Probability and Statistics

Fall 2000

LECTURE 2: CONDITIONAL PROBABILITY AND INDEPENDENCE (CB 1.3)

The second fundamental problem of probability theory deals with the updating of probabilities of events when new information comes in.

Example: You draw a two cards out of a deck of 52 and someone tells you that one of them is an ace. What is the probability that exactly one of them is an ace. Before the information came in, the marginal or unconditional probability of the event of exactly one ace is

$$\frac{\binom{48}{1}\binom{4}{1}}{\binom{52}{2}} = \frac{32}{221}.$$

□

Definition 1 *The conditional probability of an event E_2 given an event E_1 is*

$$Pr(E_2|E_1) = \frac{Pr(E_1 \cap E_2)}{Pr(E_1)},$$

provided $Pr(E_1) > 0$.

Example (ctd): The numerator, the probability of exactly one ace in combination with at least one ace, the combined event, is 32/221. The probability of the conditioning event, at least one ace is the sum of the probabilities of one ace and two aces. The latter is 1/221, so the probability of the conditioning event is 33/221. (Alternatively the probability of at least one ace is 1 minus the probability of no aces which is $1 - (48/52) \cdot (47/51)$). Then the conditional probability is the ratio (32/221)/(33/221)=32/33. □

Example: A simple example is that of two coin tosses. What is the probability of two heads given that you have at least one head in the two tosses. $E_1 = \{HH, TH, HT\}$, with probability 3/4, $E_2 = \{HH\}$, so $E_1 \cap E_2 = \{HH\}$ with probability 1/4, so

$$Pr(E_2|E_1) = \frac{Pr(E_1 \cap E_2)}{Pr(E_1)} = \frac{1/4}{3/4} = \frac{1}{3},$$

not 1/2 as many people think at first.

Example: A famous example of the type of difficulty many people have in understanding conditional probabilities is the game show problem. You are a contestant in a game show and have to choose one of three doors. Behind one of the doors is a prize; the other doors are empty. After you choose a door, the gameshow host opens one of the other two doors and shows you there is no prize behind that door. The host then offers you the opportunity to switch from the door you chose to the third door. Should you switch?

To solve this problem we first have to remove some of the ambiguities in the description above. We assume that the door behind which the prize is located is choosen randomly, with probability 1/3 for each door. More importantly, we assume that the host will always open one of the doors not chosen by the contestant and not containing the prize. If there is a choice for the host, for example if you choose door a and the prize is in fact behind door a, the host will choose one of the eligible doors (b or c) randomly. We also establish some notation. Let P be the door with the prize, Y the door you choose, and H the door the host opens, with $P, Y, H \in \{a, b, c\}$. The information then can be formulated as

$$Pr(P = a|Y) = Pr(P = b|Y) = Pr(P = c|Y) = 1/3,$$

$$Pr(H = Y) = Pr(H = P) = 0,$$

$$Pr(H = h|P, Y) = 1/2, \text{for all } h \in \{a, b, c|P \neq h, Y \neq h\}, \text{ if } Y = P,$$

$$Pr(H = h|P, Y) = 1, \text{for all } h \in \{a, b, c|P \neq h, Y \neq h\}, \text{ if } Y \neq P.$$

The question is, given the door you choose, say door a, and given that the host reveals that door b is empty whether the probability that the prize is in c is higher or lower than the probability that the prize is in a:

$$Pr(P = a|Y = a, H = b) >< Pr(P = c|Y = a, H = b).$$

By symmetry this relation is obviously the same as

$$Pr(P = a|Y = a, H = c) >< Pr(P = b|Y = a, H = c).$$

Another way of asking the question is whether the information that the host opens b is relevant. If it is not relevant (implied by no benefit from switching) then the following equality should hold:

$$Pr(P = a|Y = a, H = b) = Pr(P = a|Y = a).$$

Let us calculate the two probabilities for the last relation. First, $Pr(P = a|Y = a)$ is clearly equal to 1/3. The probability

$$Pr(P = a|Y = a, H = b) = \frac{Pr(P = a, H = b|Y = a)}{Pr(H = b|Y = a)}$$

$$= \frac{Pr(H = b|Y = a, P = a) \cdot Pr(P = a|Y = a)}{Pr(H = b, P = a|Y = a) + Pr(H = b, P = b|Y = a) + Pr(H = h, P = c|Y = a)}$$

The numerator is equal to $(1/2) \cdot (1/3) = 1/6$. The first term in the denominator is also 1/6, the second is zero because the host never opens the door with the prize, and the third is

$$Pr(H = b, P = c|Y = a)$$

$$= Pr(H = b|P = c, Y = a) \cdot Pr(P = c|Y = a) = 1 \cdot (1/3) = 1/3.$$

Hence the denominator is 3/6, and the conditional probability is 1/3. Hence probability that you win if you don't switch is 1/3, and the probability that you win if you switch is 2/3: you should always switch. □

Example: Another typical example is that of a diagnostic test for diseases. Suppose 1 in 10,000 people in the population have a particular disease. A test exists with the following properties. If you have the disease and get tested the test will come out positive 99% of the time and negative (a "false" negative) 1% of the time. If you do not have the disease, the test will come out positive (a "false" positive) 5% of the time, and negative 95% of the time. What is the probability that a randomly choosen person from the population who tests positive actually has the disease?

We are interested in the probability

$$Pr(D|P) = \frac{Pr(D,P)}{Pr(P)} = \frac{Pr(P|D)Pr(D)}{Pr(P,D) + Pr(P,D^c)} = \frac{Pr(P|D)Pr(D)}{Pr(P|D)Pr(D) + Pr(P|D^c)Pr(D^c)}$$

$$\frac{(99/100)(1/10,000)}{(99/100)(1/10,000) + (5/100)(9,999/10,000)} = \frac{99}{99 + 49,995} \approx .002.$$

Even though the test seems very good, giving the correct result at least 95% of the time, the probability of actually having the disease once you test positive is still very small since it is so small to begin with (0.0001). In fact, it has gone up by a factor 20, because the test gives the wrong answer in only one in twenty cases for healthy people. □

Note that in the example we were given conditional probabilities in one direction (test results given health status) but were interested in conditional probabilities in the other direction (health status given test results). A general result concerning this type of calculation, is the following, referred to as Bayes' theorem:

Result 1 *If $E_1, E_2, \ldots, E_k$ form a partition of Ω, then*

$$Pr(E_j|E) = \frac{Pr(E_j) \cdot Pr(E|E_j)}{\sum_{i=1}^{k} Pr(E_i) \cdot Pr(E|E_i)}.$$

Calculations involving conditional probabilities can be greatly simplified if a particular relation holds:

Definition 2 *Two events E_1 and E_2 are independent if*

$$Pr(E_1 \cap E_2) = Pr(E_1) \cdot Pr(E_2).$$

If two events E_1 and E_2 are independent and $Pr(E_1) > 0$, then $Pr(E_2|E_1) = Pr(E_2)$. The condition that $Pr(E_1) > 0$ is important, however. Note that the empty set $\emptyset$ is independent of any event because both lefthand side and righthand side probabilities are equal to zero.

Definition 3 *Three events E_1, E_2 and E_3 are jointly independent if :*

1. (a) *E_1 and E_2 are independent,*
   (b) *E_1 and E_3 are independent,*
   (c) *E_2 and E_3 are independent.*
2. $$Pr(E_1 \cap E_2 \cap E_3) = Pr(E_1) \cdot Pr(E_2) \cdot Pr(E_3).$$

Similarly joint independence of four events requires that all combinations of three events are jointly independent as well as that the probability of the intersection is equal to the product of the probabilities.

To see why both conditions 1 and 2 are necessary in the three event definition of independence consider the case where $E_1 = E_2$ and $E_3 = \emptyset$. In that case condition 2 would be satisfied but no-one would want to call the three events independent. To see that condition 2 is necessary consider a roulette wheel with 8 numbers. Event E_1 is an odd number, E_2 is a number (strictly) less than 5, and E_3 is a number in the set $\{1, 3, 6, 8\}$. In that case E_1 is independent of E_2, E_1 is independent of E_3, and E_2 is independent of E_3, but the probability of the intersection is not equal to the product of the three marginal probabilities:

$$Pr(E_1) \cdot Pr(E_2) \cdot Pr(E_3) = 1/8 \neq 1/4 = Pr(E_1 \cap E_2 \cap E_3),$$

and again this does not correspond to the intuitive notion of independence, and thus the three events are not jointly independent.