

Unique Proof

Theorem: Let $S(n) = 1^x + 2^x + 3^x + \dots + n^x$ where $x \geq 0$ and x be from the integer family. Then $S(n)$ can be written as a polynomial $f(n)$ with its highest power $x+1$ and this polynomial can be deduced in a very unique way.

Let $S(n) = 1^x + 2^x + 3^x + \dots + n^x$

Then $S(n)$ can be written as a polynomial $f(n)$ if we can find such a polynomial such that

$$S(n+1) = S(n) + (n+1)^x = f(n) + (n+1)^x$$

If this can be done, then this polynomial $f(n)$ is a solution.

This can be done for all x such that $x \geq 0$ and x integer.

We can find such a polynomial if we let $S(n) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots$

$$\text{Then } S(n+1) = a(n+1)^{(x+1)} + b(n+1)^x + c(n+1)^{(x-1)} + \dots$$

$$\text{So } S(n+1) = an^{(x+1)} + bn^x + cn^{(x-1)} + \dots + a(n+1)n^x + \dots$$

$$\text{So } S(n+1) = S(n) + (\text{a polynomial with its highest power } x) = S(n) + g(n)$$

This polynomial $g(n)$ is then equal to $(n+1)^x$ so that values can be worked out for $a, b, c, \dots$

Consider the case for $x=2$

$$\text{In our case } S(n) = 1 + 2^2 + 3^2 + \dots + n^2$$

$$\text{Let } S(n) = an^3 + bn^2 + cn + d$$

Now if we can find values for a, b, c, d such that $S(n+1) = S(n) + (n+1)^2$ then we are done.

$$\text{Then } S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)$$

$$\text{So } S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) + [3an^2 + n(3a+2b) + (a+b+c)]$$

$$\text{Therefore } [3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)$$

$$\text{So } 3a = 1 \text{ and } a = 1/3$$

So $3a+2b=2$, $1+2b=2$, $2b=1$, $b=1/2$

So $a+b+c = 1$, $1/3 + 1/2 + c = 1$, $c=1/6$

Therefore $S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d$

But $S(1) = 1 = (1/3) + (1/2) + (1/6) + d$ and therefore $d=0$

So $S(n) = f(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)$

Case proven.

The above procedure can be followed for any power. There is no need to know the previous solution to obtain the next as in the Calculus approach.
