Solutions to Econ 686 Assignment Two

by

Gordon R. Fisher*

Concordia University

School of Graduate Studies

Department of Economics

Question 1

$\hat{\beta}_i \pm SE\left(\hat{\beta}_i\right) \qquad i = 1, 2, \cdots, 9$

$s^2 \left(X^\top X\right)^{-1}$

R^2

Question 2

$y^\top y = y^\top P_0 y + y^\top (P - P_0) y + y^\top (I_n - P) y$ whereupon

1. $F = \frac{y^\top (P - P_0) y}{y^\top (I_n - P) y} \cdot \frac{n-k}{r} \qquad k = 9;\ r = 1, 2, \cdots, 8$. To test $B^\top \beta = 0$, $B^\top$ is $(r \times k)$ of rank r.

*GRF/ms

P - P_0 is the op onto $L_0^{\perp} \cap L$, $L_0 = N\left[A'\right] \cap L$, i.e. onto

$$
\begin{aligned}
& \left[L \cap N\left[A'\right]\right]^{\perp} \cap L \\
= & \left[L^{\perp} + R[A]\right] \cap L \\
= & \left[L^{\perp} + PR[A] + (I_n - P)R[A]\right] \cap L \\
= & \left[L^{\perp} \oplus PR[A]\right] \cap L \\
= & R[PA].
\end{aligned}
$$

Assume that $R[A] \cap L^{\perp} = \emptyset$. Then PA has rank r and

$$P - P_0 = PA\left(A'PA\right)^{-1}A'P.$$

But $A = QX\left(X'QX\right)^{-1}B$ for some arbitrary pd matrix. Q $\Rightarrow$ $PA = X\left(X'X\right)^{-1}B$ $\Rightarrow$ $P\text{-}P_0 = X\left(X'X\right)B\left[B'\left(X'X\right)^{-1}B\right]^{-1}B'\left(X'X\right)^{-1}X'$. Hence $y'(P\text{-}P_0)y = \hat\beta' B\left[B'\left(X'X\right)^{-1}B\right]^{-1}B'\hat\beta$ and

2.

$$F = \frac{\left(B'\hat\beta\right)'\left[\hat D\left(B'\hat\beta\right)\right]^{-1}B'\hat\beta}{r}$$

where $\hat D\left(B'\hat\beta\right) = B'\left(X'X\right)^{-1}Bs^2$, $s^2 = \frac{y'(I_n - P)y}{n-k}$. Finally, (1) can be written

3.

$$\frac{y'(I_n - P_0)y - y'(I_n - P)y}{y'(I_n - P)y}\cdot\frac{n-k}{r} = \frac{\hat\varepsilon_0'\hat\varepsilon_0 - \hat\varepsilon'\hat\varepsilon}{\hat\varepsilon'\hat\varepsilon}\cdot\frac{n-k}{r}.$$

Note that (2) is an estimated standardized quadratic form in

4. $B'\hat{\beta} = B'\left(X'X\right)^{-1}X'y \sim N\left(B'\beta; B'\left(X'X\right)^{-1}\sigma^2 B\right)$

For any vector-valued parameter θ, $H_0 : h(\theta) = 0$ is a null hypothesis of r elements on $\theta \in \Theta \subset R^k$. If $\hat{\theta}$ is the MLE of θ and $\sqrt{n}\left(\hat{\theta} - \theta\right) \sim N(0; \Sigma)$ for Σ p.d., then subject to sufficient regularity $\sqrt{n}h\left(\hat{\theta}\right) \sim N\left(0; H'\Sigma H\right)$ on H_0 $H = \frac{\partial h'(\theta)}{\partial \theta'}$. On $H_a : h(\theta) \neq 0$, $\sqrt{n}h\left(\hat{\theta}\right) \sim N\left(h(\theta); H'\Sigma H\right)$. Hence

$$W = \sqrt{n}h\left(\hat{\theta}\right)\left[H'\Sigma H\right]^{-1} h\left(\hat{\theta}\right)\sqrt{n} \sim \chi^2(r; \delta)$$

$\delta = 0$ on H_0 and W is the large-sample Wald test. Clearly (4) follows the distribution of $h\left(\hat{\theta}\right) = B'\hat{\beta}$. Thus in the case of $B'\beta = 0$,

$$W = n\left(B'\hat{\beta}\right)'\left[B'\left(\frac{X'X}{n}\right)^{-1}B\sigma^2\right]^{-1} B'\hat{\beta} \sim \chi^2(r; \delta):$$

But this may be converted into a small-sample specialization

$$W^* = \frac{\left(B'\hat{\beta}\right)'\left[B'\left(X'X\right)^{-1}s^2 B\right]^{-1} B'\hat{\beta}}{r} \sim F(r; n - k):$$

The LM statistic is based on the estimated Lagrange Multiplier λ in the Lagrangian expression

$$L^* = \|y - \mu\|^2 - 2\mu'A\lambda \qquad \mu \in L$$

selecting $\mu = \tilde{\mu}$ and $\lambda = \tilde{\lambda}$ when L^* is at a stationary point and $\|y - \mu\|^2$ is smaller with $A'\mu = 0$. On this case

$$\tilde{\lambda} = \left(A'PA\right) A'Py$$

$$= \left[B'\left(X'X\right)^{-1}B\right]^{-1}B'\left(X'X\right)^{-1}X'y$$
$$\left[B'\left(X'X\right)^{-1}B\right]^{-1}B'\hat{\beta}$$

$$D\left[\tilde{\lambda}\right]$$
$$= \left[B'\left(X'X\right)^{-1}B\right]^{-1}B'\left(X'X\right)^{-1}\left(X'X\sigma^2\right)\left(X'X\right)^{-1}B\left[B'\left(X'X\right)^{-1}B\right]^{-1}$$
$$= \left[B'\left(X'X\right)^{-1}B\right]^{-1}\sigma^2$$

Hence $\tilde{\lambda} \sim N\left(0; D\left[\tilde{\lambda}\right]\right)$ on H_0 and $N\left(\lambda; D\left[\tilde{\lambda}\right]\right)$ otherwise, and

$$\tilde{\lambda}'\left(D\left[\tilde{\lambda}\right]\right)^{-1}\tilde{\lambda} \sim \chi^1(r;\delta) \qquad \delta = 0 \text{ on } H_0.$$

However

5. $LM = \tilde{\lambda}'\left[D\left(\tilde{\lambda}\right)\right]^{-1}\tilde{\lambda} = y'X\left(X'X\right)^{-1}B\left[B'\left(X'X\right)^{-1}Bs_0^2\right]^{-1}B'\left(X'X\right)^{-1}X'y =$ $\frac{1}{n-k+r}\cdot\frac{y'(P-P_0)y}{y'(I_n-P_0)y} = \frac{y'(P-P_0)y}{s_0^2}$ and

6. $M \quad \frac{y'(P-P_0)y}{y'(I_n-P_0)y} = \frac{1}{n-k+r}\tilde{\lambda}'\left[D\left(\tilde{\lambda}\right)\right]^{-1}\tilde{\lambda} \sim \beta\left(\frac{r}{2};\frac{n-k}{2}\right)$. $LM \sim \chi^2(r)$ when n is 'large' but (6) is exact for any n. Thus the difference between F and M is that s^2 is replaced by the estimate of σ^2 under H_0 or s_0^2 and then

$$\frac{1}{n-k+r}LM = M \sim \beta\left(\frac{r}{2};\frac{n-k}{2}\right)$$

Question 3

H_{01} or $\beta_3 = 0$ is tested with a t-test

$$\frac{\hat{\beta}}{SE\hat{\beta}_3} \sim t(n-9)$$

$H_{02}: 7\beta_6 - \beta_8$ is tested also with a t-test.

$$VAR\left(7\hat{\beta}_6 - \hat{\beta}_8\right) = s^2\left[\left(7m^{66} - m^{86}\right)7 - \left(7m^{68} - m^{88}\right)\right]$$

where $m^{ij} = \left(X^\top X\right)^{-1}_{ij}$; $i, j = 1, 2, \cdots, 9$.

$$t = \frac{7\hat{\beta}_6 - \hat{\beta}_8}{SE\left(7\hat{\beta}_6 - \hat{\beta}_8\right)};$$

$SE\left(7\hat{\beta}_6 - \hat{\beta}_8\right) = \sqrt{VAR\left(7\hat{\beta}_6 - \hat{\beta}_8\right)}$ and $t \sim t(n-9)$

H_{03}

$$\begin{aligned} \beta_2 + \beta_4 &= 0 \\ 7\beta_6 - \beta 8 &= 0 \\ 2\beta_7 + \beta_9 &= 0 \end{aligned}$$

implies

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix} \beta = B^\top \beta = 0.$$

Clearly this requires an $F(3; n-9)$-test. Formula (3) above is probably the easiest to apply.

Question 4

H_{01} Set $\alpha_1 = \beta_1$ $\alpha_2 = \beta_2$ $\alpha_3 = \beta_4$ $\cdots$ $\alpha_8 = \beta_9$. Then M is (9×8);

$$B^{\top} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\mu = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}'$$

$$= M\alpha$$

$$H_{02} B^{>} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 7 & 0 & -1 & 0 \end{bmatrix}$$ and M is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{03}B^{\top} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$B^{\top}M = 0$