Solutions to Econ 686 Assignment Three

by

Gordon R. FIsher*

Concordia University

School of Graduate Studies

Department of Economics

Question 1

$L_n(x;\pi) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$ so the log of the function corresponding to one observation will be

$$\frac{1}{n}\log L_n(x;\pi) = c + \frac{1}{n}x\log\pi + \frac{n-x}{n}\log(1-\pi).$$

The derivative is

$$\begin{aligned}\frac{d\frac{1}{n}\log L_n(x;\pi)}{d\pi} &= \frac{x}{n}\cdot\frac{1}{\pi} + \frac{n-x}{n}\cdot\frac{(-1)}{(1-\pi)} = 0\\ &= \frac{x-n\pi}{n\pi(1-\pi)}\end{aligned}$$

At the ML estimate $\hat{\pi}$ of π

$$\frac{x-n\hat{\pi}}{n\hat{\pi}(1-\hat{\pi})} = 0$$

*GRF/ms

implies $\hat{\pi} = \frac{x}{n}$.

Question 2

$$\frac{d^2 \frac{1}{n} \log L_n(x; \pi)}{d\pi^2} = \frac{n\pi(1-\pi)(-n) - (x - n\pi)(n - 2n\pi)}{[n\pi(1-\pi)]^2}$$

Therefore

$$\begin{aligned} E\left[-\frac{1}{n}\frac{d^2 \log L_n(x; \pi)}{d\pi^2}\right] &= E\left[-\frac{-n^2\pi^2 - xn + 2xn\pi}{[n\pi(1-\pi)]^2}\right] \\ &= \frac{E[n^2\pi^2 + xn - 2nx\pi]}{[n\pi(1-\pi)]^2} \\ &= \frac{n^2\pi^2 + n^2\pi - 2n^2\pi}{[n\pi(1-\pi)]^2} \\ &= \frac{1}{\pi(1-\pi)} \\ &= B_\pi \end{aligned}$$

Now $\sqrt{n}(\hat{\pi} - \pi) \sim N(0, B_\pi^{-1}) \Rightarrow \hat{\pi} \sim N\left(\pi; \frac{1}{n} B_\pi^{-1}\right)$. Hence $\frac{1}{n}B_{\hat{\pi}}^{-1} = \frac{\hat{\pi}(1-\hat{\pi})}{n}$. It follows that

$$\begin{aligned} W &= \frac{n(\hat{\pi} - \theta)^2}{\hat{\pi}(1-\hat{\pi})} \\ &= \frac{n\left(\frac{x}{n} - \theta\right)^2}{\hat{\pi}(1-\hat{\pi})} \\ &= \frac{(x - n\theta)^2}{n\hat{\pi}(1-\hat{\pi})}. \end{aligned}$$

Question 3

$$L = \frac{1}{n}\log L_n(x; \pi) - \lambda(\pi - \theta)$$

The first order conditions are

$$\frac{\partial L}{\partial \pi} = \frac{x - n\pi}{n\pi(1-\pi)} - \tilde{\lambda} = 0 \Rightarrow \tilde{\lambda} = \frac{x - n\pi}{n\pi(1-\pi)}$$

$$\frac{\partial L}{\partial \lambda} = \pi - \theta = 0 \Rightarrow \pi = \theta.$$

Thus $\tilde{\lambda} = \frac{x - n\theta}{n\theta(1-\theta)}$. Now $\sqrt{n} s_\pi \sim N(0, B_\pi)$ where $s_\pi = \frac{d}{d\pi}\frac{1}{n}\log L_n(x;\pi)$. Thus $s_\pi \sim N\left(0, \frac{1}{n}B_\pi\right)$. Now $\tilde{\lambda}H = s_\pi$ where $H = \frac{dh(\pi)}{d\pi} = \frac{d}{d\pi}(\pi - \theta) = 1$. Thus $\tilde{\lambda} \sim N\left(0, \frac{1}{n}B_\pi\right)$.

Hence

$$\begin{aligned} LM &= \frac{\tilde{\lambda}^2}{V\left(\tilde{\lambda}\right)} \\ &= \frac{\left[\frac{x - n\theta}{n\theta(1-\theta)}\right]^2}{\frac{1}{n}B_\pi} \\ &= \frac{\frac{(x - n\theta)^2}{[n\theta(1-\theta)]^2}}{n\theta(1-\theta)} \\ &= \frac{(x - n\theta)^2}{n\theta(1-\theta)}. \end{aligned}$$