma^2\sigma^4\right]}\, dx \\
&= e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-[\mu+\gamma\sigma^2])^2}{2\sigma^2}}\, dx \\
&= e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2},
\end{aligned}$$

where the last equality holds because the function

$$x \mapsto \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-[\mu+\gamma\sigma^2])^2}{2\sigma^2}}$$

is a probability density function, namely that of a random variable, which is $N(\mu + \gamma\sigma^2, \sigma^2)$ distributed. □

Theorem 2 *If*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right),$$

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and $g : \mathbb{R} \to \mathbb{R}$, *then*

$$\begin{aligned} \mathrm{E}\left[e^{-Y} g(X)\right] &= \mathrm{E}\left[e^{-Y}\right] \mathrm{E}\left[g(\hat{X})\right] \\ &= e^{-\mu_Y + \frac{1}{2}\sigma_Y^2}\, \mathrm{E}\left[g(\hat{X})\right], \end{aligned}$$

where $\hat{X} = X - \sigma_{XY}$, *i.e.* $\hat{X} \sim N(\mu_X - \sigma_{XY}, \sigma_X^2)$.

Proof: We have

$$\mathrm{E}\left[e^{-Y} g(X)\right] = \mathrm{E}\left[g(X)\, \mathrm{E}\left[e^{-Y} \middle| X\right]\right]. \tag{1}$$

It is well-known that

$$Y \mid X = x \sim N\left(\mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x - \mu_X), \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}\right).$$

From Theorem 1 it follows that

$$\mathrm{E}\left[e^{-Y} \middle| X\right] = \exp\left\{-\mu_Y - \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X) + \frac{1}{2}\sigma_Y^2 - \frac{\sigma_{XY}^2}{2\sigma_X^2}\right\}.$$

Substituting this expression into Equation 1 yields

$$\begin{aligned} \mathrm{E}\left[e^{-Y} g(X)\right] &= \mathrm{E}\left[g(X) \exp\left\{-\mu_Y - \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X) + \frac{1}{2}\sigma_Y^2 - \frac{\sigma_{XY}^2}{2\sigma_X^2}\right\}\right] \\ &= e^{-\mu_Y + \frac{1}{2}\sigma_Y^2}\, \mathrm{E}\left[g(X) \exp\left\{-\frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X) - \frac{\sigma_{XY}^2}{2\sigma_X^2}\right\}\right]. \end{aligned} \tag{2}$$

Per definition, we have

$$\begin{aligned} &\mathrm{E}\left[g(X) \exp\left\{-\frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X) - \frac{\sigma_{XY}^2}{2\sigma_X^2}\right\}\right] \\ &= \int_{-\infty}^{+\infty} g(x) \exp\left\{-\frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X) - \frac{\sigma_{XY}^2}{2\sigma_X^2}\right\} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{(x - \mu_X)^2}{2\sigma_X^2}\right\} dx \\ &= \int_{-\infty}^{+\infty} g(x) \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{(x - [\mu_X - \sigma_{XY}])^2}{2\sigma_X^2}\right\} dx \\ &= \mathrm{E}\left[g(\hat{X})\right], \end{aligned} \tag{3}$$

since

$$x \mapsto \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{(x - [\mu_X - \sigma_{XY}])^2}{2\sigma_X^2}\right\}$$

is the probability density function of $\hat{X}$. Substituting (3) into (2), we get the stated result. The relation

$$\mathrm{E}\left[e^{-Y}\right] = e^{-\mu_Y + \frac{1}{2}\sigma_Y^2}$$

follows from Theorem 1, of course. □

Theorem 3 *If $X \sim N(\mu, \sigma^2)$ and $K > 0$, then*

$$\begin{aligned}
\mathrm{E}\left[\, e^X \middle|\, e^X > K \right] &= e^{\mu + \frac{1}{2}\sigma^2} N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma} + \sigma\right) \\
&= \mathrm{E}\left[e^X\right] N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma} + \sigma\right).
\end{aligned}$$

Proof: Since $e^X > K \Leftrightarrow X > \ln K$, we have by definition of the expectation

$$\begin{aligned}
\mathrm{E}\left[\, e^X \middle|\, e^X > K \right] &= \mathrm{E}\left[\, e^X \middle|\, X > \ln K \right] \\
&= \int_{\ln K}^{+\infty} e^x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \int_{\ln K}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-[\mu+\sigma^2])^2}{2\sigma^2}} e^{\frac{2\mu\sigma^2+\sigma^4}{2\sigma^2}}\, dx \\
&= e^{\mu + \frac{1}{2}\sigma^2} \int_{\ln K}^{+\infty} f_{\bar{X}}(x)\, dx,
\end{aligned}$$

where

$$f_{\bar{X}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-[\mu+\sigma^2])^2}{2\sigma^2}}$$

is the probability density function of a random variable, which is $N(\mu + \sigma^2, \sigma^2)$ distributed. Since

$$\begin{aligned}
\int_{\ln K}^{+\infty} f_{\bar{X}}(x)\, dx &= \mathrm{Prob}(\bar{X} > \ln K) \\
&= \mathrm{Prob}\left(\frac{\bar{X} \Leftrightarrow [\mu + \sigma^2]}{\sigma} > \frac{\ln K \Leftrightarrow [\mu + \sigma^2]}{\sigma}\right) \\
&= \mathrm{Prob}\left(\frac{\bar{X} \Leftrightarrow [\mu + \sigma^2]}{\sigma} < \Leftrightarrow \frac{\ln K \Leftrightarrow [\mu + \sigma^2]}{\sigma}\right) \\
&= N\left(\Leftrightarrow \frac{\ln K \Leftrightarrow [\mu + \sigma^2]}{\sigma}\right) \\
&= N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma} + \sigma\right),
\end{aligned}$$

the assertion of the theorem follows. □

Theorem 4 *If $X \sim N(\mu, \sigma^2)$ and $K > 0$, then*

$$\begin{aligned}
\mathrm{E}\left[\max\left(0, e^X \Leftrightarrow K\right)\right] &= e^{\mu + \frac{1}{2}\sigma^2} N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma} + \sigma\right) \Leftrightarrow K N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma}\right) \\
&= \mathrm{E}\left[e^X\right] N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma} + \sigma\right) \Leftrightarrow K N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma}\right).
\end{aligned}$$

Proof: Obviously, we have

$$\begin{aligned}
\mathrm{E}\left[\max\left(0, e^X \Leftrightarrow K\right)\right] &= \mathrm{E}\left[\, e^X \Leftrightarrow K \middle|\, e^X > K \right] \\
&= \mathrm{E}\left[\, e^X \middle|\, e^X > K \right] \Leftrightarrow K \mathrm{Prob}\left(e^X > K\right).
\end{aligned}$$

The first term is known from Theorem 3. For the second term, we get

$$\begin{aligned}\text{Prob}\left(e^X > K\right) &= \text{Prob}\left(X > \ln K\right) \\ &= \text{Prob}\left(\frac{X \Leftrightarrow \mu}{\sigma} > \frac{\ln K \Leftrightarrow \mu}{\sigma}\right) \\ &= \text{Prob}\left(\frac{X \Leftrightarrow \mu}{\sigma} < \Leftrightarrow \frac{\ln K \Leftrightarrow \mu}{\sigma}\right) \\ &= N\left(\Leftrightarrow \frac{\ln K \Leftrightarrow \mu}{\sigma}\right) \\ &= N\left(\frac{\mu \Leftrightarrow \ln K}{\sigma}\right).\end{aligned}$$

The claim now follows. □

Theorem 5 *If*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}\right),$$

and $K > 0$, *then*

$$\begin{aligned}\text{E}\left[e^{-Y} \max\left(0, e^X \Leftrightarrow K\right)\right] &= \text{E}\left[e^{-Y}\right] \Bigg\{ e^{-\sigma_{XY}}\, \text{E}\left[e^X\right] N\left(\frac{\mu_X \Leftrightarrow \sigma_{XY} \Leftrightarrow \ln K}{\sigma_X} + \sigma_X\right) \\ &\qquad \Leftrightarrow K N\left(\frac{\mu_X \Leftrightarrow \sigma_{XY} \Leftrightarrow \ln K}{\sigma_X}\right)\Bigg\} \\ &= \text{E}\left[e^{-Y}\right] \Bigg\{ e^{\phi} N\left(\frac{\phi \Leftrightarrow \ln K}{\sigma_X} + \frac{1}{2}\sigma_X\right) \\ &\qquad \Leftrightarrow K N\left(\frac{\phi \Leftrightarrow \ln K}{\sigma_X} \Leftrightarrow \frac{1}{2}\sigma_X\right)\Bigg\},\end{aligned} \tag{4}$$

where

$$\phi = \mu_X + \frac{1}{2}\sigma_X^2 \Leftrightarrow \sigma_{XY}.$$

Proof: From Theorem 2, we have

$$\text{E}\left[e^{-Y} \max\left(0, e^X \Leftrightarrow K\right)\right] = \text{E}\left[e^{-Y}\right] \text{E}\left[\max\left(0, e^{\hat{X}} \Leftrightarrow K\right)\right],$$

where $\hat{X} \sim N(\mu_X \Leftrightarrow \sigma_{XY}, \sigma_X^2)$. From Theorem 4, we have

$$\begin{aligned}\text{E}\left[\max\left(0, e^{\hat{X}} \Leftrightarrow K\right)\right] &= \text{E}\left[e^{\hat{X}}\right] N\left(\frac{\mu_X \Leftrightarrow \sigma_{XY} \Leftrightarrow \ln K}{\sigma_X} + \sigma_X\right) \\ &\quad \Leftrightarrow K N\left(\frac{\mu_X \Leftrightarrow \sigma XY \Leftrightarrow \ln K}{\sigma_X}\right),\end{aligned}$$

and since

$$\text{E}\left[e^{\hat{X}}\right] = \text{E}\left[e^{X - \sigma_{XY}}\right] = e^{-\sigma_{XY}}\, \text{E}\left[e^X\right],$$

we get the first expression in (4). To get the second, we use Theorem 1 to get that

$$e^{-\sigma_{XY}}\,\mathrm{E}\left[e^{X}\right] = e^{-\sigma_{XY}}\,e^{\mu_X+\frac{1}{2}\sigma_X^2} = e^{\phi}.$$

The arguments of the normal distribution function can easily be rewritten as in the second part of (4). $\square$