Quartic Equation

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A unique Solution assuming Complex roots

The general Quartic is given by

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

As in the third order polynomial we are first going to reduce the equation.

Dividing by A we therefore solve for

$x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are all

members of the real numbers.

6.1 Reducing the fourth order polynomial

$x^4 + ax^3 + bx^2 + cx + d = 0$ eq1

Let $x = x_1 + m$

Then $x^2 = x_1^2 + 2mx_1 + m^2$

and $x^3 = x_1^3 + 3mx_1^2 + 3x_1m^2 + m^3$

and $x^4 = x_1^4 + 4mx_1^3 + 6m^2.x_1^2 + 4\ x_1.m^3 + m^4$

Substitute the above into eq1 and we get

$x_1^4 + x_1^3.(4m + a) + x_1^2(6m^2 + 3am + b) + x_1(4m^3 + 3am^2 + 2bm$

$+ c) + (m^4 + am^3 + bm^2 + cm + d) = 0$ eq2

We eliminate the third order term by

Letting $4m + a = 0$ and thus $m = -a/4$

Eq2 then reduces to $x_1^4 + ex_1^2 + fx_1 + g = 0$

where $e = (6m^2 + 3am + b) = 6a^2/16 - 3a^2/4 + b = -3a^2/8 + b$
eq3

where $f = 4m^3 + 3am^2 + 2bm + c = -a^3/16 + 3a^3/16 -2ab/4 +c$

Thus $f = c - ab/2 +a^3/8$ eq4

and $g = m^4 + am^3 + bm^2 + cm + d$

Thus $g = a^4/256 - a^4/64 + ba^2/16 - ac/4 + d$

resulting in $g = d - ac/4 + ba^2/16 - 3a^4/256$ eq5

Let $x_1 = x_2/q$

Thus $x_2^4 + eq^2.x_2^2 + fx_2.q^3 + gq^4 = 0$

Now we consider two cases, the first for e>0 and the second for e<0.

Case one: e>0

Let $q^2 = 1/e$ eq6

Substitute eq6 and we get

$$x_2^4 + x_2^2 + hx_2 - K = 0 \quad \text{eq7}$$

where $h = fq^3 = \dfrac{f}{\sqrt{eee}}$ eq8

and $K = -g/e^2$ eq9

We therefore have reduced our original equation to the form as in eq7 with h and K both real. If we can get an exact solution for this equation then it will be straight forward to get x by working backwards.

End of case one

Case two: e<0

Let q^2 =-1/e , so q = 1/sqrt(|e|)

sqrt means square root of , and i = sqrt(-1)

and |e| = positive value of e

and the equation reduces to $x_2^4 - x_2^2 + hx_2 - K = 0$

where $K = -gq^4 = -g/e^2$

and $h = fq^3 = f/\sqrt{|e^3|}$

End case 2

So we now have depending on which case a quadratic term which is either unity positive or unity negative and we have both h and K real numbers.

6.2 Roots of the fourth order polynomial

We shall assume that as in the case of the cubic equation, that the root will consists of two independent terms and that each will be a root of the fourth order.

We have reduced the problem to the following equation

$x_2^4 + x_2^2 + hx_2 - K = 0$ with h and K both real for case 1

And to $x_2^4 - x_2^2 + hx_2 - K = 0$ for case 2

Now we just rewrite these equations so that it reads easier.

So x_2 becomes x , h becomes K_1 and $-K$ becomes K_2

We will consider the two cases separately

We will however see in the end that the method we use for the two cases is exactly the same

Case one

Let us then examine $x^4 + x^2 + K_1 x + K_2 = 0$ eq1 for case 1

We shall write

0.25 as .25
0.5 as .5
0.75 as .75
0.25 as .25

Let us assume a root $x = z_1^{.25} - iz_2^{.25}$ eq2

where $i = \sqrt{-1}$

Therefore $x^2 = z_1^{.5} - 2iz_1^{.25} z_2^{.25} - z_2^{.5}$ eq3

and $x^4 = z_1 - 4iz_1^{.75} z_2^{.25} - 6z_1^{.5} z_2^{.5} + 4iz_1^{.25} z_2^{.75} + z_2$ eq4

We know that for the equation to be 0, that the imaginary and the real part of the equation have to be 0.

We substitute eq's 2,3 and 4 into eq1 and write the real and imaginary parts separately as two independent equations.

Real part : $z_1 - 6z_1^{.5} z_2^{.5} + z_2 + z_1^{.5} - z_2^{.5} + K_1 z_1^{.25} + K_2 = 0$ eq5

For the imaginary part

$- K_1 z_2^{.25} - 2z_1^{.25} z_2^{.25} - 4z_1^{.75} z_2^{.25} + 4z_1^{.25} z_2^{.75} = 0$ eq6 divide eq6 with $z_2^{.25}$ because it is a factor of all the terms

then $-K_1 - 2z_1^{.25} - 4z_1^{.75} + 4z_1^{.25} z_2^{.5} = 0$ eq7

multiply eq7 with $z_1^{.25}$

then $-4z_1^{.5} z_2^{.5} = -K_1 z_1^{.25} - 2z_1^{.5} - 4z_1$ eq8

divide eq7 with $z_1^{.25}$

then $-K_1 / z_1^{.25} - 2 \; -4z_1^{.5} + 4z_2^{.5} = 0$

and $-K_1 / 4z_1^{.25} - 1/2 = z_1^{.5} - z_2^{.5}$ = delta eq9

Out of eq5 we get

$((z_1^{.5} - z_2^{.5}))^2 - 4z_1^{.5} z_2^{.5} + (z_1^{.5} - z_2^{.5}) + Kz_{11}^{.25} + K_2 = 0$ eq10

Substitute eq8 into eq10 and also substituting delta we get

$delta^2 - 2z_1^{.5} - 4z_1 + delta + K_2 = 0$ eq11

Substitute eq9 into eq11 and we get

$$\frac{K_1^2}{16z_1^{.5}} + \frac{K_1}{4z_1^{.25}} + 1/4 - 2z_1^{.5} - 4z_1 - \frac{K_1}{4z_1^{.25}} - 1/2 + K_2 = 0 \quad \text{eq12}$$

multiply eq12 with $z_1^{.5}$

then $$K_1^2/16 - .25z_1^{.5} - 2z_1 - 4z_1^{1.5} + K_2z_1^{.5} = 0 \quad \text{eq13}$$

Let $z_1 = v^2$ eq16 and substitute into eq13

then $$-4v^3 - 2v^2 + v(K_2 - .25) + K_1^2/16 = 0 \quad \text{eq14}$$

thus $$v^3 + v^2/2 + v(1/16 - K_2/4) - K_1^2/64 = 0 \quad \text{eq15}$$

We thus have a equation in v that we can solve.

Then we can get z_1 from eq16.

And from eq9 we can get z_2

and from eq2 we can get x.

We know that complex roots always have a conjugate partner and we can therefore write the other root as

$$x = z_1^{.25} + iz_2^{.25}$$

What we have done here is to demonstrate that there is an exact method by which we can determine the roots of the fourth order polynomial for case 1.

We will now do an example to demonstrate

6.3 Example

Let's look at $x^4 + x^2 + 2x - 24 = 0$

Through inspection we know that $x = 2$

and that $K_1 = 2$ and $K_2 = -24$

If we take the factor x-2 out of the equation we are left with

$x^3 + 2x^2 + 5x + 12 = 0$ which have a real root at

x = -2,2029812583

The quadratic term that's left delivers then

x = 0,10149062915 +- i 2,3317082922 as complex roots

We have done all the above to check our method that we are going to use.

Let us now go on and use eq15 to evaluate v.

Substituting we get

$v^3 + 0,5v^2 + v(1/16 +24/4) - 4/64 = 0$

Therefore $v^3 + 0,5v^2 + 6,0625v - 0,0625 = 0$

We get a real root at v = 0,010300347807

Therefore $z_1 = v^2 = 0,000106097$

and $z_1^{.25} = 0,101490625$

We also know $z_1^{.5} - z_2^{.5} = \frac{-2}{4z_1^{.25}} - 0,5$

Therefore $z_2^{.25} = 2,331708333$

The complex root is therefore

x = 0,101490625 +- i 2,331708333

which is the root we got through inspection.

Now we examine case 2

Let us then examine $x^4 - x^2 + K_1 x + K_2 = 0$ eq1 for case 2

We shall write

0.25 as .25
0.5 as .5
0.75 as .75
0.25 as .25

Let us assume a root $x = z_1^{.25} - iz_2^{.25}$ eq2

where $i = \sqrt{-1}$

Therefore $x^2 = z_1^{.5} - 2iz_1^{.25} z_2^{.25} - z_2^{.5}$ eq3 and $x^4 = z_1 - 4iz_1^{.75} z_2^{.25} - 6z_1^{.5} z_2^{.5} + 4iz_1^{.25} z_2^{.75} + z_2$ eq4

We know that for the equation to be 0, that the imaginary and the real part of the equation have to be 0.

We substitute eq's 2,3 and 4 into eq1 and write the real and imaginary parts separately as two independent equations.

Real part : $z_1 - 6z_1^{.5} z_2^{.5} + z_2 - z_1 + z_2 + K_1 z_1^{.25} + K_2 = 0$ eq5

For the imaginary part

$- K_1 z_2^{.25} + 2z_1^{.25} z_2^{.25} - 4z_1^{.75} z_2^{.25} + 4z_1^{.25} z_2^{.75} = 0$ eq6

divide eq6 with $z_2^{.25}$ because it is a factor of all the terms

then $-K_1 + 2z_1^{.25} - 4z_1^{.75} + 4z_1^{.25} z_2^{.5} = 0$ eq7

multiply eq7 with $z_1^{.25}$

then $-4z_1^{.5} z_2^{.5} = -K_1 z_1^{.25} + 2z_1^{.5} - 4z_1$ eq8

divide eq7 with $z_1^{.25}$

then $-K_1 /z_1^{.25} + 2 -4z_1^{.5} + 4z_2^{.5} = 0$

and $-K_1/4z_1^{.25} + 1/2 = z_1^{.5} - z_2^{.5} = delta$ eq9

Out of eq5 we get

$$((z_1^{.5} - z_2^{.5}))^2 - 4z_1^{.5} z_2^{.5} - (z_1^{.5} - z_2^{.5}) + Kz_{11}^{.25} + K_2 = 0 \quad eq10$$

Substitute eq8 into eq10 and also substituting delta we get

$$delta^2 + 2z_1^{.5} - 4z_1 - delta + K_2 = 0 \quad eq11$$

Substitute eq9 into eq11 and we get

$$\frac{K_1^2}{16z_1^{.5}} - \frac{K_1}{4z_1^{.25}} + 1/4 + 2z_1^{.5} - 4z_1 + \frac{K_1}{4z_1^{.25}} - 1/2 + K_2 = 0 \quad eq12$$

multiply eq12 with $z_1^{.5}$

then $K_1^2/16 - .25z_1^{.5} + 2z_1 - 4z_1^{1.5} + Kz_{21}^{.5} = 0$ eq13

Let $z_1 = v^2$ eq16 and substitute into eq13

then $-4v^3 + 2v^2 + v(K_2 - .25) + K_1^2/16 = 0$ eq14 thus $v^3 - v^2/2 + v(1/16 - K_2/4) - K_1^2/64 = 0$ eq15

We thus have an equation in v that we can solve.

Then we can get z_1 from eq16.

And from eq9 we can get z_2

and from eq2 we can get x.

We know that complex roots always have a conjugate partner and we can therefore write the other root as

$$x = z_1^{.25} + iz_2^{.25}$$

We have seen therefore that the path we follow for case 2 is exactly the same as for case one and this should not come as a surprise.

We have therefore shown that an exact solution does exist for the quartic with real coefficients.