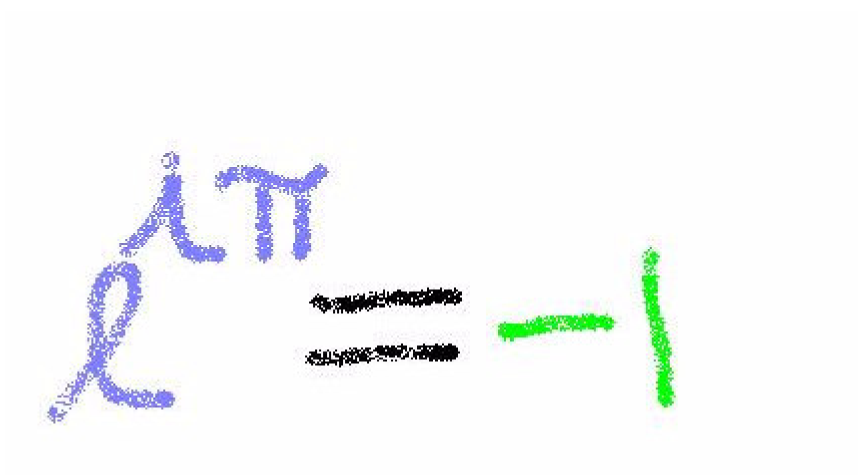
Mathematics

A simplistic approach to

Classical and other interesting

Proofs and problems

CH VD Westhuizen



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Let ^ suffice for "to the power of" , so 3^2 = 9

Let 2.3.4 or 2*3*4 be the same as 2 times 3 times 4 and let 2,4 be 2 and 4 tenths

3.1 Prove that every prime number except 2 can be written as the difference of two squares.

3.2
Proof 2 : Prove that $Y = 2^{(2^n - 1)}$ divides $Z = (2^n)!$

3.3 Prove that the list of prime numbers is infinite.

3.4 Prove that given $Z = 2.3.5.7\ldots p_j \ldots\ldots p_k$ that $p_k < \sqrt{Z}$

where Z is the product of all primes from 2 up to p_k and $p_k > 3$

3.5 Prove that given a^b - b^a = 10^x where a=1000 , b=1001 that x = 3002.998818

3.6 Prove that if $2^n - 1$ is prime ,then so is n for n>0.

3.7 Prove that $2^{32} + 1$ is not prime by showing that it has a factor 641. This is a famous proof by Euler.

3.8 Prove that $\sqrt{2}$ is not rational.

3.9 Prove that there exists infinite integer solutions for $x^2 + y^3 = z^6$

3.10 Prove that 1 + 2 + 3 + .. + n = n(n+1)/2

3.11 Prove that if a > 0 then that a + 1/a >= 2

3.12 Prove that $a^2 + b^2 >= 2ab$

3.13 Prove that $(a+b)/2 >= \sqrt{ab}$

3.14 Prove that if n is an integer number that either n^2 or $n^2 - 1$ has four as a factor?

3.15 Prove that a+1 always divides $a^u + 1$ if u is uneven.

3.16 Prove that p if prime always divide $z = a^p - a$

3.17 Prove that if p prime and bigger than 3 that p+1 divides p!

3.18 Prove that $a+ar+arr+arrr+ar^4+\ldots+ar^n = a\dfrac{r^{n+1}-1}{r-1}$

3.19 Prove that 1+2^n < 3^n for n>=2

3.20 Prove that there exist irrational numbers a , b such that a^b is rational.

3.21 Prove that aa + bb is not equal to 4n-1 where a, b and n are integers.

3.22 Prove that every integer > 1 is a prime or the product of primes.

3.23 Prove that 4n+1 can be written as aa-bb for n>0.

3.24 Prove that 8 divides n.n-1 for n odd

3.25 Prove that –1 times –1 is 1 given that 1-1=0 and that -(1) = –1 and that 1 times 0 = 0

3.26 Prove that *e*^π > π^*e* without a calculator

3.27 Prove that a-1 divides a^n - 1

3.28 Prove that p if prime always divide the binomial coefficient $\binom{p}{r}$ for 0<r<p

3.29 Prove that (1^3 + 2^3 + 3^3+….+n^3) = (1+2+3+…+n)^2

3.30 Prove that 6 divides n(n+1)(n+2)

3.31 Prove that $\sqrt{3}$ is irrational.

3.32 Prove that 1+3+5+7+….(2n-1) = n^2

3.33 Prove that 1/2 + 1/4 + 1/8 + 1/16 +…+1/(2^n) = 1 as n tends to infinity

3.34 Prove that $\sum_{k=0}^{n}$ 1/[(k+1)(k+2)] = 1 as n tends to infinity

3.35 Prove that $\sum_{k=1}^{n}$ 1/(a^k) = 1/(a-1) as n tends to infinity and a>=2

3.36 Prove that $\sum_{n=1}^{\infty}$ 1/n diverges

3.37 Prove that $\sum_{n=1}^{\infty}$ 1/(n^2) converges

3.38 Prove that 7 divides 3^(2n+1) + 2^(n+2)

3.39 Prove that 0,9999999999….. = 1

3.40 Prove that $\lim_{x \to 0}$ [x/sin(x)] = 1

3.41 Prove that all-uneven numbers bigger than zero can be written as the difference of two squares.

3.42 Prove that all numbers, which are multiples of four, can be written as the difference of two squares.

3.43 Prove that any cube can be written as the difference of two squares.

3.44 Prove a solution for x^x^x^x^x^…. = 2

3.45 Prove that $n^{(1/n)} = 1$ as n tends to infinity

3.46 Prove that $(n!)^{(1/n)}$ tends to infinity as n tends to infinity

3.47 Prove that $1^2 + 2^2 + 3^2 + 4^2 + 5^2 +\ldots+ n^2 = n(n+1)(2n+1)/6$

3.48 Prove that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 +\ldots+ n^3 = k^2(k+1)^2/4$

3.49 Prove that 1.2 + 2.3 + 3.4 + 4.5 +…+ n(n+1) = n(n+1)(n+2)/3

3.50 Prove that there exist infinite many solutions for m! = k! n!

3.51 Prove the identity *e*^(ix) = cos(x) + i.sin(x) where i = square root of -1

3.52 Prove that *e*^(π i) = -1

3.53 Prove that *e* is irrational

3.54 Prove that the last two digits of 21^{100} is 01

3.55 Prove an exact formula for sqrt(x + sqrt(x + sqrt(x + sqrt(x + sqrt(x +….)))))) if sqrt stands for square root of.

3.56 Prove an answer for 1 + 1/(1 + 1/(1 + 1/(1 + 1/….))))

3.57 Prove that $a^0 = 1$ given a>0

3.58 Prove that S = 1/2 + 1/3 +1/5 +1/6 + …+1/n is not integer for n>1

3.59 Prove that a complex number to the power of a complex number may be real by proving that i to the power of i is not complex.

3.60 Prove that the list of primes in the form 4n+3 is infinite.

3.61 Prove that if $a^2 + b^2 = c^2$ and a,b,c integers greater than 0, that a and b cannot be both uneven.

3.62 Prove that ab > a+b for a,b integer and a and b bigger than 2

3.63 Prove that there exists infinite integer solutions for $x^2 + y^2 = z^2$

3.64 Prove that 2 = sqrt(1 + sqrt(1 + 2sqrt(1 + 3sqrt(1 + 4sqrt(1+…..)))))) where sqrt stands "for square root of"

3.65 Prove that it is meaningless to divide by 0

3.66 Prove that 0! = 1 given that (n+1)! = (n+1)n!

3.67 Prove that every positive integer n>4 can be written as the sum of at least two consecutive positive integers except for the case n = 2^z

3.68 Prove that pi does not contain itself

3.69 Prove that in a circle of radius r the biggest rectangle enclosed in the circle will have an area of $2r^2$

3.70 Prove that the square root of any prime number is irrational.

3.71 Prove that given a and b integer and a,b<>0 that there is no integer solution for a^2 = b^2 - 2b

3.72 Prove that if m,n positive integers then there is no integer solution to 1/m+1/n+4/mn=3

3.73 Prove that a=b=c=d given that a^2 + b^2 + c^2 + d^2 = ab + bc +cd +da

3.74 Prove that given the general series 1,3,7,15,31…where the nth term is given by

$T_n = 2T_{n-1} + 1$ that in general $T_n = 2^n - 1$

3.75 Prove that the harmonic series 1 + 1/2 + 1/3 +1/4 +.... + 1/n >ln(n+1)

3.76 Prove that 1 + 1/4 + 1/9 + 1/16 +... + 1/n^2 < 2 - 1/n for n>1

3.77 Prove an answer for z = 1/3 + 2/9 + 3/27 + 4/81 +........

3.78 Prove that the product of four non zero consecutive integers cannot be a square

3.79 Prove x algebraically for $x^4 + x^3 + x^2 + x + 1 = 0$

3.80 Prove that it is possible to reduce the general cubic equation $Ax^3 + Bx^2 + Cx + D = 0$

to a reduced equation $u^3 + u + a = 0$ where u = f(x) and a = f(A,B,C,D)

3.81 Prove an exact solution for $x^3 + x + a = 0$

3.82 Prove that $\binom{m}{n}$ is always integer for m >=n

3.83 Prove that if a == b mod (c) and d == e mod (c) ,that (a+d) == (b+e)mod(c) with a, b, c, d, e all integer

3.84 Prove that if a == b mod(c) and d == e mod(c) , that ad == be mod(c) with a, b, c, d, e all integer

3.85 Prove that if a == b mod(c) and d | c that a == b mod(d) given a, b, c, d, e all integer

3.86 Prove that between any two different rational numbers, there is an irrational number

3.87 Proof that for a positive integer n (at least three) each of the numbers n!+2, n!+3,...n!+n is divisible by a prime which doesn't divide any of the others.

3.88 Proof that 5 divides 6^n -1 for n > 0

3.89 Prove that if $x+1$ and $x-1$ are prime and $x > 4$ that 6 divides x.

3.90 Prove that for $n>0$ it is possible to find x,y,z such that $x^2 + y^2 = z^n$

3.91 Prove that 12 divides $(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$ given a,b,c,d are integers

3.92 Prove that if p prime that p is never congruent to b mod(c) if $b \mid c$ and $p \neq b \neq 1$

3.93 Prove a solution in x for $ax^2 + bx + c = 0$

3.94 Prove that the smallest value for $x^2 + y^2 = 36/13$ under the restriction
$3y + 2x = 6$

3.95 Prove that $\ln(i) = (\pi/2)i$

3.96 Prove that the square root of i is a complex number

3.97 Prove in Boolean Algebra that $x'y + x'y' + xy' = x' + y'$

3.98 Prove that the $\ln(-3)$ exists

3.99 Prove that God has a paradoxical property

3.100 Prove that the male barber who shaves all the men who don't shave themselves went crazy

3.101 Prove an integer x such that $x^4 - 2x^3 + 4x^2 - 6x + 3$ is a square

3.102 Prove that the product of n consecutive integers is dividable by n!

3.103 Prove that given Wilson's theorem [p if prime divides $(p-1)! + 1$] that p also divides $(p-2)! - 1$

3.104 Prove if $S(n) = 1^x + 2^x + 3^x + .. + n^x$ where $x \geq 0$ and x be from the integer family. Then S(n) can be written as a polynomial f(n) with its highest power $x+1$ and this polynomial can be deduced in a very unique way.

3.105 Prove that if p prime that p divides $[n^p + (p-1)n]$ for $n \geq 1$

3.106 Prove that the product of two consecutive non zero integers can never be a power.

3.107 Prove that if $a \equiv 1 \mod 3$ and $b \equiv 2 \mod 3$, then that 9 divides $a^3 + b^3$

3.108 Prove that n! can never be equal to x^y for $y>1$ and $n>1$

3.109 Prove that for every integer n there is at least n consecutive non prime numbers

3.110 Prove that if a==b mod c and b == d mod c then a == d mod c

3.111 Prove a solution for n so that 2^1994 + 2^1999 + 2^2000 + 2^2001 + 2^n is square

3.112 Prove that the product of four consecutive integers plus one is always a square

3.113 Prove that 3 divides 10^x + 5 for all x >= 0

3.114 Prove that 1-1+1-1+1-1+1-1+1-1+1........ has no definite answer

3.115 Prove that for n odd and x == 5^n mod 8 that x == 5 mod 8

3.116 Prove that if p prime and p == 5 mod 8 , that p is not congruent to 3 mod 8

3.117 Prove a general solution for $T_{n+1} = 2T_n^2 - 1$ **, given** $T_1 = 2$

3.118 Prove an answer for a^2 - b^2 given a - b = 1 and ab = 1

3.119 Prove that 1/sqrt(1) + 1/sqrt(2) + 1/sqrt(3) +...+ 1/sqrt(n) > sqrt(n) for n>=2

3.120 Prove that sqrt(a) + sqrt(b) > sqrt(a+b) for a,b >=1

3.121 Prove that the $\text{sum}\left[\sum_{r=0}^{n} \binom{n}{r} \right] = 2^n$ **for n > 1**

3.122 Prove that n^5 == n mod 30 for n integer and n > 0

3.123 Prove that every even square is either 0 mod 8 or 4 mod 8

3.124 Prove that numbers such as 8n+7 is never the sum of three squares

3.125 Prove that 7 divides (5^[2n] + 3*2^[2n + 1]) for n >= 1

3.126 Prove that the remainder of 5^1001 divided by 6 is 5

3.127 Prove that x^n + x^-n > x^(n-1) + x^(1-n) for n integer >0 and x>1

3.128 Prove that $\lim_{n \to \infty} (1 + 1/n)^n = e = 2.718281828.....$

3.129 Prove 1/(1.2.3) + 1/(2.3.4) + 1/(3.4.5) +.....+1/(100.101.102) = 0,249951465....

3.130 Prove that 3 = sqrt(6+sqrt(6+sqrt(6+........

3.131 Prove the theorem of Pythagoras

3.132 Prove with induction Fermat' s little theorem: p | (n^p - n) if p prime

3.133 Prove that the remainder of (37^13) / 17 is 12

3.134 Prove that given a polynomial f(x) that the remainder of f(x) / (x-a) is f(a)

4.0 Facts

5.0 Unsolved Problems

6.0 Bibliography

1.0 Introduction

This book is a list of mathematical proofs and problems put as proofs. The reader should try to solve them before he or she looks at the proof. The list consists of both easy and not so easy proofs. The proofs contain aspects in a wide mathematical range but concentrates mainly on Number Theory.

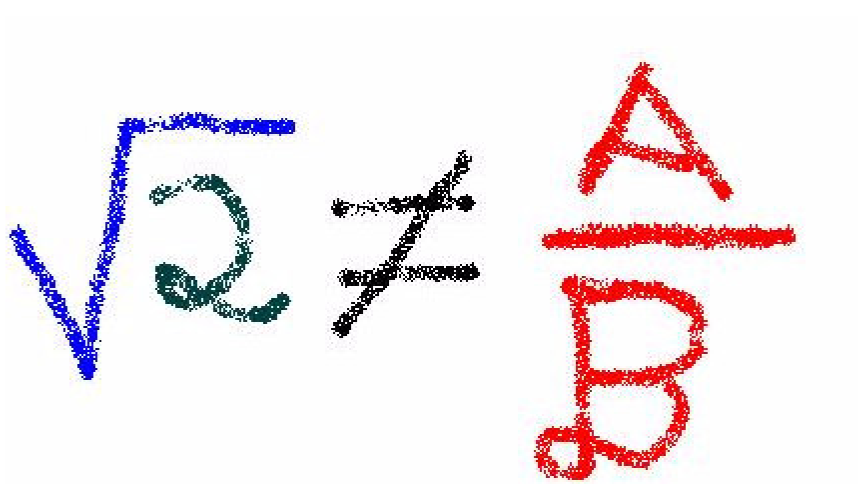
Furthermore we included some interesting facts about prime numbers and other oddities.

Included are also some axioms and elementary formulas the reader may find useful.

Last but not least we included some unsolved problems. These problems have kept some of the greatest mathematical minds busy. Solving any of these problems will make the reader famous in the mathematical archives for all eternity.

The methods used to do the proofs are standard and are widely used. The reader should not expect to solve all of these proofs. Great mathematicians have sweated on some of these proofs.

These proofs are ideal material for the math teacher to keep the students busy and make them think not only logically but also laterally.



2.0 Elementary stuff the reader should know or already knows.

Let A.B = A multiplied with B

Let A^B = A to the power of B

A+0 = A

A x 0 = 0

0/A = 0

A/0 is undefined.

A+B = B+A

A x B = B x A or A.B = B.A

0-A = -A

A-A = A + (-A) = 0

-A-B = -(A+B)

A-B = -(B-A)

A(-B) = -AB = B(-A)

(-A)(-B) = A.B

-(-A) = A

A+(B+C) = (A+B)+C

A(B+C) = A.B + A.C

(A+B) x (C+D) = A.C + A.D + B.C + B.D = A(C+D) + B(C+D)

A+A.B = A(1+B)

(A+B)(A-B)=(A.A – B.B)

A^0 = 1 , A^1 = A , A^2 = A.A , A^3 = A.A.A , A^n = A.A.A.A.A.A.A.A…..n times

$A^n = A^{n}$

$A^{(-1)} = 1/A$, $A^{(-n)} = 1/A^{n}$

$(A.B)^{n} = A^{n} B^{n}$

$A^{n} A^{p} = A^{(n+p)}$

$A^{n} / A^{p} = A^{(n-p)}$

$(a/b)^n = a^n / b^n$

$(A^n)^{p} = A^{n.p}$

$(a^n)^{(1/n)} = a$

$\sqrt[n]{a} = a^{(1/n)}$

$\sqrt[n]{a^p} = a^{(p/n)}$

$A.A = A^2 = A^{2}$

0! = 1 , 1! = 1 , 2! = 1.2=2 , 3! = 1.2.3 = 6 , n! = 1.2.3.4.5.6.7.8………n

$\binom{n}{r} = n! / [r! (n-r)!]$

$\sum_{n=1}^{10} n = 1+2+3+4+5+6+7+8+9+10$

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^n b^1 + \binom{n}{2}a^{n-1} b^2 + \ldots + \binom{n}{n-1} a\, b^{n-1} + \binom{n}{n} b^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

1 times 2 times 3 times 4 = 1.2.3.4 = 24

$$\prod_{n=1}^{r} n^2 = 1.4.9.16..r^2$$

If a.x.x + b.x + c = 0 then $x = -b/(2.a) \pm \dfrac{\sqrt{b.b - 4.a.c}}{2.a}$

The gradient of a function f(x) is given by $f'(x) = \lim_{h \to 0} [\, f(x+h) - f(x)\,] / h$

f'(x) = Dy/Dx

If f'(x) = Dy/Dx then $f(x) + c = \int f'(x)\, dx$

If f(x) = a then f'(x) = 0

If f(x) = x then f'(x) = 1

If f(x) = x.x then f'(x) = 2.x

If f(x) = x^n then f'(x) = n.x^(n-1)

If f(x) = a.g(x) then f'(x) = a.g'(x) where a is a constant

If f(x) = g(x) + h(x) then f'(x) = g'(x) + h'(x) D(x.x + x)/Dx = 2.x + 1

If f(x) = a.x.x + b.x + c then f'(x) = 2.a.x + b

If f(x) = g(x)h(x) then f'(x) = g'(x)h(x) + g(x)h'(x) D(xx)/Dx = 1.x + x.1 = 2x

If y = f(u) and u = g(x) then Dy/Dx = (Dy/Du)(Du/Dx) = f'(x)

If f(x) = *e*^x then f'(x) = *e*^x

If f(x) = *e*^g(x) then f'(x) = g'(x) *e*^g(x) D(*e*^(xx))/Dx = f'(x) = 2x *e*^(xx)

If $f(x) = \ln(x)$ then $f'(x) = 1/x$

If $f(x) = \sin(x)$ then $f'(x) = \cos(x)$

If $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$

If $f(x) = 1/g(x) = g^{-1}(x)$ then $f'(x) = -1\ g^{-2}(x)\ .\ g'(x)$ D(1/x) = -1.1/(x.x) = -1/x^2 = -x^(-2)

y = A Straight line parallel with x-axis.

X = A Straight line parallel with the y-axis.

y = ax + b Straight line with gradient of a

y = a.x.x + b.x + c Parabolic curve with gradient of f'(x) = 2.a.x + b and turning point at

x = -b/(2a) and cut the y axis at y = c

y = axxx + bxx + cx + d Cubic equation

x.x.x – y.y.y = (x-y)(x.x+x.y+y.y)

x.x.x+y.y.y = (x+y)(x.x-x.y+y.y)

(x+y)^2 = $(x+y)^2$ = x.x+2x.y+y.y

(x-y)^2 = $(x-y)^2$ = x.x-2x.y+y.y

$(x+y)^3$ = x.x.x+3x.x.y+3x.y.y+y.y.y

Area circle = πr^2

Circumference circle = $2\pi r$

Area ball = $4\pi r^2$

Volume ball = $(4/3)\ \pi r^3$

$\pi = 3,141592654....$

$e = 2,718281828...$

$\sqrt{2} = 1,414213562.....$

If $y = 10^x$ then $x = \log_{10}(y)$

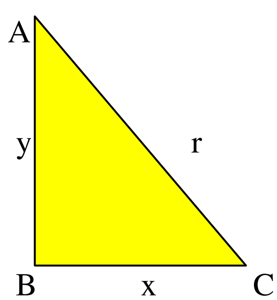
$\log_{10}(2) = 0,3010299957...$

$\ln(x) = \log_e(x)$

$\ln(ab) = \ln(a) + \ln(b)$

$\ln(a/b) = \ln(a) - \ln(a)$

$\ln(e) = 1$, $\ln(1) = \ln(e\text{^}0) = 0\ln(e) = 0$, $\ln(x\text{^}n) = n \ln(x)$



In the above figure the angle ABC = 90 degrees

The angle ACB = u degrees

Given the above the following can be deducted

x.x + y.y = r.r

tan(u) = y/x , cos(u) = x/r , sin(u) = y/r , sin(u) = tan(u) cos(u)

cot(u) = 1/ tan(u) , sec(u) = 1/cos(u) , cosec(u) = 1/sin(u)

$\sin(90-u) = \cos(u)$, $\cos(90-u) = \sin(u)$, $\tan(90-u)=\cot(u)$

$\sin(-u) = -\sin(u)$, $\cos(-u) = \cos(u)$, $\tan(-u) = -\tan(u)$

In general

$\sin(n+m) = \sin(n)\cos(m) + \sin(m)\cos(n)$

$\sin(n-m) = \sin(n)\cos(m) - \sin(m)\cos(n)$

$\cos(n+m) = \cos(n)\cos(m) - \sin(m)\sin(n)$

$\cos(n-m) = \cos(n)\cos(m) + \sin(n)\sin(m)$

$\tan(n+m) = [\tan(n)+\tan(m)]/[1-\tan(n)\tan(m)]$

$\tan(n-m) = [\tan(n)-\tan(m)]/[1+\tan(n)\tan(m)]$

$\sin(2m) = 2\sin(m)\cos(m)$

$\cos(2m) = \cos(m)\cos(m) - \sin(m)\sin(m) = 1-2\sin(m)\sin(m) = 2\cos(m)\cos(m)-1$

$\tan(2m) = [2\tan(m)]/[1-\tan(m)\tan(m)]$

If $f^{n}(x)$ is the n th derivative of f(x) in x then f(x) can be written as a polynomial in the following unique way.

$$f(x) = \sum_{r=0}^{n} [f^{r}(0)\, x^{\wedge} r]/[r!] = f(0) + f'(0)x + f''(0)xx/2 + f'''(0)xxx/6 + \ldots$$

Newton's numerical method of determining roots of a function f(x)

Let $g(x) = f'(x)$ then $x_{n+1} = x_n - f(x_n) / g(x_n)$ where x_0 is the first guess value

If $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$ then $\lim_{x\to 0} [f(x)/g(x)] = \lim_{x\to 0} [f'(x)/g'(x)]$

Prime numbers are integer positive numbers that's only devisable by themselves and one. One is by definition not a prime number. The list is therefore 2 ,3 , 5 , 7 , 11 , 13 , 17 ……..

Whole numbers are numbers like –3 or 2 or 0 or 100000.

Rational numbers can be written as R = p/q where p and q are members of the integer or whole number family and q is not 0. When p and q have no common factors, R is in its smallest form. R = 3/5 = 0.6 is a rational number in it's smallest form. R = 8/10 is not in its smallest form. It has 2 as common factor in both p and q.

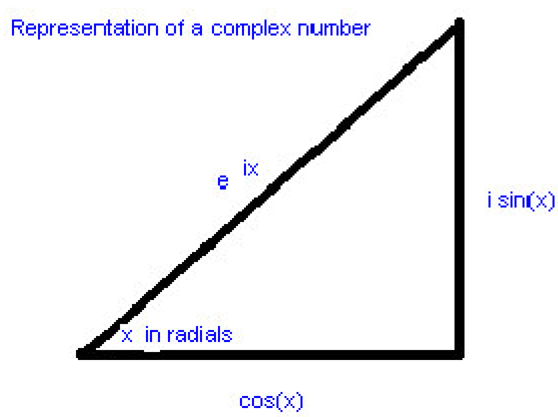
Let p and q be integer numbers and let q<>0 then irrational numbers are defined as numbers that cannot be written as p/q.

Irrational numbers are numbers that cannot be written as p/q. $\sqrt{2}$ is such an example. All the numbers defined so far are part of the real numbers.

Imaginary numbers are numbers like $\sqrt{-1}$. We don't know what to multiply with itself to give –1 as result. Usually we denote the square root of –1 as i

Then $\sqrt{-a.a}$ can be written as i times a or for short ai
Complex numbers are numbers such as z = a + bi. These numbers usually have a real and imaginary part. In this case a is real and bi is the imaginary part.

De Moivre's Theorem: $(\cos(x) + i.\sin(x))^n = \cos(nx) + i.\sin(nx)$ follows clearly from the identity $e^{ix} = \cos(x) + i.\sin(x)$



Little introduction to Congruence number theory

If (b-a)/c is integer we say that c divides (b-a) so that ck = b-a where k is integer.

If b divides c-d then the congruence (==) is given by c == d mod (b)
So t == h mod(g) means that g divides t-h and also divides h-t as h-t = -(t-h)
We know 2 divides 9-1 and this can be written as 9 == 1 mod(2)
We can also write it as 1 == 9 mod(2) because 2 divides 1-9
The following is also valid: 9 == 3 mod(2) , a == a mod(n) with n<>0 , a == 0 mod (2) for a even

a == b mod(c) if and only if b == a mod(c)
If a == b mod(c) and d == e mod(c) then (a+d) == (b+e) mod (c) We will proof this
If a == b mod(c) and d == e mod(c) then ad == be mod(c) We will proof this
If a == b mod (c) and d divides c then a == b mod (d) We will proof this as well.
If a divides b, by definition we write it as a | b
(a,b) = x means that the greatest common divider of a and b is x.

Proofs

Proof 1 : Prove that every prime number except 2 can be written as the difference of two squares.

Let p be any prime so that p>2.

Let p = zz – yy

Then zz – yy = (z + y)(z – y) = p.1

Therefore z-y = 1 and z = 1 + y

Therefore zz – yy = (1+y)(1+y) – yy = 1 + 2y + yy – yy = 1 + 2y = p

Therefore 2y = p – 1 and y = (p – 1)/2 which can be as p-1 is even.

Therefore z = 1 + y = 2/2 + (p-1)/2 = (p+1)/2 which also is a whole number as p+1 is even.

Example: Take p = 11 , then z = 6 and y = 5 so that 36-25 = 11

Proof 2 : Prove that $Y = 2^{(2^n -1)}$ divides $Z = (2^n)!$

$Z = (2^n)! = 1.2.3.4.5.6.7\ldots\ldots\ldots.2^n$

$Z = A\,(2.4.6.8.10.12\ldots\ldots\ldots2^n$ as 2^n is even.

$Z = A([2.1][2.2][2.3][2.4][2.5]\ldots..[2.2^{(n-1)}])$

$Z = A(2.2.2.2\ldots.2)(1.2.3.4\ldots\ldots2^{(n-1)})$

$Z = A(2^{(2^{(n-1)})})(2^{(n-1)})!$ This expression can be expanded further as in the manner

Above.

$Z = AB(2^{(2^{(n-1)})})(2^{(2^{(n-2)})})(2^{(n-2)})!$ We expand even further

And we then get $Z = C\,(2^{(2^{n-1})})(2^{(2^{n-2})})(2^{(2^{n-3})})\ldots\ldots.(2^{(2^{0})})$

Then $Z = C2^{[2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots.. + 1]}$

So that the [   ] term is equal to $2^{n} -1$

Therefore $Z = C(\,2^{(2^{n} - 1)})$

Case Proven.

Proof 3: Prove that the list of prime numbers is infinite.

We prove by contradiction (Reductio ad Absurdum)

Suppose the list is finite. Then we can construct a number $s = 2.3.5.7.11\ldots.p_{n}$ such that

p_{n} is the last prime number in the list. Now we construct a number $t = s + 1$

We can clearly see that none of the prime numbers in the list divides t as a remainder of 1 will always be left.

t therefore is prime or has prime factors not in our list of prime numbers. Our list is therefore not complete and thus the list of prime numbers must be infinite.

Proof4: Prove that given $Z = 2.3.5.7..p_j....p_k$ that $p_k < \sqrt{Z}$

where Z is the product of all primes from 2 up to p_k and $p_k > 3$

We use the proof of Chebychev that for any given n there is at least one prime between n and 2n For n>1.

The prime number just smaller than p_k is p_{k-1}

Now we know that $2 \times p_k \times p_{k-1} < Z$

Now we also now that $(1/2)p_k < p_{k-1} < p_k$ Chebychev' s proof

Therefore $2 \times (1/2)p_k \times p_k < Z$ so that $p_k^2 < Z$ and thus $p_k < \sqrt{Z}$

Proof 5 Prove that given a^b - b^a = 10^x where a=1000 , b=1001 that x = 3002,998818

b/a = 1001/1000 = 1,001 so that b = 1,001a

Then a^b - b^a = a^(1,001a) - (1,001a)^a = 10^x

Divide by a^a throughout and we get

a^(0,001a) - 1,001^a = (10^x)(a^-a)

1000^1 - 1,001^1000 = 1000 - 2,716923 = 997,2830761 = (10^x)(a^-a)

Take logs and we get

2,998818449 = x - a log(a) = x - 1000 log(1000) = x - 3000

So that x = 3000 + 2,998818449 = 3002,998818449

Case proven

Proof 6: Prove that if $2^n - 1$ is prime ,then so is n.

Proof

Let z = a^n -1 Clearly z = (a-1)(a^[n-1] + a^[n-2] + ..+ a + 1) when a is not 2.

Therefore z is always composite if a>2

If a = 2 and n is even then clearly n = 2v so that z=(2^v +1)(2^v -1)

If a=2 and n is uneven , but composite then let n=bc

so that z=2^(bc) -1 = f^(c) - 1 where f = 2^b

It is clear that f is not 2 and that f^c - 1 is therefore composite.

Therefore if 2^n - 1 is prime then n is not even or odd composite because then 2^n - 1 would be composite. It follows then that n must be prime if 2^n - 1 is prime.

Proof 7: Prove that $2^{32} + 1$ is not prime by showing that it has a factor 641. This is a famous proof by Euler.

Numbers in the form $2^p + 1$ where p=2^n are called Fermat numbers after the famous mathematician Pierre de Fermat.

The proof as follows.

641 = 2^4 + 5^4 = 5x2^7 + 1 so 2^4 = 641 - 5^4

2^32 = 2^4 x 2^28 = 641x2^28 - (5x2^7)^4 = 641x2^28 - (641-1)^4

Let 641x2^28 = k641

Also (641-1)^4 = 641^4 + s641 + 1 = d641 + 1

Therefore 2^32 = k641 - (d641 + 1) = K641 - 1

So that K641 = 2^32 + 1

Case proven

Proof 8: Prove that $\sqrt{2}$ is not rational.

We can prove this by assuming that it is rational.

Therefore $\sqrt{2}$ = p/q where p and q are integers and p/q cannot divide any more. P and q therefore have no common factors.

We can thus say that $2 = p^2/q^2$ where p^2 and q^2 also have no common factors except one.

We see however that we could divide p^2 by q^2 and that the answer is two.

Therefore p^2 and q^2 must have a common factor namely q^2. But this can' t be and therefore our original assumption was wrong and the square root of two cannot be rational. It is therefore irrational and can not be written as p/q

---

Proof 9: Prove that there exists infinite integer solutions for $x^2 + y^3 = z^6$

Let $x = 3k^3$ and $y = -2k^2$ and k >= 1

We substitute and get

Then $x^2 + y^3 = 9k^6 + -8k^6 = k^6$

Let k = z and our proof is done

So if k =1 then x = 3 , y = -2 and z = 1

If k = 2 then x = 24 , y = -8 and z = 2 and so on.

Case proven.

Proof10: Prove that 1 + 2 + 3 + .. + n = n(n+1)/2

Again we prove by using induction.

We see the expression holds for n=1 and assume that it also holds for n = k.

We therefore assume the sum to k terms is k(k+1)/2 and from this it follows that the sum to k+1 terms is (k+1)(k+2)/2

Deducing the above from 1 + 2 + 3 + .. + k + (k+1)

Therefore 1 + 2 + 3 + .. + k + (k+1)

= (1 + 2 + 3 + …+ k) + (k+1) = (k+1)/2 + (k+1)

= [k(k+1) + 2(k+1)]/2

= (k+1)(k+2)/2 and the case proven

Proof 11: Prove that if a > 0 then that a + 1/a >= 2

We know that (a-1)(a-1) >= 0

Therefore $a^2 - 2a + 1 >= 0$ (multiply out)

Therefore $a^2 + 1 >= 2a$ (add 2a to the left and to the right)

and thus a + 1/a >= 2 (divide by a>0 on both sides)

Proof 12: Prove that $a^2 + b^2 >= 2ab$

We know that $(a-b)(a-b) = a^2 - 2ab + b^2$

Therefore $a^2 + b^2 = (a-b)^2 + 2ab$

Therefore $a^2 + b^2 >= 2ab$ because $(a-b)^2 >= 0$ and when taken away from the right side of our equation , the right side would become less if a-b is not zero.

Proof 13: Prove that $(a+b)/2 >= \sqrt{ab}$

We know that $a^2 + b^2 >= 2ab$ from the previous problem.

Therefore $a^2 + 2ab + b^2 >= 2ab + 2ab = 4ab$

Therefore $(a+b)^2 >= 4ab$

Therefore $a+b >= 2\sqrt{ab}$

and thus $(a+b)/2 >= \sqrt{ab}$ and our case is proven

Proof 14: Prove that if n is an integer number that either n^2 or $n^2 - 1$ has four as a factor?

First we prove the case for when n is even. If n is even , it could be written in the form n=2*p where p is some whole number.

Then n^2 is $2*p*2*p = 4*p^2$

Therefore if n is even then n^2 has 4 as a factor.

If n is uneven then n-1 and n+1 is even. Therefore we could write

that n-1 = 2*e and n+1 = 2*r ,so that $(n-1)(n+1) = n^2 -1$

= 4*e*r and thus if n is uneven then $n^2 -1$ has four as a factor.

We have thus proved our case.

Proof 15: Prove that a+1 always divides $a^u + 1$ if u is uneven.

Proof follows

For u=1 the result follows clearly. Let u > 1 then

Let a + 1 = c and let $z = a^u + 1 = (c - 1)^u + 1$

Using the binomial theorem we see that $(c-1)^u = c^u + k_1 c^{u-1} + ..$

$+ k_j c^{u-j} + ...-1 = Kc -1$

Therefore z = Kc -1 + 1 = Kc = K(a+1) and therefore a+1 divides K(a+1)

Case proven

Proof 16: Prove that p if prime always divide $z = a^p - a$

Proof

For p=2 the result follows because then z = a(a-1) and either a or a-1 must be even so that 2 will divide z

When p>2 let $c^p = (b_1 + b_2 + b_3 + .. + b_a)^p$ and let all the b' s be equal to each other so that c = ab

According to the binomial theorem $\binom{p}{r}$ is dividable by p if p is prime and for

p>r>0 . Because c has a b' s we have then that $c^p = ab^p\binom{p}{0} + k_1\binom{p}{1}$

$$+ k \binom{p}{2} + .. = ab + pK$$
2 2

Now let b = 1 and we have c = a so that z = a +pK - a = pK

pK can be divided by p and our proof is therefore finished.

Case proven.

Proof 17: Prove that if p prime and bigger than 3 that p+1 divides p!

Proof
Let z = p! = 1.2.3.4.5.6.7.8.9..p

Now p+1 is not prime and therefore has prime factors smaller than p. p+1 is also even , because p is uneven. p+1 is therefore an even composite number. Therefore 2 divide p+1.
First the case when p+1=2^n
If p+1 = 2^n then let p+1 = 2x = 2.2^(n-1) Clearly x=(p+1)/2 <p
If x<>2 then x is an even factor bigger than 2 of p! and therefore 2 and x are factors of p!. But 2x=p+1 and therefore p+1 is a factor of p! and therefore divides p! If x=2 then p+1=4 which cannot be because then p = 3 and we know p>3

Secondly the case where p+1 has more than 1 prime factor.
We know 2 will always be a factor of p+1.
Therefore P+1 = 2.v and v = (p+1)/2 < p
Therefore 2 and v are factors of p! and therefore 2v are a factor of p! and p+1 must then be a factor of p!

Proof 18: Prove that a+ar+arr+arrr+ar^4+…+ ar^n = $a \dfrac{r^{n+1} - 1}{r - 1}$

Let S = a + ar + arr + arrr +…….ar^(n)

Then rS = ar + arr + arrr+….+ ar^(n+1)

Then rS - S = S(r-1) = ar^(n+1) - a = a[r^(n+1) - 1]

So that S = a[r^(n+1) - 1]/[r-1] and our case proven.

Proof 19: Prove that 1+2^n < 3^n for n>=2

We prove using induction

For n=2 it follows that 1+2^n = 1+2^2 = 5 <3^2 = 9

For n=k+1 follows that Z = 1+2^n = 1+2^(k+1) = 1+2.2^k = (1+2^k) + 2^k

So that Z < 3^k + 2^k because 1+2^k < 3^k

And Z < 3^k + 2^k +1 < 3^k + 3^k adding one will make Z even more smaller

And Z < 3^k + 3^k +3^k = 3.3^k = 3^(k+1) adding 3^k will make Z still more smaller

Therefore Z = 1 + 2^(k+1) < 3^(k+1)

Case proven

Proof 20: Prove that there exist irrational numbers a , b such that a^b is rational.

Let z = $\sqrt{2}$ so that z is irrational and let a = b = z . Now either S = a^b is rational or not. If it is our proof is done.

If S is not rational then let a = S = z^z and b = z . Then L = a^b = (z^z)^z = 2 so that L is rational.

If S is irrational we have an a and b both irrational so that L = S^b is rational.

If S is rational we also have an a and b both irrational so that S = a^b is rational.

Case proven.

Proof 21: Prove that aa + bb is not equal to 4n-1 where a, b and n are integers.

Let <> suffice for not equal to.

We use contradiction to prove that aa+bb<>4n-1. We therefore assume that aa+bb=4n-1

Now 4n-1 is uneven and therefore either a or b must be uneven and the other even.

Let b be uneven and a be even.

Let a = 2v and then aa = 4vv

Then aa+bb = 4vv+bb

So that 4vv + bb + 1 - 4n = 0

bb+1 = (b+1)^2 - 2b but b+1 is even, so let b+1 = 2c so that bb+1 = 4cc -2b

Then 4vv + 4cc - 4n - 2b = 4(vv+cc-n) - 2b = 4k - 2b = 0

So 4k = 2b so that b=2k and b is therefore even

But this cannot be as be is uneven. We have therefore a contradiction and our original assumption must be wrong.

Case proven.

Proof 22: Prove that every integer > 1 is a prime or the product of primes.

Let us assume that a number N exists such that this number is not prime and also not the product of primes and is the least of such numbers. Then by definition this number is composite.

So N=ab , but then a and b must be smaller than N and must be therefore the product of primes or must be prime. N is therefore the product of primes.

Our assumption is therefore wrong.

Case proven.

Proof 23: Prove that 4n+1 can be written as aa-bb for n>0.

Let a = 2n+1 , then aa = 4nn+4n+1

Let b = 2n, then bb = 4nn

Then aa – bb = 4nn + 4n + 1 – 4nn = 4n + 1

Case proven

Proof 24: Prove that 8 divides nn-1 for n odd

Let n = 2x-1 for x>=1 so that n will always be odd.

For x=1 the result clearly follows as 8 divides 0

For x>1 the following reason.

nn = (2x-1)^2 = 4xx – 4x + 1 so that nn – 1 = 4xx – 4x = 4x(x-1)

Now either x or x-1 is even.

If x is even let x = 2v so that nn – 1 = 8v(x-1) It is clear that 8 divides nn-1 when x is even.

If x-1 is even let x-1=2w so that nn-1 = 4x(2w) = 8xw. It is clear that 8 divides nn-1 when x is uneven.

Therefore 8 divides nn-1 when n odd.

Case proven.

Proof 25: Prove that –1 times –1 is 1 ,given that 1-1=0 and that -(1) is –1 and that 1 times 0 = 0

1 times 0 is 0

1 times (1-1) is 0

(1 times 1) + (1 times –1) = 0

1 + (1 times –1) – (1 times –1) = - (1 times –1)

1 + 0 = 1 = - (1 times –1) = -1 times –1

Proof 26: Prove that $e^\pi > \pi^e$ without a calculator

e^x can be written as a polynomial series in the following way.

$e^x = 1 + x + xx/2 + xxx/6 + [x^4]/4! + \ldots [x^n]/n!$

Therefore $e^x > 1 + x$ for $x>0$

Let $x = \pi/e - 1 > 0$

Then $e^x = e^{(\pi/e - 1)} > 1 + \pi/e - 1 = \pi/e$

So that $e^{(-1)} . e^{(\pi/e)} > \pi/e$

And $e^{(\pi/e)} > \pi$

Therefore $[e^{(\pi/e)}]^e > \pi^e$

Which results in $e^\pi > \pi^e$

Case proven

Proof 27: Prove that a-1 divides $a^n - 1$

According to the binomial expansion rule

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Now let a = x+1 then x = a-1

$$\text{Then } a^n = \sum_{r=0}^{n} \binom{n}{r} (a-1)^r = (a-1)^n + n(a-1)^{n-1} + \ldots + n(a-1) + 1 = k(a-1) + 1$$

So that $a^n - 1 = k(a-1)$

Therefore a-1 is a factor of $a^n - 1$

Case proven

Proof 28: Prove that p if prime always divides the binomial coefficient $\binom{p}{r}$ for 0<r<p

Let z = $\binom{p}{r}$ = [1.2.3.4.5.6.7.8‥p]/[(1.2.3.4.5.6.7‥r)(1.2.3.4.5.6‥p-r)]

Because p is prime a = 1.2.3.4‥r will not have p as a factor

Because p is prime b = 1.2.3.4.5..p-r will also not have p as a factor

But 1.2.3.4.5..p has p as factor

Therefore z = p!/[ab] = p(p-1)! / (ab) = pK

So it follows that p divides Z

Proof29: Prove that (1^3 + 2^3 + 3^3+…+n^3) = (1+2+3+…+n)^2

We prove using induction.

Let ∑ n suffice for sum of r from r = 1 to r = n

(1^3 +2^3 + 3^3….n^3) = ∑ (n^3)

(1+2+3+4….n)^2 = (∑ n)^2

For n=1,n=2,n=3 the result clearly follows.

We assume the result holds for n = k , that is that ∑ (k^3) = (∑ k)^2

We now prove for n = k+1

The following is clear ∑ (k+1)^3 = ∑ (k^3) + (k+1)^3

Secondly

(∑ (k +1))^2 = [1+2+3+4‥+k + k+1]^2 = ([1+2+3+‥k] + (k+1))^2

= $[\sum k + (k+1)]^2 = (\sum k)^2 + 2(k+1)(\sum k) + (k+1)^2$

= $(\sum k)^2 + 2(k+1)(k)(k+1)/2 + (k+1)^2$

= $(\sum k)^2 + k(k+1)^2 + (k+1)^2$

= $\sum (k^3) + (k+1)(k+1)^2 = \sum (k^3) + (k+1)^3$

Case proven

Proof 30: Prove that 6 divides n(n+1)(n+2)

We prove using induction.

Let z = n(n+1)(n+2)

If n=1 then z = n(n+1)(n+2) = 1.2.3 = 6 , so 6 divides 6

We assume that 6 divides z = n(n+1)(n+2) = n(n^2 + 3n +2) = n^3 + 3n^2 + 2n

Let z=6k because 6 is a factor of z

We now prove that 6 divides w = (n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n +6

Now w = (n^3 + 3n^2 +2n) + 6 + (9n + 3n^2)

Therefore w = (z + 6) + 3n(3+n) = 6(k+1) + 3n(3+n)

It is clear 6 divides 6(k+1). We prove now that 6 divides 3n(3+n)

3 divides 3n(n+3) and if n even then 2 divides n or if n uneven then 3+n is even and 2 divides n+3

Therefore 6 divides 3n(n+3)

Six divides w and thus our case is proven.

Proof 31: Prove that $\sqrt{3}$ is irrational.

We prove the theorem using contradiction.

Let's suppose that $\sqrt{3}$ is rational and can be written as p/q where q<>0 and where p/q is in its smallest form. That is p and q have no common factors.

Therefore p^2 / q^2 = 3

So that p^2 = 3q^2

This means that p^2 has 3 as factor. If it has 3 as factor then p has 3 as factor so that p^2 has 9 as factor.

p^2 can therefore be written as p^2 = 9k

So 3q^2 = 9k and q^2 = 3k

Therefore q^2 must have 3 as a factor. This however is impossible because then both p and q has 3 as factor and p/q is then not in it's smallest form.

Our assumption that $\sqrt{3}$ is rational is therefore wrong.

Case proven.

Proof 32: Prove that 1+3+5+7+….(2n-1) = n^2

We prove using induction.

For n = 1 the result is 1

For n = 2 the result is 1+3 = 4

The assumption holds true for n=1 and n=2. We assume the result holds for n=k.

Therefore for n=k the result is k^2

We prove now the result holds for n=k+1

For k+1 the result therefore should be $(k+1)^2$

Now $1+3+5\ldots+ 2k-1 + 2(k+1)-1 = k^2 + 2(k+1)-1 = k^2 + 2k +2-1 = k^2 + 2k + 1 = (k+1)^2$

The result holds true and we have proven our case.

Proof 33: Prove that 1/2 + 1/4 + 1/8 + 1/16 +….+1/(2^n) = 1 as n tends to infinity

We prove using plain old algebra.

Let z = 1/2 + 1/4 +…1/(2^n) = [2^(n-1) + 2^(n-2) + 2^(n-3) +…+ 1]/[2^n]

Now 2^(n-1) + 2^(n-2) + … + 1 = 2^n - 1

So z = [2^n - 1]/2^n = 1 - 1/2^n , but 1/2^n tends to 0 as n tends to infinity

Therefore z = 1

Case proven

Proof 34: Prove that $\sum_{k=0}^{n} 1/[(k+1)(k+2)] = 1$ as n tends to infinity

Let z equals the sum of the terms.

Then z = 1/(1.2) + 1/(2.3) + 1/(3.4) + 1/(4.5)….

Writing the above in another way.

z = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5)…

It should now become clear that the result is

z = 1/1 - 1/n = 1 - 1/n , but n tends to infinity so 1/n becomes 0

Therefore z = 1

Case proven.

Proof 35: Prove that $\sum_{k=1}^{n}$ 1/(a^k) = 1/(a-1) as n tends to infinity and a>=2

First we prove that (a-1)[a^(k-1) + a^(k-2) +…+ a +1] = a^k - 1

1 + (a-1)[a^(k-1) + a^(k-2) +…+ a +1] = 1 + 1 .(a-1) + (a-1)a + (a-1)a^2 + (a-1)a^3 +…

= 1 + a - 1 + a^2 - a + a^3 - a^2 + a^4 - a^3 +…. a^k

= (1-1) + (a-a) + (a^2 - a^2) +….+ [(a^(k-1) - a^(k-1)] + a^k

= a^k

Now 1/a + 1/a^2 + 1/a^3 +…+ 1/a^n

= [1 + a + a^2 + a^3 +…+ a^(n-1)]/a^n = (a-1)[1 + a + a^2 + a^3 +…+ a^(n-1)]/[(a-1)a^n]

= [a^n - 1]/[(a-1)a^n] = [1/(a-1)][1 - 1/a^n] , but as n tends to infinity so does 1/a^n tends to zero

= 1/(a-1)

Case proven

Proof 36: Prove that $\sum_{n=1}^{\infty}$ 1/n diverges

We could prove using the integral test or do the following

Let z = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + 1/11 + 1/12 + 1/13 + .. + 1/n

Then z > 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/16 +1/16+1/16 + 1/16 + 1/16 + …+1/2^p

So z > 1/(2^1) + (2^1)/(2^2) + (2^2)/(2^3) + (2^3)/(2^4) + (2^[p-1])/(2^p)

So z > 1/2 + 1/2 + 1/2 + 1/2 +….p times

So z > p/2 , but n = 2^p so p = (ln(n))/ln(2) so as n tends to infinity so does p

Therefore p/2 tends to infinity and so does z

Case proven.

Proof 37: Prove that $\sum_{n=1}^{\infty}$ **1/(n^2) converges**

Again we can use the integral test , but we will proceed as in Proof 36.

So let z = 1/1 + 1/(2.2) + 1/(3.3) + 1/(4.4) + 1/(5.5) + …+ 1/(n.n)

So z < 1/1 + 1/(2.2) + 1/(2.2) + 1/(4.4) + 1/(4.4) + …

So z < 1/1 + 2/2^2 + (2^2)/(2^4) + (2^3)/(2^6) + … + (2^p)/(2^2p)

So z < 1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + …+ 1/2^p

So z < [2^p + 2^(p-1) + 2^(p-2) +….+ 1]/2^p

So z < [2^(p+1) - 1]/2^p = 2 - 1/2^p

So z < 2 because 1/2^p tends to zero as p tends to infinity

Therefore z converges and our case is proven.

Proof 38: Prove that 7 divides 3^(2n+1) + 2^(n+2)

We use induction to prove the theorem.

For n=1 3^(2n+1) + 2^(n+2) = 3^3 + 2^3 = 27 + 8 = 35

Clearly 7 divides 35

We now assume that the theorem is true for n=k.

We now prove for n=k+1

Let 7 divides z = 3^(2k+1) + 2^(k+2) as in our assumption

Now Let a = 3^(2[k+1]+1) + 2^(k+1+2) = 3^(2k+3) + 2^(k+3) for k+1

Therefore a = 9.3^(2k+1) + 2.2^(k+2)

So a = 9[3^(2k+1) + 2^(k+2)] -7.2^(k+2)

So a = 9z - 7.2^(k+2)

Now 7 divides 9z because 7 divides z and 7 also divides 7.2^(k+2) so 7 divides a

Case proven

Proof 39: Prove that 0,9999999999….. = 1

Let S = 0,9999999999999….

Then S = 9/10 + 9/100 + 9/1000 + 9/10000 + …..

So 10S = 9 + 9/10 + 9/100 + 9/1000 + 9/10000 + …..

So 10S – S = 9S = 9

Therefore S = 9/9 = 1

Case proven

Proof 40: Prove that $\lim_{x \to 0} [x/\sin(x)] = 1$

Let f(x) = x and g(x) = sin(x)

Then f'(x) = 1 and g'(x) = cos(x)

Because $\lim_{x \to 0} x = 0$ and $\lim_{x \to 0} \sin(x) = 0$, $\lim_{x \to 0} [x/\sin(x)] = \lim_{x \to 0} [1/\cos(x)] = 1/\cos(0) = 1/1 = 1$

Proof 41: Prove that all uneven numbers bigger than zero can be written as the difference of two squares.

Let z = 2n -1 where n>0 so that z represents the positive uneven numbers.

Let $z = z.1 = a^2 - b^2 = (a+b)(a-b)$

If a-b = 1 then a+b = 2n-1

Substituting a = 1+b into a+b gives a+b = 1+b+b = 2n-1

So 1+2b = 2n-1

2b = 2n - 2

b = n - 1

a = 1+b = 1+n-1 = n

We test our deductions by back substituting

So $a^2 - b^2 = n^2 - (n-1)^2 = n^2 - (n^2 -2n +1) = 2n - 1 = z$

Case proven

Proof 42: Prove that all numbers which are multiples of four can be written as the difference of two squares.

Let z =4v so that z represents all numbers which are multiples of four.

Let $z = 4v = 2(2v) = a^2 - b^2 = (a+b)(a-b)$

Now let a-b = 2 and a+b = 2v so that (a+b)(a-b) = 2.2v = 4v = z

Clearly a = 2+b so that a+b = 2+b+b = 2+2b = 2v so that b = v-1 and a = 2+b = 2+v-1 = v+1

Back substituting to test then delivers

$a^2 - b^2 = (v+1)^2 - (v-1)^2 = v^2 + 2v + 1 - (v^2 -2v - 1) = 2v + 2v = 4v = z$

Case proven

Proof 43: Prove that any cube can be written as the difference of two squares.

Let the cube be $z = a^3$

Now either a is even or uneven.

If a is even then a = 2v so that $z = 8v^3$

Clearly if a is even then 4 divides z so that according to proof 42 it can be written as the difference of two squares.

If a is uneven then z = 2n-1 with n>0 so that according to proof 41 it can be written as the difference of two squares.

Case proven.

Proof 44: Prove a solution for x^x^x^x^x^…. = 2

$x^{x^{x^{x^{.^{.}}}}} = 2$ Let x=a then $a^{x^{x^{x^{.^{.}}}}} = 2$, but $x^{x^{x^{.^{.}}}} = 2$ so $a^2 = 2$, so $a = x = \sqrt{2}$

Case proven

Proof 45: Prove that $n^{(1/n)} = 1$ as n tends to infinity

Let $S = n^{(1/n)}$ then Z = ln(S) = (1/n) ln(n) = [ln(n)]/n

Now n tends to infinity and also ln(n)

Now let f(n)=ln(n) and g(n)=n

Then f' (n) = 1/n and g' (n) = 1

So Z = f(n)/g(n) = f' (n)/g' (n) = 1/n/1 = 1/n

So Z -> 0 as n tends to infinity

Then $S = e^{Z} = e^{0} = 1$

Case proven

Proof 46: Prove that $(n!)^{(1/n)}$ tends to infinity as n tends to infinity

Let $S = (n!)^{(1/n)}$

Let Z = ln(S) = (1/n)ln(n!) = (1/n)ln(1.2.3.4.5.6.7.8.9…n)

So Z = (1/n)[ln(1) + ln(2) + …+ln (n) + n.ln(n) - n.ln(n)]

Z = ln(n) + (1/n)[ln(1/n) + ln(2/n) + ln(3/n) +…+ ln(1)]

Z = ln(n) + R

But R is a Riemann sum

So $R = \int_{1/n}^{1} \ln(x)dx = [x.\ln(x) - x]_{1/n}^{1} = [1.\ln(1) - 1] - [(1/n)\ln(1/n) - (1/n)]$

R = [0-1] - [(1/n)ln(1/n)] = -1 - [ln(1/n)]/[n] = -1 - [(n)(-1/{n.n})]/1 = -1 + 1/n = -1

So Z = ln(n) -1 , so Z tends to infinity as n tends to infinity

But $S = e^{Z}$, so S tends to infinity as Z tends to infinity.

Case proven.

Proof 47: Prove that $1 + 2^2 + 3^2 + 4^2 + 5^2 +…+ n^2 = n(n+1)(2n+1)/6$

We proof using induction

For n=1 $n(n+1)(2n+1)/6 = 1(2)(3)/6 = 1$

We suppose that the equation holds for n=k

The sum is therefore $1 + 4 + 9 + \ldots = z = k(k+1)(2k+1)/6 = (2k^3 + 3k^2 + k)/6$ for k=n

We prove now for n = k+1

For n = k+1 the sum is $v = (k+1)(k+2)(2k+3)/6 = (2k^3 + 9k^2 + 13k + 6)/6$

$v = [(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)]/6 = (2k^3 + 3k^2 + k)/6 + (6k^2 + 12k + 6)/6$

$v = z + (k+1)^2$

So the sum to k+1 is the sum to k terms plus k+1 squared.

Case proven.

Alternate proof to Proof 47

Let S(n) = 1 + 2^2 + 3^2 + 4^2 +….n^2

Now S(n+1) = S(n) + (n+1)^2

Now S(n) can be written as a polynomial if we can find such a polynomial

such that S(n+1) = S(n) + (n+1)^x

We can find such a polynomial if we let S(n) = an^(x+1) + bn^x + cn^(x-1)+.....

Then S(n+1) = a(n+1)^(x+1) + b(n+1)^x + c(n+1)^(x-1) + ...

So S(n+1) = an^(x+1) + bn^x + cn^(x-1) +...... + a(x+1)n^x +..........

So S(n+1) = S(n) + a polynomial with its highest power x

This polynomial is then equal to (n+1)^x so that values can be worked out for a,b,c.....

In our case S(n) = 1 + 2^2 + 3^2 +…+n^2

Let S(n) = an^3 + bn^2 + cn + d

Now if we can find values for a, b, c, d such that S(n+1) = S(n) + (n+1)^2 then we are done.

Then S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)

So S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) + [3an^2 + n(3a+2b) + (a+b+c)]

Therefore [3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)

So 3a = 1 and a = 1/3

So 3a+2b=2 , 1+2b=2 , 2b=1 , b=1/2

So a+b+c = 1 , 1/3 + 1/2 + c = 1 , c=1/6

Therefore S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d

But S(1) = 1 = (1/3) + (1/2) + (1/6) + d and therefore d=0

So S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)

Case proven.

Proof 48: Prove that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3 = n^2(n+1)^2/4$

We prove using induction.

For n=1 $k^2(k+1)^2/4 = 1(2)^2/4 = 4/4 = 1$

The sum to k terms is therefore $z = k^2(k+1)^2/4$

The sum to k+1 terms is then given by $s = (k+1)^2(k+2)^2/4 = (k+1)^2(k^2 + 4k + 4)/4$

So $s = k^2(k+1)^2/4 + (k+1)^2(4k+4)/4 = z + (k+1)^3$

So the sum to k+1 terms is the sum to k terms + the cube of k+1

Case proven.

Proof 49: Prove that 1.2 + 2.3 + 3.4 + 4.5 +…+ n(n+1) = n(n+1)(n+2)/3

We proof using induction.

For n=1 we have n(n+1)(n+2)/3 = 1.2.3/3 = 1.2

We assume it is true for n=k

Now z(k) = 1.2 + 2.3 + …+ k(k+1) = k(k+1)(k+2)/3

So z(k+1) = z(k) + (k+1)(k+2)

Now the sum to k+1 terms is therefore z(k+1) = (k+1)(k+2)(k+3)/3

Therefore z(k+1) – z(k) = [(k+1)(k+2)][(k+3) – k]/3 = (k+1)(k+2)

So z(k+1) = z(k) + (k+1)(k+2)

Case proven.

Proof 50: Prove that there exist infinite many solutions for m! = k! n!

Let m = x! then m! = (m-1)! m = (m-1)! x! = (x! –1)! x! = k! n!

So m! = (x!)! , k! = (x! – 1)! , n! = x!

So m = x! , k = x! –1 , n = x

Example: Let n = x = 3

Then k = x! – 1 = 6-1 = 5
m = x! = 6

So m! = 6! = 720

k! n! = 5! 3! = 120 . 6 = 720

Case proven.

Proof 51: Prove the identity *e*^(ix) = cos(x) + i.sin(x) where i = square root of -1

Let z = cos(x) + i.sin(x)

Then $dz/dx = -\sin(x) + i.\cos(x)$

So $dz/dx = i(\cos(x) + i.\sin(x)) = i.z$

Therefore $dz/z = i.dx$

Integrate on both sides $\ln(z) + k = i.x + m$

Let $m-k = n$

Then $\ln(z) = i.x + n$

If $x = 0$ then $z = 1 + i.0 = 1$
Therefore $\ln(1) = 0 = i.0 + n = n$

So $n = 0$ and $\ln(z) = ix$ so that $z = e^{(ix)} = \cos(x) + i.\sin(x)$

Therefore $e^{(ix)} = \cos(x) + i.\sin(x)$

Case proven.

Proof 52: Prove that $e^{(\pi i)} = -1$

We use the identity we proved in proof 51.

$e^{(\pi i)} = \cos(\pi) + i.\sin(\pi) = -1 + 0.i = -1$

Case proven.

Another way to use the identity and De Moivre's theorem is to solve for $\sqrt{i}$

Clearly $\cos(\pi/2) + i.\sin(\pi/2) = 0 + i.1 = i$

Let π = pi

Therefore $\sqrt{i} = [\cos(pi/2) + i.\sin(pi/2)]^{1/2} = \cos(pi/4) + i.\sin(pi/4) = \dfrac{(1 + i)}{\sqrt{2}}$

Problem Solved.

Proof 53: Prove that *e* is irrational

e can be written as a Taylor expansion in the following way.

e = 1 + 1/1! + 1/2! + 1/3! +1/4! +….+1/n! where n tends to infinity.

Now let us suppose that *e* is rational and can be written as e=a/b where a and b is integer and a/b is in the smallest form possible.

Therefore *e* = a/b , Now we form *e*b! which is integer so that

*e*b! = b! + b!/1 +b!/2! + b!/3! + … b!/b! + b!/(b+1)! +…..+ b!/n!

Let z = b! + b!/1 + b!/2! + b!/3! + … b!/b! which is clearly integer.

Let p = b!/(b+1)! + b!/(b+2)! +…+ b!/n!

So p = 1/(b+1) + 1/[(b+1)(b+2)] + 1/[(b+1)(b+2)(b+3)] +…

Clearly p < 1/(b+1) +1/[(b+1)(b+1)] + 1/[(b+1)^3] + 1/[(b+1)^4] +….

So p < (1 - (b+1)^(-n-1))/b = 1/b because as n tends to infinity so does (b+1)^(-n-1) tends to 0

So p < 1/b , but b>1 so that p <1 and thus *e*b! = z + p

But *e*b! is integer and z+p is clearly not integer because p <1.

This is a contradiction and *e* is therefore not rational , but irrational.

100

Proof 54: Prove that the last two digits of 21 is 01

This is quite easy.

21^1 = 21
21^2 = ..41
21^3 = ..61
21^4 = …81
21^5 = ..01
21^6 = …21
21^7 = …41

We see therefore that every 5^{th} power will end on 01 and because 100 is 5 times 20 it is also a 5^{th} power. 21 to the power of 100 will therefore end on 01.

Proof 55: Prove an exact formula for sqrt(x + sqrt(x + sqrt(x + sqrt(x + sqrt(x +….)))))) if sqrt stands for square root of.

Let z = sqrt(x + sqrt(x + sqrt(x +…))))

Then z^2 = x + sqrt(x + sqrt(x + sqrt(x + ….)))))

So $z^2 = x + z$

Therefore $z^2 - z - x = 0$

Therefore using the kwadratic formula we get $z = (1 + \sqrt{1 + 4x})/2$

Testing let x=2

Then z = (1 + 3)/2 = 4/2 = 2

2^0,5 = 1,41421

2+1,41414 = 3,414213562

3,414…^0,5 = 1,847759…

1,847759..+2 = 3,847759…

3,847759….^0,5 = 1,96157….

1,96157…+2 = 3,96157…

3,96157…^0,5 = 1,990 ….

and we see that we rapidly approach 2 our answer we got with the exact formula.

Proof 56: Prove an answer for 1 + 1/(1 + 1/(1 + 1/(1 + 1/….))))

Let z = 1 + 1/(1 + 1/(1 + 1/(1 + 1/….))))

Then z -1 = 1/(1 + 1/(1 + 1/(1 + 1/….))))

So 1/(z-1) = 1 + 1/(1 + 1/(1 + 1/(1 + 1/….)))) = z

So 1 = z(z-1) = z^2 - z

Therefore $z^2 - z - 1 = 0$ and $z = (1 + \sqrt{1 + 4})/2$ = 1,618033989…..

Proof 57: Prove that $a^0 = 1$ given a > 0

Elementary proof

$a^0 = a^{(1-1)} = a^1$ times a^{-1}, but a^{-1} is 1/a so that

a^0 = a times 1/a = a / a = 1

Proof 58: Prove that S = 1/2 + 1/3 +1/5 +1/6 + …+1/n is not integer for n>1

Let us assume that S is integer.

Let a = 3.5.7.9.11.13.15.17….n if n is uneven

Let a = 3.5.7.9.11.13.15.17….(n-1) if n is even

We determine that 2^k <= n < 2^(k+1)

Let b = 2^(k-1)

We now form Z = abS which is clearly integer if S is integer.

Now in the expansion (1/2 + 1/3 + 1/4 + 1/5 + 1/6+…+ 1/j + ...1/n)ab it is clear that the terms in the parenthesis where the numerators are uneven must be integer if multiplied by ab

Now j <= n

If j is uneven then a/j = (3.5.7.11…)/j which is integer

If j is even but not purely a power of two then $j = 2^m v$ where v is uneven. Clearly m < k because

The minimum of v = 3 so that $j = 2^m . 3 = 2^m (2+1) = 2^{m+1} + 2^m <= n < 2^{k+1}$

So $j = 2^{m+1} + 2^m < 2^{k+1}$, so $2^{m+1} < 2^{k+1}$,so m+1< k+1 , so m < k , so m <= k-1

Therefore if j even ,but not purely a power of two then clearly ab/j is integer

If j is purely a power of two and j <= n then b/j is integer except for j = 2^k

When j=2^k then a/j = 1/2 which is not integer

Therefore for all ab/j the result will be integer except for j=2^k

Therefore (1/2 +1/3 + 1/4 + 1/5 + 1/6 +….+1/n)ab is not integer

But Sab is integer and we have a contradiction. S is therefore not integer.

(1/2 + 1/3 + 1/4 + 1/5 + 1/6 +….+1/n) is therefore not integer.

Proof 59: Prove that a complex number to the power of a complex number may be real by proving that i to the power of i is not complex.

We know that e^{ix} = cos(x) + i sin(x)

Let x = pi/2

Then $e^{i\,pi/2}$ = 0 + i = i

Therefore $i^i = e^{i(i\,pi/2)} = e^{-pi/2}$ which is real and has an approximate value of 0,2078795764..

Proof 60: Prove that the list of primes in the form 4n+3 is infinite.

We assume that the list of primes in the form 4n+3 is finite.

Let z = 4n+3 be the last prime in this form.

Form m = 5.7.11.13..p_j ….(4n+3) where m is equal to the product of all primes bigger than 3 and up to and including 4n+3 the last prime in this form.

Neither 2 or 3 divide m as can clearly be seen

Now we form w = 4m+3

We note that all primes smaller or equal to (4n+3) divide 4m except the prime 3.
We also note that neither 4 nor any of the factors of m divides 3
4m and 3 therefore has no common factors. W is therefore either prime or the product of primes .
If w is prime, this prime is bigger than 4n+3 and we are finished.
If we are not finished w is the product of primes and we note that these primes must also be bigger than 4n+3. Now all prime factors of w can be written as 2x + 1. If x uneven then we can write it as 4y + 3 by noting 2x + 1 = 2(x-1) + 1+2 .
Now w = (2d+1)(2e+1)…(2j+1) = 2f+1 , so if all d,e...j are even then f is even. So some must be uneven so that f is uneven so that 2f+1 = 4m+3. Therefore some prime factors of w are in the form 4k+3

W must therefore have a factor 4k+3 bigger than m in w if w is composite.

Our assumption in the beginning is therefore wrong and the list of primes in the form 4n+3 is infinite.

Proof 61: Prove that if $a^2 + b^2 = c^2$ and a,b,c integers greater than 0, that a and b cannot be both uneven.

Suppose a and b are both uneven.

We know that c is even and can therefore be written as $c = 2^n v$ so that $c^2 = 2^{2n} v^2$ where v is uneven and n >= 1.

a+b is also even and may be written as $a+b = 2^m w$, so that $(a+b)^2 = w^2 2^{2m}$ where w is uneven and m >=1

Then $(a + b)^2 = c^2 + 2ab$ and it is clear that a+b>c

So that $w^2 2^{2m} = v^2 2^{2n} + 2ab$

Therefore $w^2 2^{(2m-1)} = v^2 2^{(2n-1)} + ab$

But 2m-1>=1 so that the left hand side of the equation is even.

We see 2n-1>=1 and ab is uneven. Therefore the right hand side of the equation is an even integer plus an uneven integer ab giving an uneven result.

We have therefore that an even = uneven which is impossible and our original assumption that a and b are both uneven is therefore false.

Our original equation will therefore have that either a or b is uneven and the other even which will make c uneven.

Notice also that if a and b both even that all the terms in the equation can be divided by 4 until we get a term or terms that is/are uneven. We will end up again with an equation where either a or b is even and the other uneven so that c will always be uneven.

Proof 62: Prove that ab > a+b for a,b integer and a and b bigger than 2

The proof consists of three parts. The proof for when a>b , when b>a and when a=b

If a > b then a = b + x where x > = 1

So $a + b = 2b + x$ and $ab = (b+x)b = b^2 + bx$

Now b > 2 so $b^2 > 2b$, also b > 1 so bx > x

Therefore $b^2 + bx > 2b + x$ so that ab > a+b

The same argument suffice when b > a

If a = b then a+b = 2a and $ab = a^2$

Now $a > 2$ so that $a^2 > 2a$ and thus $ab > a+b$

In all three cases $ab > a+b$ and our case is proven.

Proof 63: Prove that there exists infinite integer solutions for $x^2 + y^2 = z^2$

We use plain old Algebra to prove the theorem.

$x^2 = (z-y)(z+y)$

Let $z-y = 1$ then $z+y = 2y+1$ so that $2y + 1 = x^2$

Therefore choose x^2 any uneven square > 1 Then $2y = x^2 - 1$ so that $y = (x^2 - 1)/2$

y is clearly integer as $x^2 - 1$ is even and 2 will therefore divide it.

Now $z = 1 + y = (x^2 + 1)/2$

z is clearly integer as $x^2 + 1$ is even and 2 will therefore divide it.

Therefore by choosing any uneven square $s > 1$ so that $x = \sqrt{s}$

We can deduce then y and z so that the equation $x^2 + y^2 = z^2$ holds true.

We know the list of squares is infinite and the solutions to the above equation must therefore also be infinite.

Case proven.

Example: Choose $x = 9$, then $x^2 = 81$ so that $y = 80/2 = 40$ and $z = 82/2 = 41$

Thus $9^2 + 40^2 = 41^2$

Proof 64: Prove that 2 = sqrt(1 + sqrt(1 + 2sqrt(1 + 3sqrt(1 + 4sqrt(1+…..)))))) where sqrt stands "for square root of"

Proof follows

2 = sqrt(4) = sqrt(1 + 1.3)

3 = sqrt(9) = sqrt(1 + 8) = sqrt(1 + 2.4)

4 = sqrt(16) = sqrt(1 + 15) = sqrt(1 + 3.5)

5 = sqrt(25) = sqrt(1 + 24) = sqrt(1 + 4.6)

6 = …

So in general we must proof that a = sqrt(1 + (a-1)(a+1))

Clearly this is so as sqrt(1 + (a-1)(a+1)) = sqrt(a.a) = a

Case proven.

Proof 65: Prove that it is meaningless to divide by 0

Let's suppose it is not meaningless to divide by 0 and that the division will deliver a number.

Now let x<>0.

Then x/0 = b so that 0(x/0) = 0.b

Now 0(x/0) = x(0/0) = x = 0.b = 0

(0/0) = 1 because any number divided by itself is one

0.b = 0 because any number multiplied by zero is zero.

But x = 0 cannot be because x<>0.

Our assumption is therefore wrong and it is meaningless to divide by 0

Proof 66: Prove that 0! = 1 given that (n+1)! = (n+1)n!

(n+1)n! = (n+1)!

Let n = 0

So (0+1)0! = (0+1)!

So (1)0! = 1! = 1

So 0! = 1/1 = 1

Case proven

Proof 67: Prove that every positive integer n > 4 can be written as the sum of at least two consecutive positive integers except for the case where n = 2^z

If n is uneven then n = 2k+1 = k + (k+1) so that n is clearly the sum of two consecutive positive integers.

If n is even then n = $w\,2^{v}$ where w is uneven and v >= 1

Let us form the sum S = a + (a+1) + (a+2) + .. (a+b)

Then 2S = (2a+b)(b+1) so that S = (2a+b)(b+1)/2

Let b be uneven so that b+1 is even and 2a+b uneven.

Now let 2^{v} = (b+1)/2 and let 2a+b = w so that S = n

So b = 2^{v+1} - 1 which is uneven. Clearly also b is integer and positive.

And a = (w-b)/2 so that w-b is even and two therefore divide w-b so that a is integer. In this case a may be negative as well as b may be bigger than w.

We can therefore form a summation of consecutive terms so that the sum will equal n.

If we have negative terms we can cancel them with the positive terms. The only problem will be when the negative terms cancel all the positive terms except the last term which in this case will obviously be a+b = n.

Let us suppose we have such a case. Then a + (a+b-1) = 0

So 2a +b = 1 , but w = 2a+b so w = 1 , but this cannot be because then n = 2^{v} and this is a contradiction because our proof stipulates that n cannot be power of two only.

Therefore we will always have that n is at least the sum of two consecutive positive integers as the last term a+b <> n if n is not purely a power of two.

Example

n = 10

Then n = 2.5 so that b = 2.2 - 1 = 3 , a = 1

Then 10 = a + (a+1) +…(a+b) = 1 + 2 + 3 + 4

Example

n = 12

Then n = 2.2.3 so that b = 2.2.2 -1 = 7 , a = (3-7)/2 = -2

So 12 = -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 = 3 + 4 + 5

Proof 68: Prove that pi does not contain itself

If pi contains itself it would look something like 3,1415……….31415 ……

We suppose that pi does contain itself.

Then 10^n times pi = 314159265……..3,1415926 …… where n is an integer > 1

So 10^n times pi = z + pi where z is also integer

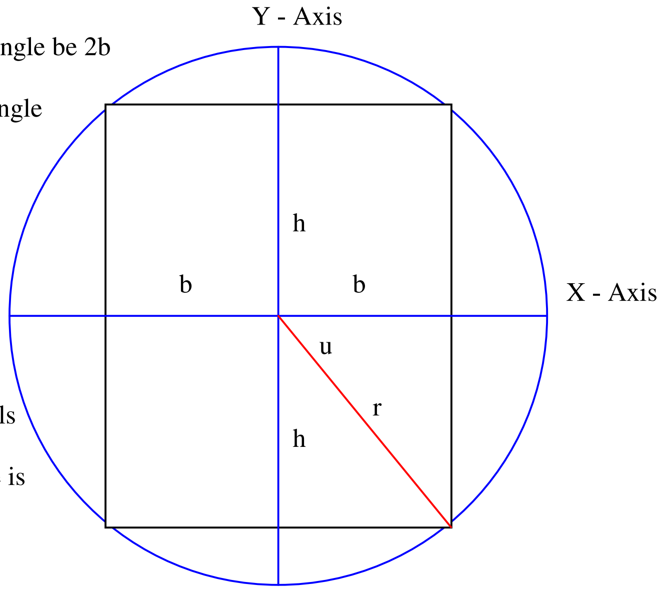
Therefore pi(10^n - 1) = z

And pi = z / (10^n - 1) which is a rational result.

We know however that pi is irrational and thus our assumption must be wrong and pi does not contain itself.

Proof 69: Prove that in a circle of radius r the biggest rectangle enclosed in the circle will have an area of $2r^2$

In the diagram let the radius of the circle be r.



Let the width of the rectangle be 2b

And the high of the rectangle

be 2h. The area of the

rectangle A = 4bh

Let the angle between

The x-axis and the red

line be u degrees in radials

The length of the red line is

r the radius of the circle.

Then cos(u) = b/r

Also sin(u) = h/r

So the area of the rectangle is $A = 4\,r^2 \cos(u)\sin(u)$

Now the maximum area of the rectangle will be where dA/du = 0

Now $dA/du = 4\,r^2 (\cos(u)\cos(u) - \sin(u)\sin(u)) = 4\,r^2 (1 - 2\sin(u)\sin(u)) = 0$

So 1-2sin(u)sin(u) = 0

So $\sin(u) = \sqrt{1/2}$

But if so then $\cos(u) = \sqrt{1/2}$

And thus $A = 4\,r^2 \sqrt{1/2}\ \sqrt{1/2} = 2\,r^2$

Case proven.

The percentage of area of the circle covered by this rectangle is therefore 2/pi = 63,66%

Proof 70: Prove that the square root of any prime number is irrational.

Let p be any prime. We assume that $\sqrt{p}$ is rational for some prime.

That is $\sqrt{p}$ = a/b where a and b are both integer and have no common factors and b is not 0.

Therefore $pb^2 = a^2$

Now a must have p as factor and a^2 must then at least have p^2 as factor. This however implies that b must have p as factor. If so then both a and b have p as factor and this contradicts our assumption in the beginning that a and b have no common factors. Therefore our assumption was wrong and the square root of any prime must be irrational.

Proof 71: Prove that given a and b integer and a,b<>0 that there is no integer solution for a^2 = b^2 - 2b

We assume that we have an integer solution.

Rewriting the equation we get that b^2 - a^2 = 2b

So b^2 - a^2 = (b+a)(b-a) = 2b

Therefore because 2b is even then b and a are both even or both are uneven.

This means b+a will always be even and also that b-a will always be even.

If so then (b+a)(b-a) = 4v = 2b , so b must be even and so must be a.

If b-a is even then we can write b-a = 2n where n<>0

Now if n = 1 then b-a = 2 so that b+a = b which cannot be as a<>0

So 1< n < -1

If b-a = 2n then a = b-2n

Also b+a = b/n so that b+a = b+b-2n = 2b-2n = b/n

Therefore n is a factor of b so that b/n is integer

So 2bn - 2n^2 = b , 2bn - b = 2n^2 , b(2n-1) = 2n^2

So b = 2n^2 /(2n-1)

Because n is a factor of b let b = nd where d is integer

So nd = 2n^2 / (2n-1)

Therefore d = 2n/(2n-1) = g/(g-1) = (j+1)/j = 1 + 1/j which is not integer

Except if j =1 so that g = 2 so that n=1 which n cannot be

We therefore have a contradiction and there is no integer solution for a^2 = b^2 - 2b

---

Proof 72: Prove that if m,n positive integers then there is no integer solution to 1/m+1/n+4/mn=3

There are none m,n to satisfy the equation

Proof

1/n +1/m + 4/(mn) = 3

let us suppose there are solutions for m=k

Then 1/k + 1/n + 4/(kn) = 3 so (4+k)/(kn) = 3 - 1/k = (3k-1)/k

So (4+k)/n = 3k-1 so that n = (4+k)/(3k-1)

Therefore 3k-1 must divide 4+k because n is integer

Now we find for which k is 3k-1>4+k

Therefore 3k-1>4+k so 2k>5 so k>5/2 so k>=3 because k is integer

Therefore for all k>=3 there is no solution because n will then be smaller than one

If k=2 then n = 6/5 which is not integer

If k=1 then n = 5/2 which is not integer

So there is no solution for any m

Case proven

Proof 73: Prove that a=b=c=d given that a^2 + b^2 + c^2 + d^2 = ab + bc +cd +da

Let z = a^2 + b^2 + c^2 + d^2 = ab + bc +cd + da

Let m = (a-b)^2 = a^2 -2ab + b^2

Let n = (b-c)^2 = b^2 -2bc + c^2

Let p = (c-d)^2 = c^2 -2cd + d^2

Let q = (d-a)^2 = d^2 -2da + a^2

Now m+n+p+q + 2(ab+bc+cd+da) = 2(a^2 + b^2 + c^2 + d^2) = 2z

So m+n+p+q + 2z = 2z

So m+n+p+q = 0

But m,n,p and q are all either 0 or positive because each of them is square

Therefore each of them must be zero

Therefore m = 0 = (a-b)^2 , so a-b=0 so a=b

The same argument follows for n,p and q so that b=c , c=d ,d=a

Which results in a=b=c=d

Case proven

Proof 74: Prove that given the general series 1,3,7,15,31…where the nth term is given by

$T_n = 2T_{n-1} + 1$ that in general $T_n = 2$^n - 1

Now T_1 = 2^1 -1 = 1 for n=1

Assume $T_k = 2^k - 1$ for n=k

Now we prove for n=k+1

So $T_{k+1} = 2^{k+1} - 1 = 2(2^k - 1) + 1 = 2T_k + 1$

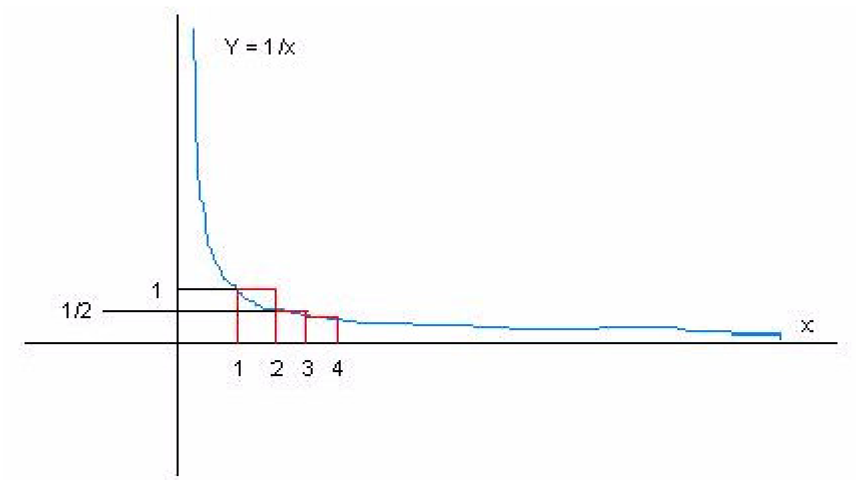
Let k+1=m

So $T_m = 2T_{m-1} + 1$

Case proven

Proof 75: Prove that the harmonic series 1 + 1/2 + 1/3 +1/4 +.... + 1/n >ln(n+1)

First have a look at the drawing below



The drawing is a representation of the function f(x) = 1/x

As can be seen the areas represented by the red rectangles is the start of the sum of the harmonic series 1 + 1/2 + 1/3 +....+ 1/n where each rectangle has a width of 1 and a high of 1/r at point x = r The last rectangle at point x = n has a width of 1 and stops therefore at x = n+1

The integral of y = 1/x from point x = 1 to point x = n+1 is the area under the curve y = 1/x from x = 1 to x = n+1.

As can be seen the integral area is smaller than that of the harmonic sum

So $1 + 1/2 + 1/3 + 1/4 + \dots 1/n > \int_1^{n+1} 1/x \, dx = \ln(n+1)$

So $1 + 1/2 + 1/3 + 1/4 + .. + 1/n > \ln(n+1)$

Numerical aproximation

N	sum	integral
10	2,9	2,3
100	5,1	4,6
1000	7,4	6,9
10000	9,7	9,2
100000	12,0	11,5
1000000	14,3	13,8

Clearly also is the fact that the harmonic series diverges as n tends to infinity as ln(n+1) tends to infinity as n tends to infinity

---

Proof 76: Prove that 1 + 1/4 + 1/9 + 1/16 +... + 1/n^2 < 2 - 1/n for n>1

For n = 2 the sum S = 1 + 1/4 = 1,25 which is smaller than 2 - 1/2 = 1,5

Let $S_n = 1 + 1/4 + 1/9 + 1/16 + \dots + 1/n^2$

We therefore assume that $S_n < 2 - 1/n$

We therefore have to prove that $S_{n+1} < 2 - 1/(n+1)$

Now $S_{n+1} = S_n + 1/(n+1)^2 < 2 - 1/n + 1/(n+1)^2$

But $1/[n(n+1)] > 1/(n+1)^2$

So $S_{n+1} < 2 - 1/n + 1/[n(n+1)] = 2 - [n+1 - 1]/[n(n+1)] = 2 - n/[n(n+1)] = 2 - 1/(n+1)$

Case proven

Proof 77: Prove an answer for z = 1/3 + 2/9 + 3/27 + 4/81 +........

We rewrite z as follows

z = (1/3 + 1/9 + 1/27 + 1/81+....) + (1/9 + 1/27 + 1/81 =......) + (1/27 + 1/81 +...) + ...

So $z = z_1 + z_2 + z_3 + z_4 + ...$

Where z_1 = 1/3 + 1/9 + 1/27 + 1/81 + ... = 1/2 using the formula for the geometric series

Where z_2 = 1/9 + 1/27 + 1/81 + ... = 1/2 - 1/3 = 1/6

Where z_3 = 1/6 - 1/9 = 1/18 and z_4 = 1/18 - 1/27 = 1/54

So z = 1/2 + 1/6 + 1/18 + 1/54 +... = 0.5(1 +1/3 + 1/9 + 1/27 +....) = 0,5(1 + z_1) = 1/2 + 1/4

So z = 3/4

So 1/3 + 2/9 + 3/27 + 4/81 + ... = 3/4

Case proven

Numerical Aproximation

Terms	z
1	0,333333
5	0,736
10	0,7499
15	0,7499999

Proof 78: Prove that the product of four consecutive integers where non of them are zero cannot be a square

Let the product be z = n(n+1)(n+2)(n+3)

So $z = n^4 + 6n^3 + 11n^2 + 6n$

But $z+1 = (n^2 + 3n + 1)^2 = n^4 + 6n^3 + 11n^2 + 6n + 1$

So z+1 is square and therefore clearly z cannot be square.

Lets assume that z can be square if z+1 is square.

Let $z+1 = m^2$ and let $z = v^2$ where m and v non zero integers

So $1 = m^2 - v^2 = (m+v)(m-v)$

In this case both (m+v) and (m-v) must then be 1 or -1

So m+v = m-v

But this cannot be and our assumption that z is square must then be wrong

Case proven

Proof 79: Prove x algebraically for $x^4 + x^3 + x^2 + x + 1 = 0$

Divide by x^2

Then $x^2 + x + 1 + x^{-1} + x^{-2} = 0$ eq1

Now let $u = x + 1/x$, so $u^2 = x^2 + 2 + x^{-2}$

Writing eq1 in another way: $(x^2 + 2 + x^{-2}) -1 + (x + 1/x) = 0$

Substituting we get $u^2 + u -1 = 0$

Now solve for u we get that $u = (-1 \pm \sqrt{5})/2$

This we can then use to solve for x + 1/x = u to get x

Case proven

Another approach is to introduce an extra root as follows.

We introduce x-1 , so that $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

Now the above results into $x^5 - 1 = 0$, so that $x^5 = 1$

This then can be easily solved using complex algebra.
We will get five roots of which the root x=1 will then not be part of the answer as this root was introduce.

Proof 80: Prove that it is possible to reduce the general cubic equation $Ax^3 + Bx^2 + Cx + D = 0$ to a reduced equation $u^3 + u + a = 0$ where u = f(x) and a = f(A,B,C,D)

First divide by A , so we have $x^3 + (B/A)x^2 + (C/A)x + D/A = 0$

Let B/A = q , C/A = w , D/A = e

Then we have $x^3 + qx^2 + wx + e = 0$ eq1

Let $x = x_1 + r$, then $x^2 = x_1^2 + 2rx_1 + r^2$ and $x^3 = x_1^3 + 3rx_1^2 + 3x_1r^2 + r^3$ eq2

Substitute eq2 into eq1

The second power term then changes to $qx_1^2 + 3rx_1^2$

We then make this term zero by letting q + 3r = 0
So r = -q/3 , so $x = x_1 - q/3$

We are then left with $x_1^3 + tx_1 + p = 0$

Let $x_1 = su$ where u is the variable and s the constant

Then we have that $(su)^3 + tsu + p = 0$

So $u^3 + tsu/s^3 + p/s^3 = 0$

Let $ts/s^2 = 1$, so $s = \sqrt{t}$

And let $p/s^3 = a$

We then have that $u^3 + u + a = 0$

So $x_1 = u\sqrt{t}$ and $x = x_1 + r = u\sqrt{t} - q/3$

So by letting $x = u\sqrt{t} - q/3$

We get the reduced form of $u^3 + u + a = 0$

It is left to the readers to get the value of t and of a

Proof 81: Prove an exact solution for $x^3 + x + a = 0$

Let $x = A + B$

Then $x^3 = A^3 + B^3 + 3AB(A+B)$

So $A^3 + B^3 + 3AB(A+B) + A+B + a = 0$

Let $A^3 + B^3 = -a$ so that $3AB(A+B) + (A+B) = 0$

Divide by $A+B$, then $3AB + 1 = 0$

So $A = -1/B$ and therefore $(-1/B)^3 + B^3 = -a$

So $B^6 + aB^3 - 1 = 0$

So $B^3 = [-a \pm \sqrt{a^2 + 4}\,] / 2 = -a/2 \pm \sqrt{(a/2)^2 + 1}$

So $B = \left[-a/2 + \sqrt{(a/2)^2 + 1} \right]^{(1/3)}$ choosing the plus in place of the minus. Choosing the minus will deliver the same results in the end.

But $A^3 + B^3 = -a$, so $A^3 = -a - B^3 = -a + a/2 - \sqrt{(a/2)^2 + 1} = -a/2 - \sqrt{(a/2)^2 + 1}$

So $x = A+B = \left[-a/2 - \sqrt{(a/2)^2 + 1}\right]^{(1/3)} + \left[-a/2 + \sqrt{(a/2)^2 + 1}\right]^{(1/3)}$

Proof 82: Prove that $\binom{m}{n}$ is always integer

Now $\binom{m}{n} = m!/[n!\,(m-n)!]$

To prove this we are going to prove that for any prime p , that m! always has more prime factors p than n! (m-n)! so that n! (m-n)! will always divide m!

We now define two functions

The first function I(x) where I(x) returns the integer part of x

Example : I(2) = 2 , I(3,24) = 3 and so on

Now it is clear that $I(a-b) \le I(a) - I(b)$, if not proof follows

Let $a = c + d/10$ and let $b = e + f/10$ where $c > e$

If we have that $d<f$ we have that $I(a-b) = I(c-1-e + (10d - f)/10) = c-e-1$ whereas $I(a) - I(b) = c-e$

In this case we see that $I(a-b) < I(a) - I(b)$

If we have the case where $d>f$ we have that $I(a-b) = I(c-e + (d-f)/10) = c-e$ whereas

$I(a) - I(b) = c-e$

In this case we see that $I(a-b) = I(a) - I(b)$

Example I(3,3 – 1,4) = I(1,9) = 1 and I(3,3) – I(1,4) = 3 – 1 = 2

The second function we introduce is $X_p(k)$ and this function delivers the number of prime factors p in k

For example $X_3(8) = 0$, but $X_2(8) = 3$ and $X_3(6) = 1$

Now $X_p(k!) = I(k/p) + I(k/p^2) + I(k/p^3) + \ldots$ until $p^j > k$ because all $I(k/\ldots)$ after that will deliver 0

If the above is not clear proof follows

$k! = 1*2*3*4*5*6\ldots k$

Therefore we check if p divides k! and then if 2p divides k! and then 3p and 4p and so on, so that we have only to see how many times p divides k to get all the p's and that we get from $I(k/p)$. If $w*p < k$
And $(w+1)p>k$ then we know that p, 2p, 3p.. up to wp is in k!

But we also have p^2 factors and the same goes for them. We see that any p^2 factor has already been detected by $I(k/p)$ because $p^2 = p*p$, and so to add the primes p in p^2 factors up we take $I(k/p^2)$
And not $2*I(k/p^2)$. The same argument goes for higher order exponents of p

Now back to the original proof

We must therefore proof that $X_p(m!) >= X_p(n!) + X_p[(m-n)!]$

$X_p(m!) = I(m/p) + I(m/p^2) + I(m/p^3) + \ldots$

$X_p(n!) = I(n/p) + I(n/p^2) + I(n/p^3) + \ldots$

$X_p[(m-n)!] = I([m-n]/p) + I([m-n]/p^2) + \ldots = I(m/p - n/p) + I(m/p^2 - n/p^2) + \ldots$

So $X_p[(m-n)!] <= I(m/p) + I(m/p^2) + \ldots - [I(n/p) + I(n/p^2) + \ldots]$

So $X_p[(m-n)!] + X_p(n!) <= X_p(m!)$

Case proven

Proof 83: Prove that if a == b mod (c) and d == e mod (c) ,that (a+d) == (b+e)mod(c) with a, b, c, d, e all integer

Let M = (a-b)/c and N = (d-e)/c wher M and N integer

So M+N = [(a-b) + (d-e)]/c = [(a+d) - (b+e)]/c which is also integer

So c divides (a+d) - (b+e) and thus (a+d) == (b+e) mod(c)

Case proven

Proof 84: Prove that if a == b mod(c) and d == e mod(c) , that ad == be mod(c) with a, b, c, d, e all integer

Let M = (a-b)/c and N = (d-e)/c where M and N integer

Now ad/c = (a-b+b)d/c = d(a-b)/c + bd/c = Md + bd/c

But (ad - be)/c = ad/c - be/c = Md + bd/c - be/c = Md + b(d-e)/c = Md + bN which is integer, so

ad == be mod (c)

Case proven

Proof 85: Prove that if a == b mod(c) and d | c that a == b mod(d) given a, b, c, d, e all integer

Let M = (a-b)/c and N = c/d where M and N integer

Then MN = (a-b)/c times c/d = (a-b)/d which is integer

So a == b mod (d)

Case proven

Proof 86: Prove that between any two different rational numbers there is an irrational number

We prove by construction

Let a and b be both rational

Let b > a

Then b-a > 0

So b-a > (b-a)/ $\sqrt{2}$ > 0

So b > a + (b-a)/ $\sqrt{2}$ > a

But the middle term is clearly irrational

Case proven

Proof 87: Proof that for a positive integer n (at least three) each of the numbers n!+2, n!+3,...n!+n is divisible by a prime which doesn' t divide any of the others.

.

Let n >= v > w

We know that n!+v has a factor v and also prime factor(s) bigger than n

Let this prime factor be n+m

We know that n!+w has a factor w and also prime factor(s) bigger than n

Now suppose this prime factor is also n+m

Then we can say that n! = -w mod(n+m) and n! = -v mod(n+m)

So 0 = (-w+v) mod (n+m)

So n+m must divide v-w, but both v and w is smaller than n w-v < n ,so (w-v)/(n+m) is not integer

Thus the prime factors cannot be the same. Case proven

Proof 88: Proof that 5 divides 6^n -1 for n > 0

How about the following induction proof

We know 5 | 6^1 - 1

We suppose then that 5 | 6^n -1 and we have to prove that 5 | 6^(n+1) -1

We therefore have that 6^n = 1 mod 5
and 6 = 1 mod 5

So we have that (6*6^n) = (1*1) mod 5

And thus 6^(n+1) = 1 mod 5 and it follows that 5 divides 6^(n+1) - 1

Case proven

Proof 89: Prove that if $x+1$ and $x-1$ are prime and $x > 4$ that 6 divides x.

2 always divide x as x is even , so we must show that 3 divides x

If $x-1$ is divided by 3 it will either have a remainder of 2 or of 1

So $x-1 == 1 \bmod 3$ or $x-1 == 2 \bmod 3$

We know $2 == -1 \bmod 3$

If $x-1 == 1 \bmod 3$ then $x-1+2 == 1-1 \bmod 3$

So $x+1 == 0 \bmod 3$ which is impossible as $x+1$ is prime<>3 and cannot therefore be divided by 3

So $x-1$ <> $1 \bmod 3$ and the only possibility is $x-1 == 2 \bmod 3$

We know $1 == -2 \bmod 3$

So $x-1+1 == 2-2 \bmod 3$ so $x == 0 \bmod 3$ and thus $3 | x$

So 2 and 3 divides x and thus 6 divides x

Case proven

Proof 90: Prove that for $n>0$ it is possible to find x,y,z such that $x^2 + y^2 = z^n$

Let $c^2 = z$

Clearly $x^2 + y^2 = z^1$ and it is true for $n = 1$

We assume it is true for $n = k$

So $x_1^2 + y_1^2 = z^k$

So we must prove that for some x,y we have $x_2^2 + y_2^2 = z^{(k+1)}$

Now $c^2 (x_1^2 + y_1^2) = z*z^k = z^{(k+1)}$

Let $x_2 = cx_1$ and let $y_2 = cy_1$

So $x_2^2 + y_2^2 = z^{(k+1)}$

Case proven

Proof 91: Prove that 12 divides (b-a)(c-a)(d-a)(c-b)(d-b)(d-c) given a,b,c,d are integers

Let N = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)

Note that each integer is involved with each other integer in subtractions and represented as factors of N. So if a factor is subtracted from another factor we again get a factor represented allthough not always with the same sign.

If we divide the factors by 4 we could have remainders of 0 , 1 ,2 and 3

We have 6 factors and only 4 remainders, so at least 2 factors must have the same remainders.

Let us name the factors f,g,h,I ,j and k and let f and g have the same remainder

So we have f == x mod 4 and g == x mod 4

So f-g == 0 mod 4 , but f-g is on of the other factors , maybe not with the same sign and thus 4 divides that factor and so 4 divides N

If any integer a,b,c,d are divided by 3 we have a choice of remainders of 0,1 and 2
So of these four integers at least two have the same remainder when divided by 3
Lets suppose its b and c

So b == x mod 3 and c == x mod 3
Then b-c == 0 mod 3 , but as we noted each integer is involved with each integer as a factor and therefore b-c is a factor and is divided by 3 so that N is divided by 3

Therefore 12 divides N

Case proven

Proof 92: Prove that if p prime that p is never congruent to b mod(c) if b | c , p<>b<>1

We see b divides c , so let bd = c

Suppose $p == b \bmod(c)$
Let $M = (p-b)/c$, so $Mc = p - b$

So $p = Mc + b = Mbd + b = b(Md + 1)$

So p is composite, but that is not true and so p is not congruent to b mod(c)

Case proven

Proof 93: Prove a solution in x for $ax^2 + bx + c = 0$

We define two real numbers $u<>0$ and $s <>0$ and let i = square root of -1

Let $x = u + is$
So $x^2 = u^2 - s^2 + 2usi$

Substitute and we get that $a(u^2 - s^2 + 2usi) + b(u + is) + c = 0$

It is clear that the real part of the equation = 0 and that the imaginary part =0

So we have two equations

$a(u^2 - s^2) + (bu + c) = 0$ eq 1

and $2aus + bs = 0$ eq 2

So $2aus = -bs$, $2au = -b$, $u = -b/(2a)$ from eq 2

And $as^2 = au^2 + bu + c$ from eq 1

So $s^2 = u^2 + bu/a + c/a = b^2/(4a^2) - b^2/(2a^2) + c/a$

So $-s^2 = b^2/(2a^2) - c/a = (1/4a^2)(b^2 - 4ac)$

So $is = \pm(1/2a)\sqrt{b^2 - 4ac}$

So $x = [-b \pm \sqrt{b^2 - 4ac}\ ] / (2a)$

Case proven

Proof 94: Prove that the smallest value for $x^2 + y^2 = 36/13$ under the restriction $3y + 2x = 6$

The restriction delivers then that $y = 2 - 2x/3$

So $f(x) = x^2 + y^2 = x^2 + (2 - 2x/3)^2 = x^2 + 4 + 4(x^2)/9 - 8x/3$

Now for a minimum of f(x) we let $f'(x) = 0$

So $2x + 8x/9 - 8/3 = 0$

Thus $x = 12/13$ and $y = 2 - 2x/3 = 18/13$

So $x^2 + y^2 = (12/13)^2 + (18/13)^2 = 36/13$

Case proven

Proof 95: Prove that ln(i) = (pi/2)i

Clearly $i = e^{(i \pi/2)}$

So $\ln(i) = \ln[e^{(i \pi/2)}] = i \cdot \pi/2$

Case proven

Proof 96: Prove that the square root of i is a complex number

Now $i = e^{(i \cdot \pi/2)}$

So $\sqrt{i} = (i)^{(1/2)} = e^{(i \cdot \pi/4)}$

We know that $e^{(ix)} = \cos(x) + i \sin(x)$

So $e^{(i \cdot \pi/4)} = \cos(\pi/4) + i \sin(\pi/4) = 1/\sqrt{2} + i/\sqrt{2}$

So $\sqrt{i}$ is complex

Case proven

Proof 97: Prove in Boolean Algebra that $x'y + x'y' + xy' = x' + y'$

We know that $x + x' = 1$, $xx' = 0$, $xx = x$, $x + 1 = 1$ and so on

$x'y + x'y' + xy' = x'(y + y') + xy' = x' + xy'$

So if $x = 0$ then $x' + xy' = 1 = x'$

And if $x = 1$ then $x' + xy' = 1*y' = y'$

So for all x then $x' + xy' = x' + y'$

Proof 98: Prove that the ln(-3) exists

Let Z = ln(-3)

So Z = ln(-3) = ln(-1 * 3) = ln(3) + ln(-1)

Now -1 = e^(i*pi)

So Z = ln(3) + ln[e^(i*pi)] = ln(3) + i*pi

So Z is a complex number and therefore exists

Case proven

Proof 99: Prove that God has a paradoxical property

God is Almighty.

God can make anything and God can do anything.

God can make a very big stone named X.

God can make X so big, that he cannot move it.

If God cannot move X then its something He cannot do. God however can do anything.
So we have a paradox.

Case proven.

Proof 100: Prove that the male barber who shaves all the men who don't shave themselves went crazy

If the barber did not shave himself , then he had to shave himself.

If the barber shaved himself, he did not shave himself.

The barber went crazy.

Proof 101: Prove an integer x such that $x^4 - 2x^3 + 4x^2 - 6x + 3$ is a square

Let $Z = x^4 - 2x^3 + 4x^2 - 6x + 3$

Then we can rewrite Z as $(x-1)^2 * (x^2 + 3)$

Therefore the first factor is always square and so the second factor $x^2 + 3$ must also be always square for Z to be square.

For $x>1$ clearly $x^2 + 3$ is never square

For $x = 1$ the factor = 4 , but then the square factor = 0

For $x = 0$, $Z = 3$

For $x = -1$ we have the only solution so that $Z = 16$

For all $x < -1$ the second factor $x^2 + 3$ is always not square

So the only solution is when $x = -1$

Case proven

Proof 102: Prove that the product of n consecutive integers is dividable by n!

Let $z = (s+1)(s+2)(s+3).....(s+n)$

The $z = (s+n)! / (s!)$

So $z / n! = (s+n)! / [(n!) (s!)] = \binom{s+n}{n}$ which is a integer

So $n! \mid z$

Proof 103: Prove that given Wilson' s theorem[p if prime divides (p-1)! + 1] that p also divides (p-2)! - 1

Let z = (p-1)! + 1

Now we know that p | z

So z = (p-1)(p-2)! + 1 = (p-1)(p-2)! - (p-1) + 1 + (p-1)

So z = (p-1)[(p-2)! - 1] + p

Clearly p divides p and p does not divide p-1 , so p must divide (p-2)! - 1 as p divides z

Case proven

Proof 104: Let S(n) = 1^x + 2^x + 3^x +..+n^x where x >= 0 and x be from the integer family. Then S(n) can be written as a polynomial f(n) with its highest power x+1 and this polynomial can be deduced in a very unique way.

Let S(n) = 1^x + 2^x + 3^x + ... n^x

Then S(n) can be written as a polynomial f(n) if we can find such a polynomial such that

S(n+1) = S(n) + (n+1)^x = f(n) + (n+1)^x

If this can be done , then this polynomial f(n) is a solution.

This can be done for all x such that x>=0 and x integer.

We can find such a polynomial if we let S(n) = an^(x+1) + bn^x + cn^(x-1)+.....

Then S(n+1) = a(n+1)^(x+1) + b(n+1)^x + c(n+1)^(x-1) + ...

So S(n+1) = an^(x+1) + bn^x + cn^(x-1) +...... + a(x+1)n^x +..........

So S(n+1) = S(n) + (a polynomial with its highest power x) = S(n) + g(n)

This polynomial g(n) is then equal to (n+1)^x so that values can be worked out for a,b,c.....

Consider the case for x=2

In our case S(n) = 1 + 2^2 + 3^2 +..+n^2

Let S(n) = an^3 + bn^2 + cn + d

Now if we can find values for a, b, c, d such that S(n+1) = S(n) + (n+1)^2
then we are done.

Then S(n+1) = S(n) + (n+1)^2 = S(n) + (n^2 + 2n + 1)

So S(n+1) = a(n+1)^3 + b(n+1)^2 + c(n+1) + d = (an^3 + bn^2 + cn + d) +
[3an^2 + n(3a+2b) + (a+b+c)]

Therefore [3an^2 + n(3a+2b) + (a+b+c)] = (n^2 + 2n + 1)

So 3a = 1 and a = 1/3

So 3a+2b=2 , 1+2b=2 , 2b=1 , b=1/2

So a+b+c = 1 , 1/3 + 1/2 + c = 1 , c=1/6

Therefore S(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n + d

But S(1) = 1 = (1/3) + (1/2) + (1/6) + d and therefore d=0

So S(n) = f(n) = (1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6)(n+1)(2n+1)

Case proven.

Proof 105: Prove that if p prime that p divides [n^p + (p-1)n] for n >= 1

Let z(n) = n^p + (p-1)n

For n = 1 we have that z(1) = 1 + p -1 = p and p clearly divides p

So we assume that p divides z(k) and we now prove that p also divides z(k+1). If so then are proof is done.

Now z(k) = k^p + (p-1)k

So z(k+1) = (k+1)^p + (p-1)(k+1)

Using the binomial theorem (k+1)^p = k^p + mp + 1

Therefore $z(k+1) = k^p + mp + 1 + k(p-1) + (p-1) = k^p + k(p-1) + mp + p$

So $z(k+1) = z(k) + (m+1)p$

But p divides z(k) and p divides (m+1)p and so p divides z(k+1)

And our case is proven.

Proof 106: Prove that the product of two consecutive non zero integers can never be a power.

Let the two numbers be x and x+1

Now let us suppose that their product is a power.

So $x(x+1) = c^z$

Let y divide x , then y does not divide x+1. So x and x+1 have no common factors.

Clearly then $x = a^z$ and $x+1 = b^z$, so that ab = c

Now if $x = a^z$ then $x+1 = a^z + 1 = b^z$

Clearly b > a , so b >= a + 1 , so $b^z = a^z + 1 >= (a+1)^z = a^z + ka + 1$

So 0 >= ka which is not possible

Our assumption is therefore wrong and $x(x+1)<> c^z$

Case proven

Proof 107: Prove that if a==1 mod 3 and b==2 mod 3 , then that 9 divides $a^3 + b^3$

First note that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Now a+b == 3 mod 3 , so a+b == 0 mod 3 , so 3 divides a+b

Also a^2 ==1 mod 3 and b^2 == 4 mod 3 and ab == 2 mod 3

So $a^2 + b^2 - ab$ == (5-2) mod3 , so $a^2 + b^2 -ab$ == 3 mod 3 , so $a^2 + b^2 - ab$ == 0 mod 3

So 3 divides $a^2 -ab + a^2$

But then 9 divides $(a+b)(a^2 +b^2 - ab) = a^3 + b^3$

Case proven

Proof 108: Prove that n! can never be equal to x^y for y>1 and n>1

We know n! = 1 times 2 times 3 times ...times p times....times n

If n is prime the proof is done. If not then n is composite.
So if n is composite let p be the biggest last prime in the product series. There are no multiples kp of p which is bigger than p and smaller than n. If so then a prime would exist between p and kp according to the theorem that there exists at least one prime between a number and its double. But p is the largest prime in the product series. So p divides n! only once and thus n! is not a power.

Proof 109: Prove that for every integer n there is at least n consecutive non prime numbers

We prove by construction.

Let z_1 = (n+1)! + 2
Let z_2 = (n+1)! + 3
.
.
.Let z_n = (n+1)! + (n+1)

We see that for each z_i that i+1 divides it.
But there are n such numbers and all of them are composite
Case proven

Proof 110: Prove that if a==b mod c and b == d mod c then a == d mod c

If b == d mod c then clearly d == b mod c

So (a-d) == (b-b)mod c

Therefore (a-d) == 0 mod c

and thus a == d mod c

Case proven

Proof 111: Prove a solution for n so that 2^1994 + 2^1999 + 2^2000 + 2^2001 + 2^n is square

Let us factor the summation first and let this summation be z

So z = 2^1994 (1 + 2^5 + 2^6 + 2^7 + 2^(n-1994))

Let n -1994 = x

Clearly 2^1994 is square

So z = 2^1994 (225 + 2^x)

Therefore we want 225 + 2^x to be square

So let y^2 = 225 + 2^x , so 2^x = y^2 - 225 = (y+15)(y-15)

So both y+15 and y-15 must have only 2 or 1 as factors

If we let 2 = y -15 then y = 17 and then y+15 = 32

So y = 17 is a solution. If y =17 then 2^x = 32 * 2 = 64 and then x = 6

If x =6 then n = x + 1994 = 2000

So n = 2000 is a solution

With n = 2000 we see that 225 + 2^x = 225 + 2^6 = 225 + 64 = 289 = 17*17

Case proven

Proof 112: Prove that the product of four consecutive integers plus one is always a square

Let z = n(n+1)(n+2)(n+3) + 1

Then z = n^4 + 6n^3 + 11n^2 + 6n + 1

Now let us try to find a product of two factors so that the result is z

Let x = n^2 + an + 1

Then $x^2 = n^4 + (2a)n^3 + (2+a^2)n^2 + (2a)n + 1$

Now let a = 3 , then $x^2 = n^4 + 6n^3 + 11n^2 + 6n + 1 = z$

So $z = x^2$ and thus z is the square of x or the square of $n^2 + 3n + 1$

So the product of four consecutive integers plus one is always a square.

This result can then also be used to prove that the product of four consecutive integers can never be a square.

Case proven.

Proof 113: Prove that 3 divides 10^x + 5 for all x >= 0

We proof by induction

We first note that for x = 0 that 3 divides 1+5 and for x = 1 that 3 divides 10+5

So now we assume that 3 divides 10^x + 5 and we prove that 3 divides 10^(x+1) + 5

Therefore using our assumption we have that 10^x == -5 mod 3

Also 10 == 10 mod 3 , so 10^(x+1) == -50 mod 3

But 3 divides 45 so 0 == 45 mod 3

So 10^(x+1) == (-50+45) mod 3 , so 10^(x+1) == -5 mod 3

So 3 divides 10^(x+1) + 5

Case proven

So 3 divides 6 , 15 , 105 , 1005 , 10005 , 100005 and 1000........0000005 and so on

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Proof 114: Prove that 1-1+1-1+1-1+1-1+1-1+1........ has no definite answer

First we write the sequence as follows

S = (1-1) + (1-1) + (1-1) + (1-1) + ... = 0 + 0 + 0 + ... = 0

But the sequence can also be write as

S = 1 + (-1+1) + (-1+1) + (-1+1) = 1 + 0 + 0 + 0 = 0 = ... = 1
So S has two values and this cannot be. So this sequence has no definite answer.

Proof 115: Prove that for n odd and x == 5^n mod 8 that x == 5 mod 8

Given the above it follows that x = 5^n + 8k

So x - 5 = 5^n - 5 + 8k

So x - 5 = 5(5^(n-1) -1) + 8k = 5*4(5^(n-2) + 5^(n-3) + ...+ 5 + 1) + 8k = 5*4*h + 8k

Clearly h is even because n is odd . So h = 2g

So x -5 = 5*4*2g + 8k = 5*8g + 8k = 8(5g + k) = 8f

So 8 divides x-5 , so x == 5 mod 8

Case proven

Proof 116 Prove given p prime and p == 5 mod 8 , that p is not congruent to 3 mod 8

Let us assume that p == 3 mod 8

So we have that p == 5 mod 8 and p == 3 mod 8

So 2p == (5+3) mod 8 , which implies that 2p == 8 mod 8 , so that 2p == 0 mod 8

Therefore p == 0 mod 4

Therefore p = 2*2*k , but this cannot be as p is prime and is therefore not composite.

So our assumption is wrong and p is therefore not congruent to 3 mod 8

Case proven

Proof 117: Prove a general solution for $T_{n+1} = 2T_n^2 - 1$, given $T_1 = 2$

We notice that if $T_1 = 2$ then $T_2 = 7$ and $T_3 = 97$

Let $T_n = \cosh(U_n)$ be a guess as to the solution where $\cosh(x) = (e^x + e^{-x})/2$

So $T_{n+1} = 2T_n^2 - 1 = 2[\cosh(U_n)]^2 - 1 = \cosh(2U_n)$

But $T_n = \cosh(U_n)$, so $T_{n+1} = \cosh(U_{n+1})$

Therefore $U_{n+1} = 2U_n$

So we now have a solution for T in terms of U
If we now can get a solution for U in terms of n we are finished.

Now $T_1 = 2 = \cosh(U_1)$, so $U_1 = \text{arccosh}(2)$

$U_2 = 2U_1 = 2\text{arccosh}(2)$

$U_3 = 2U_2 = 4\text{arccosh}(2)$

and so on

So in general $U_{n+1} = (2^n)\text{arccosh}(2)$

and we have found a solution for U in terms of n and therefore also for T in general

We know that $T_n = \cosh(U_n)$

So $T_{n+1} = \cosh(U_{n+1})$

So $T_{n+1} = \cosh(2^n[\text{arccosh}(2)])$

and $T_n = \cosh(2^{[n-1]} \cdot \mathrm{arccosh}(2))$

Therefore $T_3 = \cosh(4\mathrm{arccosh}(2)) = \cosh(5{,}26783..) = 97$ and so on

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Proof 118: Prove an answer for a^2 - b^2 given a - b = 1 and ab = 1

Note that a^2 - b^2 = (a+b)(a-b) = a+b

Note that (a-b)^2 = a^2 - 2ab + b^2 = 1^2 = 1

So a^2 + b^2 = 1 + 2ab = 1 + 2 =3

Note also that (a+b)^2 = a^2 + b^2 + 2ab = 3 + 2 = 5

So $a+b = \sqrt{5}$

Therefore $a^2 - b^2 = a+b = \sqrt{5}$

Proof 119: Prove that 1/sqrt(1) + 1/sqrt(2) + 1/sqrt(3) +...+ 1/sqrt(n) > sqrt(n) for n>=2

We are going to use induction to prove the above.

For n = 2 we have that 1/sqrt(1) + 1/sqrt(2) = 1,707..

And sqrt(2) = 1,414.. and it is clear that 1,707 > 1,414

So we assume the relation holds for n and thus S_n = 1/sqrt(1) + 1/sqrt(2) +.. + 1/sqrt(n) > sqrt(n)

We therefore have to proof that $S_{n+1} = S_n$ + 1/sqrt(n+1) > sqrt(n+1)

Using our assumption we have $S_{n+1} = S_n$ + 1/sqrt(n+1) > sqrt(n) + 1/sqrt(n+1)

So if we can prove that sqrt(n) + 1/sqrt(n+1) > sqrt(n+1) we are done.

Clearly as n is integer $n^2 + n > n^2$

So $\sqrt{n^2 + n} > n$

And $\sqrt{n}\,\sqrt{n+1} > n$

Thus $\sqrt{n}\sqrt{n+1} + 1 > n+1$

And $\sqrt{n} + 1/\sqrt{n+1} > (n+1)/\sqrt{n+1}$

So $\sqrt{n} + 1/\sqrt{n+1} > \sqrt{n+1}$ and we are done

Proof 120: Prove that $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$ for $a,b \geq 1$

$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} > a+b$

So $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$

Case proven

Proof 121: Prove that the sum $\left[\sum_{r=0}^{n} \binom{n}{r} \right] = 2^n$ for $n > 1$

Let S_n be the sum that' s being asked

For S_2 we have $1 + 2 + 1 = 4 = 2^2$ with $n = 2$

For S_3 we have $1 + 3 + 3 + 1 = 8 = 2^3$ with $n=3$

For S_4 we have $1 + 4 + 6 + 4 + 1 = 16 = 2^4$ with $n=4$

The above triangle of numbers is also known as pascal' s triangle.

So the theorem looks true for $n \leq 4$. We now assume that the relation holds for n-1 and we prove that it holds for n

We therefore assume that $S_{n-1} = 2^{(n-1)}$

When we look at n=4 we see that each term in the row is given by the sum of the two terms in the previous row n=3 to the left and to the right of the term. That is 4 = 1+3 , 6 = 3+3 , 4=3+1 etc. Note however that the end terms in each row are always one.

So for a term i in row n we have that $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$

We can now write down the summation for row n

$$S_n = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n-1} + 1$$

But the above term can be written in terms of the previous terms as follows

$$S_N = 1 + \left[\binom{n-1}{0} + \binom{n-1}{1} \right] + \left[\binom{n-1}{1} + \binom{n-1}{2} \right] + \left[\binom{n-1}{2} + \binom{n-1}{3} \right] + ... + \left[\binom{n-1}{n-2} + \binom{n-1}{n-1} \right] +1$$

Note that $1 = \binom{n-1}{0} = \binom{n-1}{n-1}$

So we see that each of the terms of row n-1 appears twice in the summation and

Thus $S_n = 2 S_{n-1}$

But S_{n-1} = 2^(n-1) , so S_n = 2 times 2^(n-1) = 2^n

Case proven

Proof 122: Prove that n^5 == n mod 30 for n integer and n > 0

Note that 30 = 2 times 3 times 5

So if we can prove that firstly n^5 == n mod 2 and secondly that n^5 == n mod 3 and lastly that n^5 == n mod 5 then we are done , because 2 , 3 and 5 all have no common factors as they are all prime numbers.

So firstly we look if n^5 == n mod 2

If n is even or uneven then S = n^5 - n is even and then 2 divides S

So n^5 == n mod 2

S = n^5 - n = n(n^4-1) = n(n^2+1)(n^2-1) = n(n^2 + 1)(n -1)(n+1)

Secondly we must have investigate n^5 == n mod 3

So if 3 divides any of the factors of S then indeed n^5 == n mod 3

There are three cases to consider here

Case 1 Either n==0 mod 3
Case2 or n==1 mod 3
Case3 or n==2 mod 3

Case 1: 3 divides n , so 3 divides the factor n , so 3 divides S

Case 2: n==1 mod 3 , so n-1 == 0 mod 3 , so 3 divides the factor n-1 , so 3 divides S

Case 3: n==2 mod 3 , so n+1 == 3 mod 3 , so n+1 == 0 mod 3 , so 3 divides the factor n+1 , so 3 divides S

Therefore in all three cases 3 divides S , so n^5 == n mod3

Thirdly we must investigate n^5 == n mod 5

There are 5 cases to consider here

Case 1 Either n ==0mod5
Case 2 or n==1mod5
Case3 or n==2mod5
Case4 or n==3mod5
Case5 or n==4mod5

Case 1: 5 divides n , so 5 divides S

Case 2: n==1mod5 , so n-1==0mod5 , so 5 divides n-1 , so 5 divides S

Case 3: n==2mod5 , so n^2==4mod5 , so n^2+1==5mod5 , so n^2 + 1==0mod5 , so 5 divides the factor n^2 + 1

Case4: n==3mod5 , so n^2==9mod5 , so n^2==4 mod 5 and we revert back to case 3 and thus 5 divides the factor n^2 + 1

Case5: n==4mod5 , so n+1==5mod5 , so n+1==0mod5 , so 5 divides n+1 , so 5 divides S

Therefore in all the cases it is clear that 5 divides S , so n^5==n mod5

Thus because 2 , 3 and 5 divides S , 30 divides S and so n^5 == n mod 30

Case proven

Proof 123: Prove that every even square is either 0 mod 8 or 4 mod 8

Let this even square be x^2

So we must prove that x^2==0mod 8 or x^2 ==4 mod 8

First note that if x^2 is even then so is x

So x = 2v where v is even or odd

If v is even then v = 2w , so x = 4w , so x^2 = 16w^2

Therefore 8 divides x^2 and in this case x^2 == 0 mod 8

If v is uneven , then v = 2w+1 , so that x = 2(2w+1) , so that x^2 = 16w^2 + 16w + 4

So x^2 - 4 = 16(w^2 + w)

Clearly 8 divides 16(w^2 + w) and thus 8 divides x^2 - 4

And so x^2 == 4 mod 8

Note that if we have said that the even squares must be bigger than four we could have proved the above for mod 16

Case proven

Proof 124: Prove that S =8n+7 is never the sum of three squares

We have already proved in proof 24 that for q odd that q^2 ==1mod 8

In proof 123 we have proved that for even q that q==0 mod 8 or q==4mod 8

So let us suppose that we can form three squares x^2 , y^2 and z^2 such that

S = x^2 + y^2 + z^2 and that S==7mod8

We have to study four cases

Case1: x even , y even , z even
Case2: x even , yuneven , z even
Case3: x even , y uneven, z uneven
Case4: x uneven , y uneven , z uneven

Let us bring in the following terminology: t ==(r,e,s)mod f

The meaning being t == r mod f or t == e mod f or t == s mod f

So for Case one we have that x^2 ==(0,4)mod8 , y^2 ==(0,4)mod8 , z ==(0,4)mod8

So S ==(0,4,8,12)mod8 and thus S is not congruent to 7 mod 8

For case two we have that x^2==(0,4)mod8 , y^2 ==1 mod 8 , z^2 ==(0,4) mod8

So S ==(1,5,9)mod8 and thus S is not congruent to 7 mod 8

For case 3 we have that x^2==(0,4)mod8 , y^2 ==1mod8 , z^2 == 1mod8

So S ==(2,6)mod8 and thus S is not congruent to 7 mod 8

For case 4 we have that x^2 , y^2 and z^2 are all congruent to 1 mod 8

So S==3mod8 and thus S is not congruent to 7mod 8

So for all possible combinations of even and uneven numbers for x,y and z we have that the sum of the three squares are never congruent to 7 mod 8

So numbers in the form of 7+8n are never the sum of three squares.

Case proven

Proof 125: Prove that 7 divides (5^[2n] + 3*2^[2n + 1]) for n >= 1

Terminology update : x | y means x divide y , so x is a factor of y

S_1 means S_1

x^y means x to the power of y

We prove using induction

For n=1 we have that S_1 = 5^2 + 3*2^3 = 25+24 = 47 and 7 | 49

So now we assume that 7 | S_n = **(5^[2n] + 3*2^[2n + 1])**

We now have to prove that 7 | S_n+1 using our assumption

So we have to prove that 7 | **(5^[2(n+1)] + 3*2^[2(n+1) + 1])**

Now S_n+1 = 5^(2n+2) + 3*2^(2n+3)

So S_n+1 = 25*5^(2n) + 4*3*2^(2n+1)
S_n+1 = (21+4)* 5^(2n) + 4*3*2^(2n+1) = 4*(5^[2n] + 3*2^[2n+1]) + 21*5^(2n)

So S_n+1 = 4*S_n + 21*5^(2n)

But 7 | 4*S_n and 7 | 21*5^(2n)

So 7 | S_n+1

Case proven

Proof 126: Prove that the remainder of 5^1001 divided by 6 is 5

Terminology update: x ==y mod z means z divide(x-y)

Clearly 5 == -1 mod 6

So 5*5 == (-1*-1) mod 6 , so 5^2 == 1 mod 6

And 5*5^2 == (1*-1) mod 6 so 5^3 == -1 mod 6

So 5^n == 1 mod 6 if n even and 5^n == -1 mod 6 if n uneven

But n=1001 is uneven , so 5^1001== -1 mod 6

Because 0 ==6 mod6 we have that(0+5^1001) == (-1+6) mod 6

So 5^1001 == 5 mod 6

Therefore if 6 is divided into 5^1001 we get a remainder of 5

Case proven

Proof 127: Prove that $x^n + x^{-n} > x^{(n-1)} + x^{(1-n)}$ for n integer >0 and x > 1

Clearly $x^{(2n)} - x = x(x^{(2n-1)} - 1)$

So $x^{(2n)} - x > x^{(2n-1)} - 1$

Therefore $x^{(2n)} + 1 > x^{(2n-1)} + x$

So $x^n + x^{(-n)} > x^{(n-1)} + x/x^n$

So $x^n + x^{(-n)} > x^{(n-1)} + x^{(1-n)}$

Case proven

3.128 Prove that $\lim_{n \to \infty} (1 + 1/n)^n = e = 2.718281828.....$

First note that a polynomial Maclaurin expansion of e^x can be used to determine that

$e = 1 + 1 + 1/2! + 1/3! + 1/4! + 1/5! + ...$

Now using the binomial expansion we can clearly see that

$$(1+1/n)^n = 1^n + \binom{n}{1}*1/n + \binom{n}{2}*(1/n^2) + \binom{n}{3}*(1/n^3) + \binom{n}{4}(1/n^4) + ...$$

So $(1+1/n)^n = 1 + 1 + n!/[(2!)\{(n-2)!\}(n)(n)] + n!/[(3!)\{(n-3)!\}(n)(n)(n)] + ...$

So $(1+1/n)^n = 1+1 + (n-1)(n)/[(n)(n)(2!)] + (n-2)(n-1)(n)/[(n)(n)(n)(3!)] + ...$

Note that $\lim_{x \to \infty} [(x-i)/x] = \lim_{x \to \infty}[1/1] = 1$ accordingly to L' Hopital' s rule

Also note that $\lim(f(x)*g(x)) = \lim(f(x)) * \lim(g(x))$

So $\lim_{x \to \infty} (1+1/n)^n = 1 + 1 + 1/2! + 1/3! + 1/4! + ...$

So lim(1+1/n)^n = *e*
x→∞

Proof 129: Prove 1/(1.2.3) + 1/(2.3.4) + 1/(3.4.5) +.....+1/(100.101.102) = 0,249951465....

We see that the nth term T_n = 1/[(n)(n+1)(n+2)]

So T_(n) = (1/2)[1/[n(n+1)] - 1/[(n+1)(n+2)]]

So T_(n-1) = (1/2)[1/[(n-1)n] - 1/[n(n+1)]]

So T_(n-2) = (1/2)[1/[(n-2)(n-1)] - 1/[(n-1)n]
.
.
.
and T_1 = (1/2)[1/(1.2) - 1/(2.3)]

We see that summing the terms we see that we get elimination of all terms except for

The term - (1/2)/[(n+1)(n+2)] of T_n and the term (1/2)(1/(1.2)) of T_1

So the sum to n terms is : S_n = 1/4 - (1/2)(1/ [(n+1)(n+2)])

Therefore the sum to 100 terms S_100 = 1/4 -(1/2)(1/101)(1/102) = 1/4 - 1 / 20604 = 2575/10302

So S_100 = 0,249951465.....

Proof 130: Prove that 3 = sqrt(6+sqrt(6+sqrt(6+........

Let x = sqrt(6+sqrt(6+sqrt(6+.....

Then x + 6 = 6+sqrt(6+sqrt(6+sqrt(6+....

So sqrt(x+6) = sqrt(6+sqrt(6+sqrt(6+.......

So sqrt(x+6) = x

So x+6 = x^2

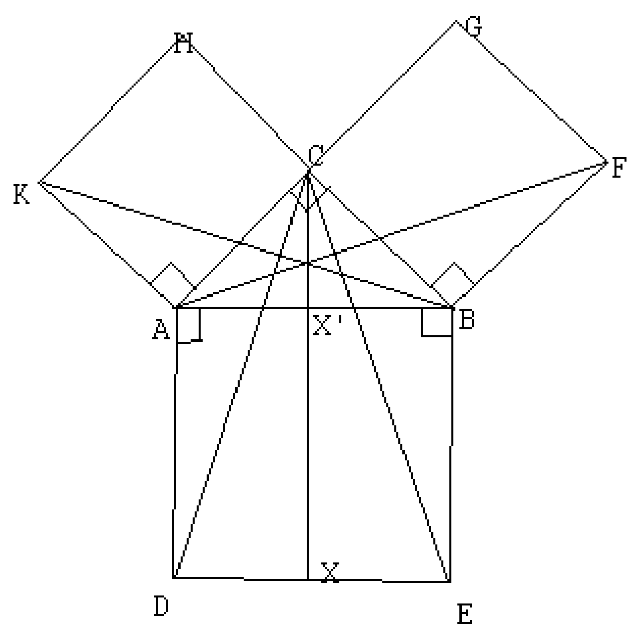
So x^2 - x -6 = 0 and thus (x-3)(x+2) = 0 , so that x = 3 or x = -2

We know x>0 therefore x = 3

Case proven

Proof 131: Prove the theorem of Pythagoras

See drawing below



The drawing consists of a triangle abc and 3 cubes abed , achk and cbfg.
Angle acb , dxx' and all the angles of the cubes are right angles.
Lines kb, af , cd , ce and cx are constructed.

Line lengths ac is also B , cb is A and ab is C

Now Triangle akb has base length ak and a height length of kh = ak

So the area of triangle akb = (ak)(kh)/2

We see that the cube khca has an area of (ak)(kh)

So the area of the triangle akb is half of the cube khca

Looking at triangles akb and acd we see that ak = ac , ad = ab ,
angle kab = 90 + angle cab = angle dac
Therefore triangle akb is congruent to triangle acd and their areas are therefore the same

We see that the area of triangle acd = half of the cube adxx' because of the base ad and the height of dx

So the area of triangle akb which is equal to that of triangle acd is also half of the cube adxx' , which is half of the cube akhc

So the area of the cube akhc = area cube adxx'

The same arguments can be used to prove that area cube cbfg = area bexx'

So the area abed = area adxx' + area bexx' = area akhc + area cbfg

But area abed = C^2 and area akhc = B^2 and area cgfg = A^2

So C^2 = A^2 + B^2

Case proven

Proof 132: Prove Fermat' s little theorem: p | (n^p - n) if p prime

We prove using induction

First note that p always divide $\binom{p}{i}$ for p > i > 0 { Proof 28}

Let p be a fixed prime. So if we prove for p fixed then we have proven for all p

Let z = n^p - n

For n = 1 we have that z = 1-1 = 0 and thus p | 0

We now assume that p | n^p -n and we must therefore prove that p | [(n+1)^p - (n+1)]

So p | z

Let S = (n+1)^p - (n+1)

Using the binomial expansion we get that S = (n^p + kp + 1) - (n+1)

We get the term kp because of proof 28

So S = (n^p - n) + kp = z + kp

But p divides z and p divides kp , so p | S

Case proven

Proof 133: Prove that the remainder of (37^13) / 17 is 12

We use congruence options to proof.

Note 13 = 1 + 4 + 8

37 == 37 mod 17

37 == (37-17-17) mod 17

37 == 3 mod 17

37^2 == 9 mod 17

37^2 == -8 mod 17 (-8 = 9-17)

37^4 == 64 mod 17

37^4 == -4 mod 17 (-4 = 64 -17-17-17-17)

37^8 == 16 mod 17

37^8 == -1 mod 17

37^(1+4+8) == (3)(-4)(-1) mod 17

37^13 == 12 mod 17

So the remainder is 12

This procedure is easy to learn and fun to use to impress an audience.

Let' s use another example

Get the remainder of (41^31) / 11

Note that 31 = 16 + 8 + 4 + 2 + 1 or 31 = 32-1

So 41 == 41 mod 11 , 41 == -3 mod 11

41^2 == 9 mod 11 , 41^2 == -2 mod 11

41^4 == 4 mod 11

41^8 == 16 mod 11 , 41^8 == 5 mod 11

41^16 == 25 mod 11 , 41^16 == 3 mod 11

41^32 == 9 mod 11

41^(1+2+4+8+16) == (-3)(-2)(4)(5)(3)mod 11

41^31 == 360 mod 11

41^31 == 8 mod 11 ,remainder is 8

or 41^(32-1) == (9/-3) mod 11

41^31 == -3 mod 11

41^31 == (11-3)mod11 , 41^31==8mod11

So the remainder is 8

Proof 134: Prove that given a polynomial f(x) that the remainder of f(x) / (x-a) is f(a)

The theorem described is the remainder theorem for polynomials.

Let $f(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + ... + c_n x^n$

Then clearly f(x)/(x-a) = Q(x) + R/(x-a) where R is some constant

Then f(x) = Q(x)(x-a) + R

So f(a) = Q(a)(a-a) + R = 0 + R = R

And thus f(a) is the remainder

Therefore if the remainder is zero then (x-a) is a factor of f(x)

Case proven

Example : f(x) = x^2 -2x + 1 = (x-1)^2 , so x-1 is a factor and x=1 is therefore a root of f(x)

f(2) = 4 - 4 + 1 = 1 , so in this case the remainder is 1

$f(1) = 1 - 2 + 1 = 0$, and we see that $x=1$ is a root of $f(x)$ because $f(1) = 0$

4.0 Facts

4.1 Arithmetic progression of primes

y is prime for y = 1419763024680x + 142072321123 for 0<= x <= 20

y is prime for y = x^2 - 79x + 1601 is prime for 1<= x < 80

4.2 y is not prime for y = 9973! + x for 2<= x <= 1006

4.3 *e* = 1/0! + 1/1! + 1/2! + /3! + 1/4! +.....

e = 2,71828182845904523536028 74.......

e = lim (1 + 1/n)^n
n-> ∞

4.4 A perfect number is a number which is equal to the sum of its proper divisors.

6 = 1+2+3
28 = 1+2 +4+7+14
496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248
8128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064

Euclid showed that if the number 2^n - 1 is prime then the number 2^{n-1} (2^n - 1) is a perfect number

If n=2 then 2^n – 1 = 3 which is prime and 2^(n-1) = 2 so that 2.3=6 is perfect.

If n=5 then 2^5-1 = 31 which is prime and 2^(5-1)=16 so that 16.31 = 496 is perfect.

4.5 A *pair of amicable numbers* is a pair like 220 and 284. The proper divisors of one number sum to the other and vice versa.

4.6 Numbers in the form $M = 2^n - 1$ where n is prime, are called Mersenne numbers. A lot of these numbers are prime themselves .

M(3) = 7, M(5) = 31, M(7) = 127 , M(13) = 8191, M(17) = 131071, M(19) = 524287 are all prime.

M(11) = 2047 = 23 x 89 is not prime.

There is currently a big search for new Mersenne primes.

One of the biggest primes found is M(3021377) which has 909526 decimal digits.

4.7 One of the roots of x^3 +x – K = 0 is as follows.

$$x = \sqrt[3]{\sqrt{KK/4 + 1/27} + K/2} - \sqrt[3]{\sqrt{KK/4 + 1/27} - K/2}$$

4.8 The number of primes smaller than n namely $\pi(n) = \dfrac{n}{\ln(n) - 1{,}08366}$

4.9 There is always a prime between n and 2n for n>1

n	2n	p
2	4	3
3	6	5
5	10	7
7	14	11, 13
11	22	13, 17, 19 and so on

4.10 A solution for the equation $-4x^3 + 3x - a = 0$

is x = sin([arcsin(a)]/3)

Let a = 0,5

Then x = sin([arcsin(0,5)]/3) = sin(0,52359../3) = sin(0,174532…) = 0,173648177..

Testing our answer -4(0,1736..)^3 + 3(0,1736…) –0,5 = 0,00000000009 which is nearly zero

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4.11 The largest known factorial prime number is 3610! – 1

4.12 Calculation of pi.

2/π = (1.3.3.5.5.7.7.9.…)/(2.2.4.4.6.6.8.8…)

π/4 = 1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11+ ….

π.π /6 = 1 + ¼ + 1/9 + 1/16 + 1/25 + 1/36 + 1/49 + … + 1/[n.n] as n tends to infinity

π to 2000 places is

```
3.14159265358979323846264338327950288419716939937510582097494459
 2307816406286208998628034825342117067982148086513282306647093844
 6095505822317253594081284811174502841027019385211055596446229489
 5493038196442881097566593344612847564823378678316527120190914
 5648566923460348610454326648213393607260249141273724587006606315
 5881748815209209628292540917153643678925903600113305305488204665
 2138414695194151160943305727036575959195309218611738193261179
 3105118548074462379962749567351885752724891227938183011949129833
 6733624406566430860213949463952247371907021798609437027705392171
 7629317675238467481846766940513200056812714526356082778577134
 2757789609173637178721468440901224953430146549585371050792279689
 2589235420199561121290219608640344181598136297747713099605187
```

721134**999999**83729780499510597317328160963185950244594553469083
02642522308253344685035261931188171010003137838752886587533208 3
8142061717766914730359825349042875546873115956286388235378759 37
51957781857780532171226806613001927876611195909216420198938095 2
57201065485863278865936153381827968230301952035301852968995773 6
22599413891249721775283479131515574857242454150695950829533116 8
61727855889075098381754637464939319255060400927701671139009848 8
24012858361603563707660104710181942955596198946767837449448255 3
79774726847104047534646208046684259069491293313677028989152104 7
52162056966024058038150193511253382430035587640247496473263914 1
99272604269922796782354781636009341721641219924586315030286182 9
74555706749838505494588586926995690927210797509302955321165344 9
87202755960236480665499119881834797753566369807426542527862551 8
18417574672890977772793800081647060016145249192173217214772350 1
41441973568548161361157352552133475741849468438523323907394143 3
34547762416862518983569485562099219222184272550254256887671790 4
94601653466804988627232791786085784383827967976681454100953883 7
86360950680064225125205117392984896084128488626945604241965285 0
22210661186306744278622039194945047123713786960956364371917287 4
67764657573962413890865832645995813390478027590 1

4.13 Any prime bigger than two in the form 4n+1 can be written in a unique way as the sum of two squares.

For n=1 , 4n+1 = 5 = 4+1

For n=3, 4n+1 = 13 = 9+4

For n=4, 4n+1 = 17 = 16+1

For n=7, 4n+1 = 29 = 25+4

4.14 An angle cannot be trisected with a ruler and compass.

4.15

$84 = 0^3 + 41639611^3 + (-41531726)^3 + (-8241191)^3$

$1^3 + 1^3 + 1^3 = 4^3 + 4^3 + (-5)^3$

$1^3 + 12^3 = 9^3 + 10^3$

$3^3 + 4^3 + 5^3 = 6^3$

$1^3 + 6^3 + 8^3 = 9^3$

$3^3 + 10^3 + 18^3 = 19^3$

7^3 + 14^3 + 17^3 = 20^3

4^3 + 17^3 + 22^3 = 25^3

18^3 + 19^3 + 21^3 = 28^3

11^3 + 15^3 + 27^3 = 29^3

6^3 + 32^3 + 33^3 = 41^3

2^3 + 17^3 + 40^3 = 41^3

16^3 + 23^3 + 41^3 = 44^3

133^4 + 134^4 = 59^4 + 158^4

1^2 + 2^2 + 2^2 = 3^2

3^3 + 4^3 + 5^3 = 6^3

2682440^4 + 15365639^4 + 18796760^4 = 20615673^4

4.16 If a^2 + b^2 = c^2 then ab cannot be square.

4.17

$e^x = 1 + x + (1/2)x^2 + (1/3!)x^3 + (1/4!)x^4 + ..$

Therefore *e* = 1 + 1 + ½ + 1/3! + ¼! + 1/5! + 1/6! + …. when letting x = 1

sin(x) = x - (1/3!)x^3 + (1/5!)x^5 - (1/7!)x^7 +…

Cos(x) = 1 - (1/2!)x^2 + (1/4!)x^4 - (1/6!)x^6 +…

Ln(x+1) = x - (1/2)x^2 + (1/3)x^3 - (1/4)x^4 +… For -1<x<=1

Arctan(x) = x - (1/3)x^3 + (1/5)x^5 - (1/7)x^7 + … For -1<= x <= 1

4.18 The Fibonacci series is given by $x_{n+1} = x_n + x_{n-1}$ where $x_0 = 0$, $x_1 = 1$

An exact formula is $x_n = \frac{a^{n+1}}{\sqrt{5}} - \frac{b^{n+1}}{\sqrt{5}}$

where $a = (1+\sqrt{5})/2$ and $b = 1 - a = (1 - \sqrt{5})/2$, the golden mean

---

4.19 Naming of SI prefixes

10^k So k are the number of zeros

k	American	European	SI--Prefix
-33			revo
-30			tredo
-27			syto
-24			fito
-21			ento
-18	quintillionth		atto
-15	Quadrillionth		femto
-12	trillionth		pico
-9	Billionth		nano
-6	Millionth		micro
-3	Thousandth		milli
-2	Hundredth		centi
-1	Tenth		deci
1	Ten		deca
2	Hundred		hecto
3	Thousand		kilo
4	Myriad		
6	Million	Million	mega
9	Billion	Milliard	giga
12	Trillion	Billion	tera
15	Quadrillion	Billiard	peta
18	Quintillion	Trillion	exa
21	Sextillion	Trilliard	hepa
24	Septillion	Quadrillion	otta
27	Octillion	Quadrilliard	nea
30	Nonillion	Quintillion	dea

33	Decillion	Quintilliard	una
36	Undecillion	Sextillion	
39	Duodecillion	Sextilliard	
42	tredecillion	Septillion	
45	quattuordecillion	Septilliard	
48	quindecillion	Octillion	
51	sexdecillion	Octilliard	
54	septendecillion	Nonillion	
57	octodecillion	Nonilliard	
60	novemdecillion	Decillion	
63	VIGINTILLION	Decilliard	
100	Googol	Googol	
303	CENTILLION		
600	CENTILLION		
10^100	Googolplex	Googolplex	

4.20 In general the series $1/1^n + 1/2^n + 1/3^n + 1/4^n + \ldots + 1/a^n < 1 + 1/[2^{(n-1)} - 1]$ for n>1 and where **a** tends to infinity

For n=2 the series < 1 + 1 = 2

For n=3 the series < 1 + 1/3 = 4/3 = 1,33333333…

For n=4 the series < 1 + 1/7 = 8/7 = 1,142857143…..

4.21 Given x and y where x and y are bigger than 0 and x<>y then there exists an integer

Square N such that N = (x^2 + y^2)/(1+xy)

Examples:

If x = 5832 , y = 18 then N = 324

If x = 14 , y = 2744 then N = 196

If x = 8 , y = 2 then N = 4

If x = 125 , y = 5 then N = 25

4.22 There exists infinite many solutions as stated in 4.20

Let $x = Ky$

Then $x^2 + y^2 = (Ky)^2 + y^2 = (K^2 + 1)y^2$

And $xy+1 = Ky^2 + 1$

Let $Ky^2 + 1 = K^2 + 1$, so $Ky^2 = K^2$

For this to be so , $y^2 = K$

Therefore $x = Ky = y^3$ will deliver infinite many solutions.

Choose $y=3$, then $x = 27$ so that $x^2 + y^2 = 738$

Then $xy+1 = 82$

Then $(x^2 + y^2)/(xy+1) = 738/82 = 9$ which is square

4.23 $a^4 + b^3 = c^2$ has many solutions.

A few are

$a = 1, b = 2 , c = 3$
$a = 26, b = 224 , c = 3420$
$a = 7, b = 15, c = 76$
$a = 97, b = 3135, c= 175784$

4.24 There is no exact formula to solve the equation $x - \ln(x) = 0$

$x + \ln(x) = 0$ has a solution for $x = 0{,}5671432904$ using a numerical approach like Newton's gradient formula.

4.25 There exists a test the "Lucas Lehmer test" to search for big prime numbers.

If $n = (2^p) - 1$ then n is prime if $n \mid S_{p-1}$ where $S_{k+1} = (S_k)^2 - 2$ with $S_1 = 4$

(note: x | y means x divides y)

Example : Let p=3 , then n = 2^3 –1 = 8-1=7 which is prime

Now using the test we get $S_{p-1} = S_2$

$S_1 = 4$, $S_2 = (S_1)^2 - 2 = 4^2 - 2 = 16-2=14$

But 7 | 14 so $n \mid S_2$ and so n must be prime

4.26 Every polynomial of uneven degree with real coefficients , has at least one real root.

The above was deduced by the math genius FC Gauss.

For example: y = x^3 + x^2 + x + 1 has a real root at x = -1

4.27 The Euclidean Algorithm can be used to get the greatest common divider (GCD) of two numbers.

Demonstrating the Euclidean Algorithm

The GCD(100,70) can be deduced as follows

100 = 70.1 + 30

70 = 30.2 + 10

30 = 10.3

So 10 is the GCD of 70 and 100

The GCD(345,255)

345 = 255.1 + 90
255 = 90.2 + 75
90 = 75.1 + 15
75 = 15.5

So 15 is the GCd of 345 and 255

The GCD of 93 and 323

323 = 93.3 + 44
93 = 44.2 + 5
44=5.8 + 4
5= 4.1 +1
4 = 1.4

Therefore 1 is the GCD of 323 and 93

4.28 One of Euler's famous equations as follow.

$$\sum_{n=1}^{\infty} (1/n\hat{\ }s) = \prod_{p} [\ 1/(1 - 1/p\hat{\ }s)]$$ where p is the list of primes from 2 upwards

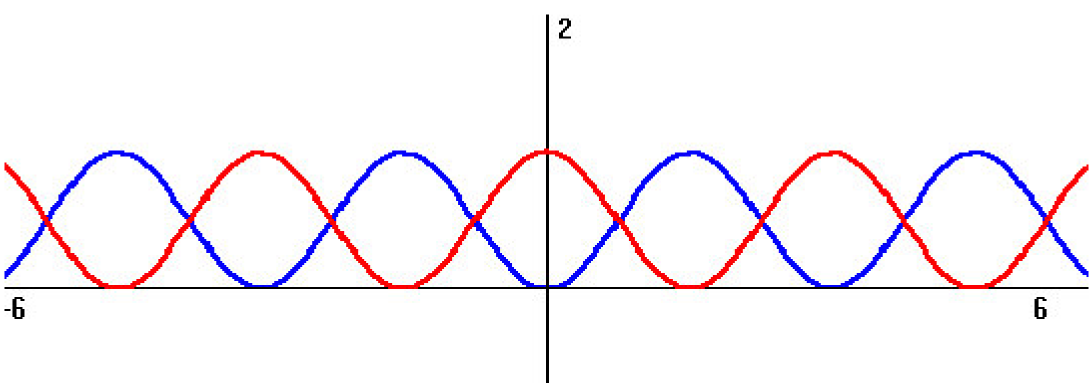
For example choosing s = 2

The sum = 1/1 + 1/4 +1/9 + 1/16 + 1/25 + 1/36 + 1/49 + ….

The product = (1/[1-1/4])(1/[1-1/9])(1/[1-1/25])(1/[1-1/49])(1/[1-1/121])…..

Both deliver 1,644934067.. which is pi squared divided by 6 by the way.

4.29 The following is a unique way to solve $\int_{0}^{2pi} \sin^2(x)dx$ without integrating sin(x)



The blue graph is of $\sin(x)^2$. The red graph of $\cos(x)^2$ looks the same except that it is shifted on the x-axis to the left with pi/2 radials. As can be seen the area from 0 to 2pi = 6,2831 is the same for both.

Let A be the area or the integral of sin(x)sin(x) from 0 to 2pi. Then A is also the area or integral of cos(x)cos(x) from 0 to 2pi.

So $2A = \int_0^{2pi} [\sin(x)^2 + \cos(x)^2]dx = \int_0^{2pi} 1.dx = 2pi$

So A = 3,141592654..

And the integral of sin(x)sin(x) from 0 to 2pi is therefore pi.

4.30 $2^n - 7 = x^2$ for n and x integer has only solutions for n = 3 or 4 or 5 or 7 or 15

4.31 There are exactly 23000 primes smaller than 2^{18}

4.32 There is a unique and simple method to work out the square root of any positive number without using the square root function of a calculator.

We guess a value as the square root of a number and then gets a better estimate. We use this estimate to get an even better estimate and so on.

Let our estimate be v and our better estimate be w. let the number we want the square root of be z.

Then w = 0,5(v + z/v)

Example: Let z = 100, so we know the square root is 10

Now we use our method to test if our estimate approaches 10.

Let our first estimate be v = 5

v	w = 0,5(v+100/v)
5	2,525
2,525	21,064
21,064	12,905
12,905	10,327
10,327	10,00518
10,00518	10,00000134

So our last value 10,00000134 is already very close to 10.

The formula is deduced from Newton' s numerical method to determine roots.

4.33 A solution to $y - x = e^{y} - e^{x}$ where y > x is given by

y = ln(ln(4)) and x = ln(ln(2))

Let z = ln(2) then x = ln(z) and y = ln(2z) = ln(2) + ln(z)

So y-x = ln(2)

And $e^{y} - e^{x} = e^{(\ln(2) + \ln(z))} - e^{\ln(z)} = 2*z - z = z = \ln(2)$

4.34

$$\frac{1 + \sqrt{5}}{2} = \text{sqrt}(1 + \text{sqrt}(1 + \text{sqrt}(1 + \text{sqrt}(1 + \ldots$$, also known as the golden mean

Four is the minimum colors needed to color any country map so that no two neighboring regions have the same color.

81 is the smallest square that can be written as the sum of three integer squares.

81^2 = 1^2 + 4^2 + 8^2

If you write the number 11111111..... with 317 one' s , you also get a prime

1234567891 is prime

39 is the first number with no remarkable property. This in itself is remarkable

5 Unsolved Problems

Can every even number >4 be written as the sum of two prime numbers ?

Is there always a prime between n^2 and (n+1)^2 ?

Are there infinite many primes of the form n!+1 ?

Are there infinite many primes in the form n!-1 ?

Let #n be the product of all primes up to and including n. Are there infinite primes in the form #n + 1 ?

Are their infinite primes in the form #n – 1 ?

Are there any numbers 2^(m) + 1 where m=2^n for n>4 that are prime?

Are there infinite many twin primes? Primes where the difference between them are 2, like 3 and 5 or 5 and 7 or 11 and 13?

Are there infinite many primes in the form 2^n – 1 where n is prime?

Are there infinite many primes in the form 2^n + 1 where n is prime?

Write 148 as w^3 + x^3 + y^3 + z^3 where w,x,y,z are integers (-10 , 0 , 6 etc.)?

Are there infinite many perfect numbers?

Are there any uneven perfect numbers?

Are there integers any x and n > 7 so that $n! = x^2 - 1$?
Currently there are only 3 known solutions
25 = 4! +1
121 = 5! + 1
5041 = 7! + 1

Are there any integers not equal to each other such that $a^5 + b^5 = c^5 + d^5$?

Is there any value for n>4 such that $n^n + 1$ is prime?

Solutions for $x^n + y^n + z^n = c^n$ for x, y, z, c, n integers and $0 =< n < 5$ exists. Are there solutions for n>4 ?

Solutions for primes in the form $2^{k-1} + k$ exist for k=1,3,7,237 and 1885. Do there exist more?

Numbers in the form F(n) = 2^p + 1 where p = 2^n are called Fermat numbers. F(n) is prime for

n= 1,2,3 and 4.

641 divides F(5) = 2^32 + 1. No other Fermat primes have been found. Are there any other?

(1,1,1) and (4,4,-5) are integer solutions for x^3 + y^3 + z^3 = 3 Are there other solutions?

Are there any four squares in Arithmetic Progression such that a^2 + b^2 = c^2 + d^2
Obviously choosing 0 and 1 will deliver some results

Is $e + \pi$ irrasional?

Is π / e irrasional?

Is $\ln(\pi)$ irrasional?

Is π^{e} irrasional?

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