Spektralanalyse physiologischer Signale

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zum Übungsblatt

$$T = 10s \rightarrow \Delta f = 0.1Hz$$

$$x(t) = 2\sin(6\pi t) = 2\sin(2\pi \cdot 3 \cdot t)$$

$$\rightarrow 3Hz = 30\Delta f$$

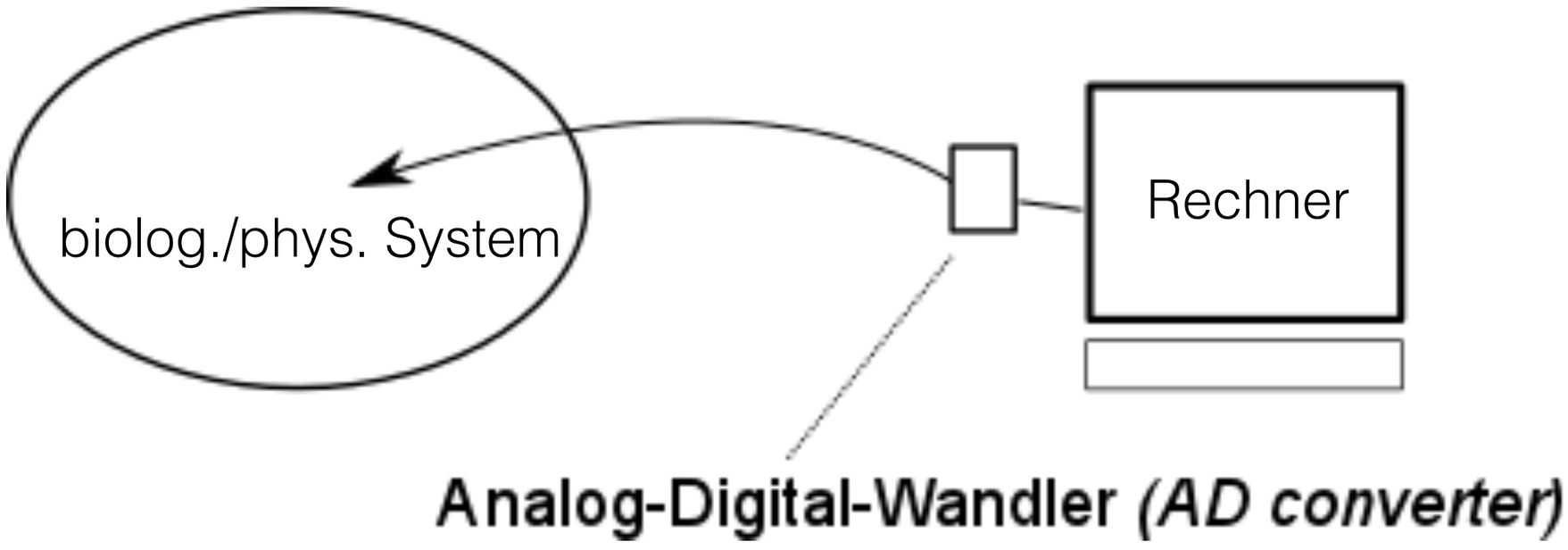
$$\rightarrow c_n = \frac{2}{2i}(\delta_{n,30} - \delta_{n,-30})$$

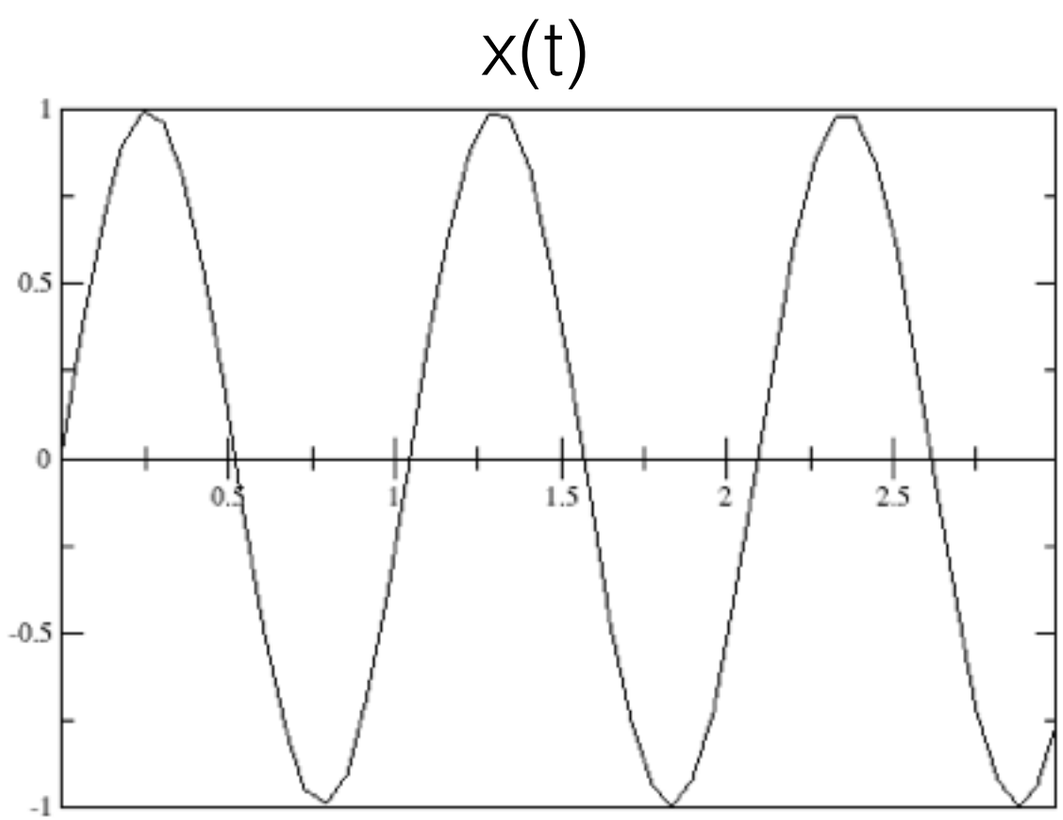
$$x(t) = 2\sin(4\pi t) + 8\cos(\pi t) = 2\sin(2\pi \cdot 2 \cdot t) + 8\cos(2\pi \cdot \frac{1}{2} \cdot t)$$

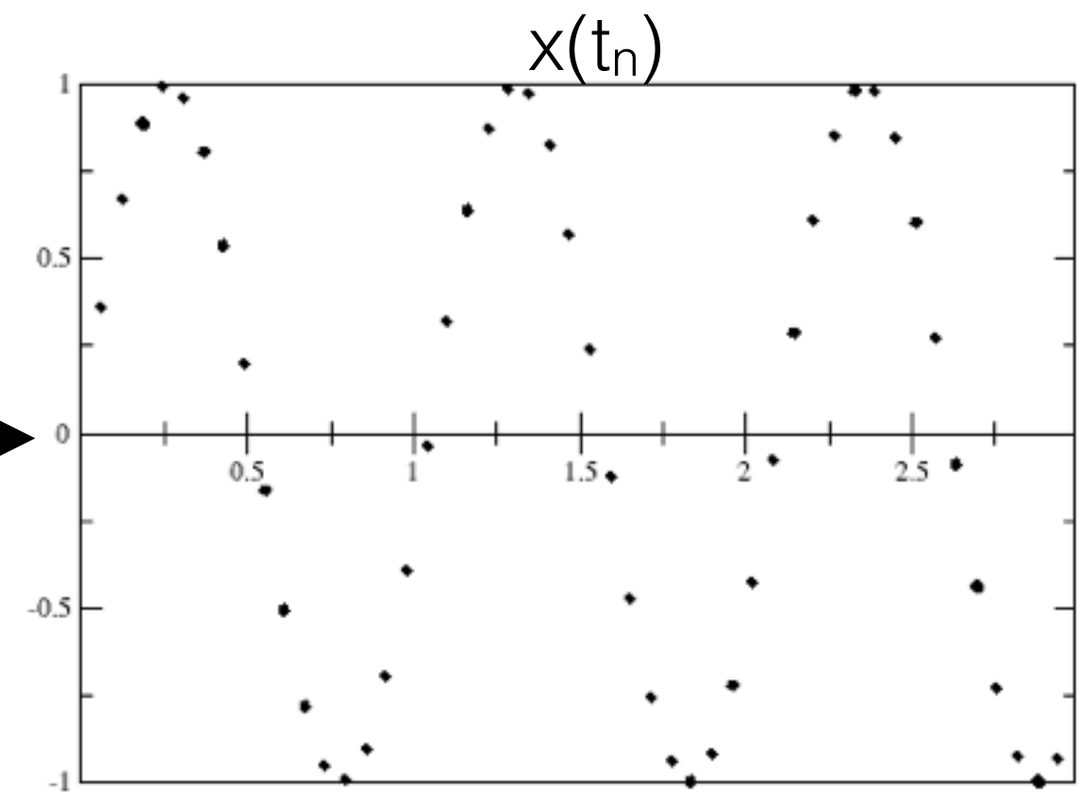
$$\rightarrow c_n = \frac{2}{2i}(\delta_{n,20} - \delta_{n,-20}) + \frac{8}{2}(\delta_{n,5} + \delta_{n,5})$$

Vorlesung 2

Prinzip des Abtastens:







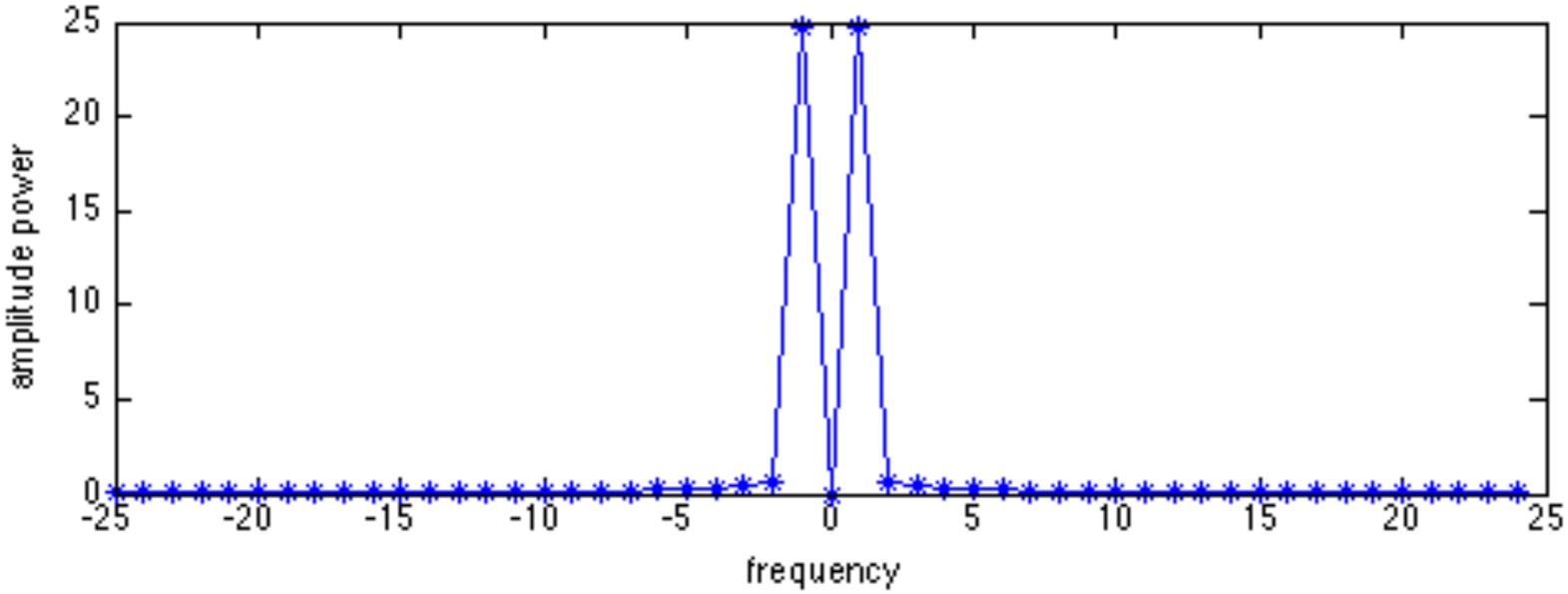
wichtige Eigenschaften eines abgetasteten Signals:

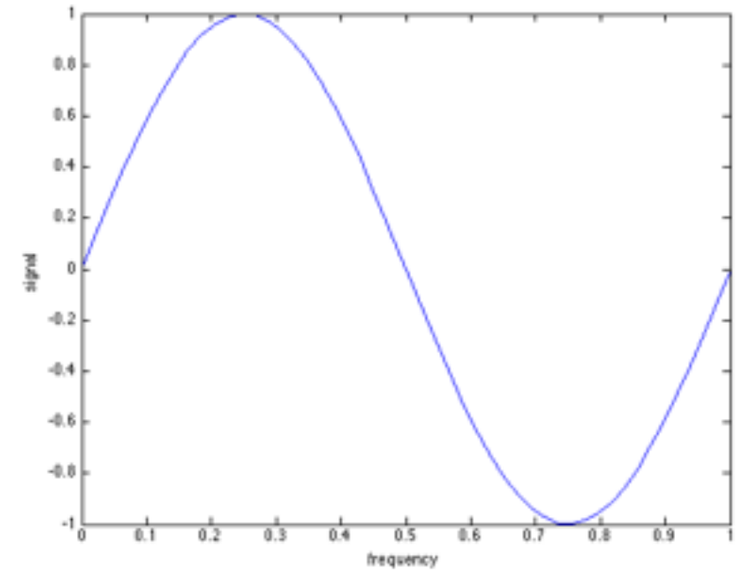
- für ein Signal der Dauer T ist die kleinste auflösbare Frequenz $f_{min}=1/T$

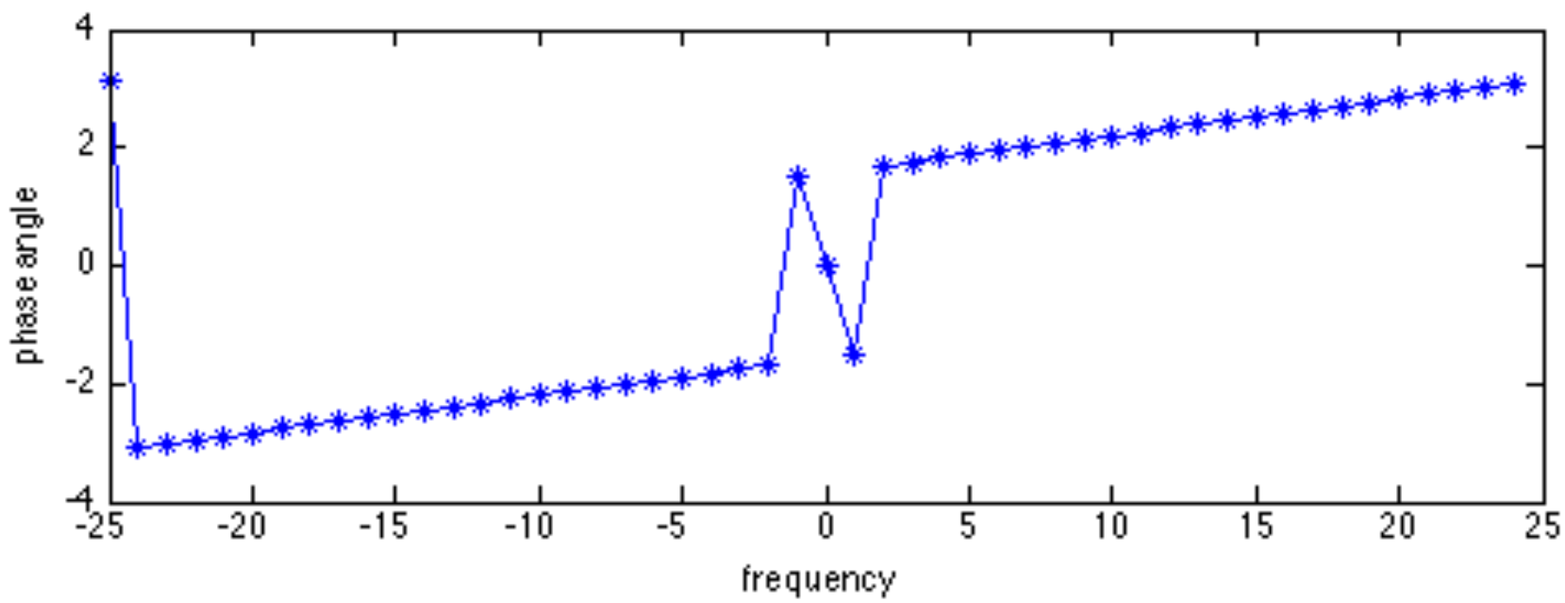
- für eine Abtastfrequenz f_s ist die Anzahl der Datenpunkte $N=Tf_s$ und es gibt N Frequenzen

- die Fourieranalyse ergibt N Frequenzen zwischen $-f_s/2$ und $f_s/2$ mit Frequenzinterval $\Delta f=f_{min}=1/T$

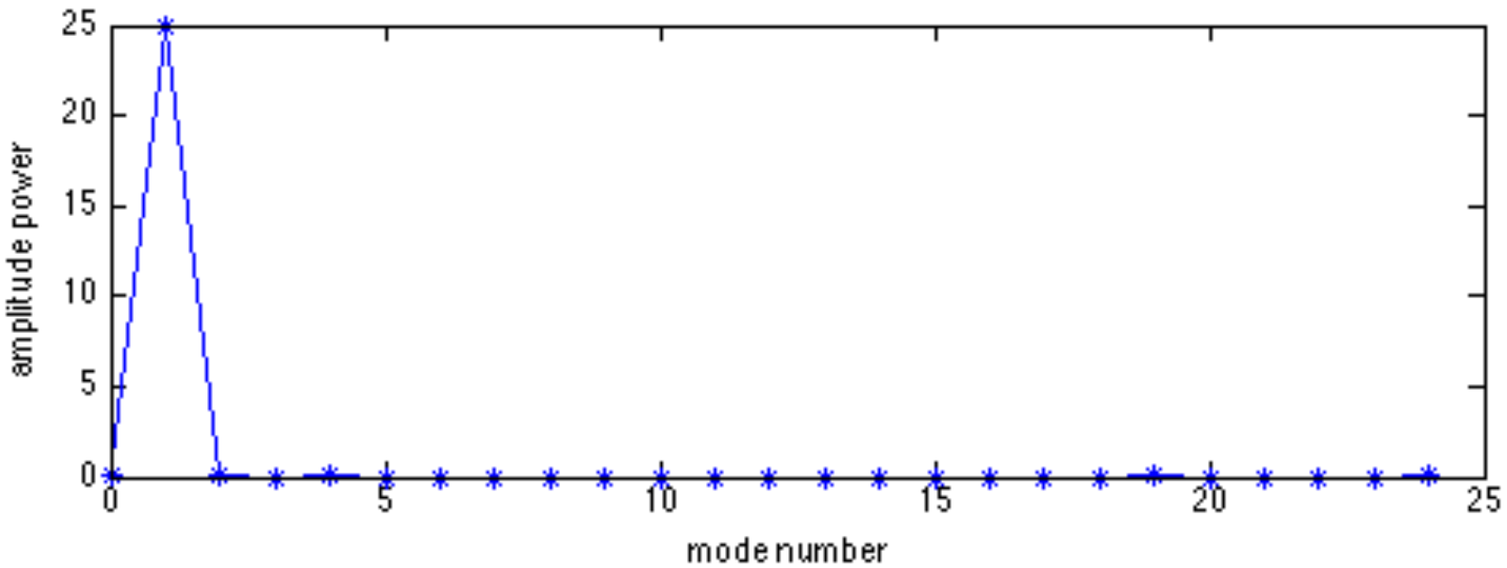
Zusammenfassung





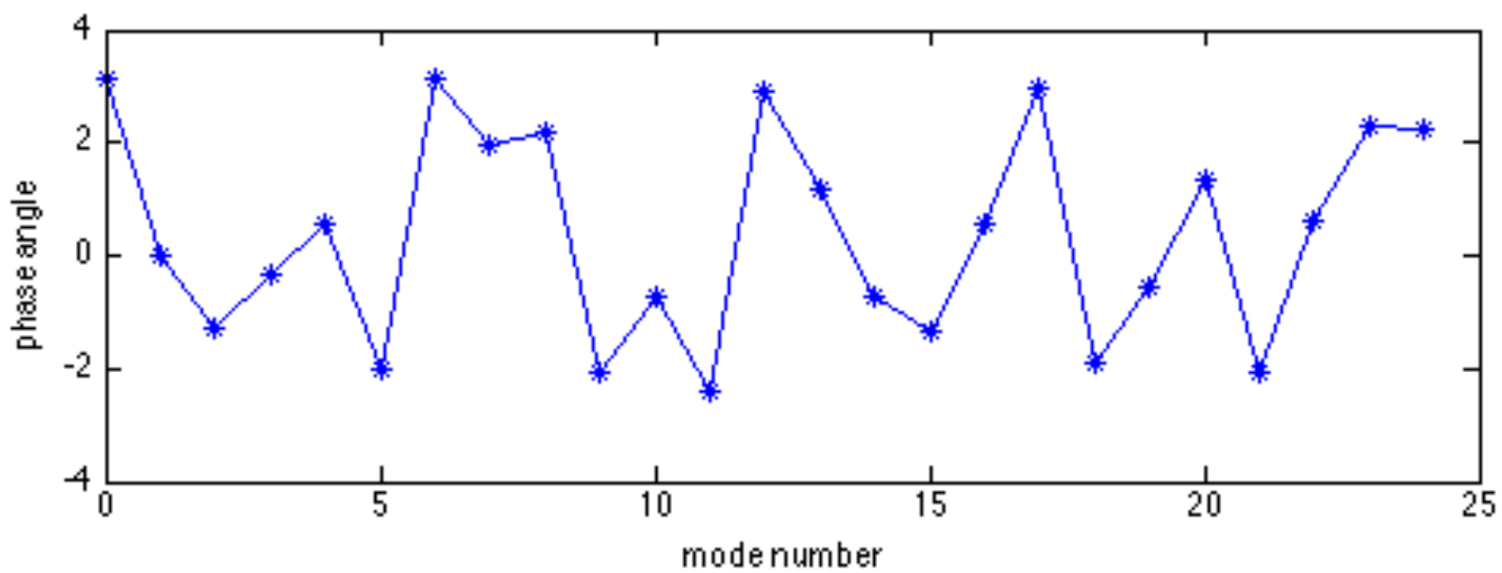


50 Frequenzen, 25 Fouriermoden, negative Frequenzen



power

$$|a_n| = \frac{2 \cdot |DFT_n|}{N}$$

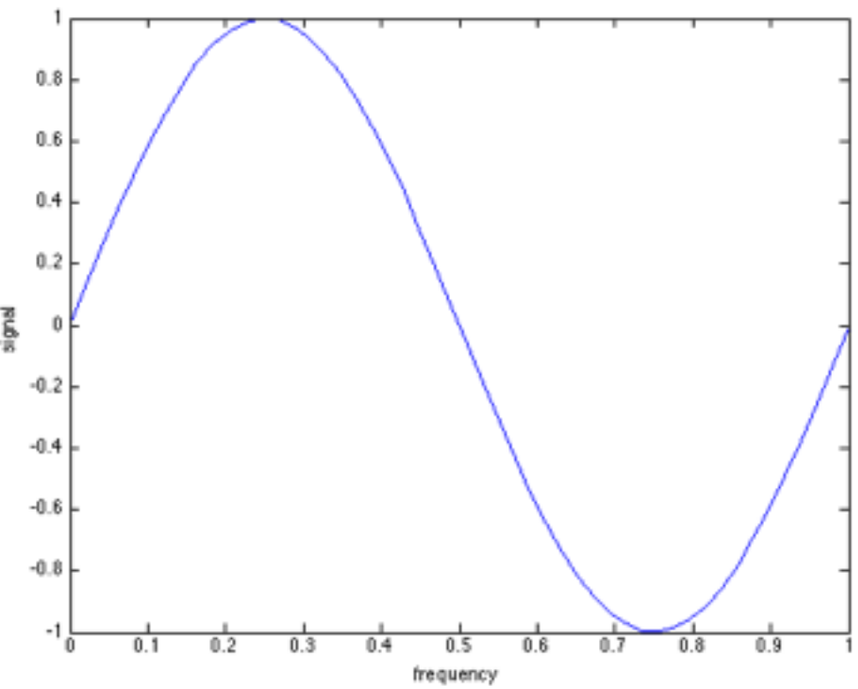


Phasenwinkel

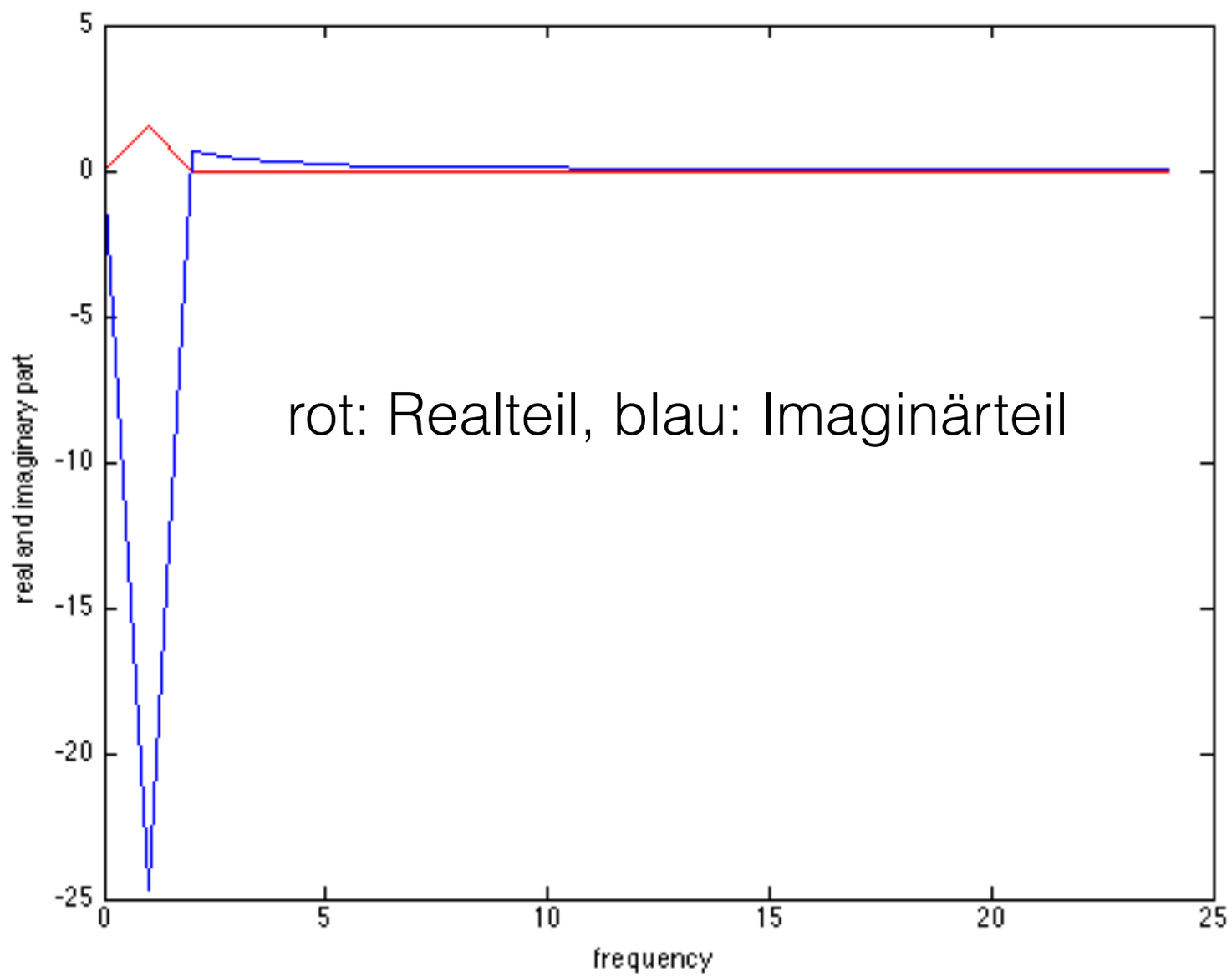
$$\tan \phi_n = \frac{Imag(DFT_n)}{Real(DFT_n)}$$

$$|DFT_n| = \sqrt{Imag^2(DFT_n) + Real^2(DFT_n)}$$

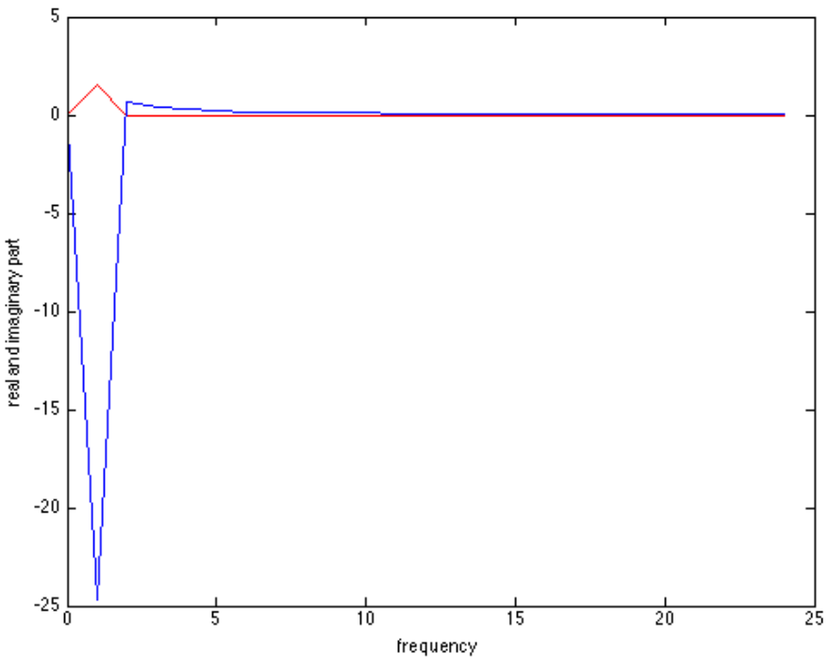
Beispiel: Sinus Funktion

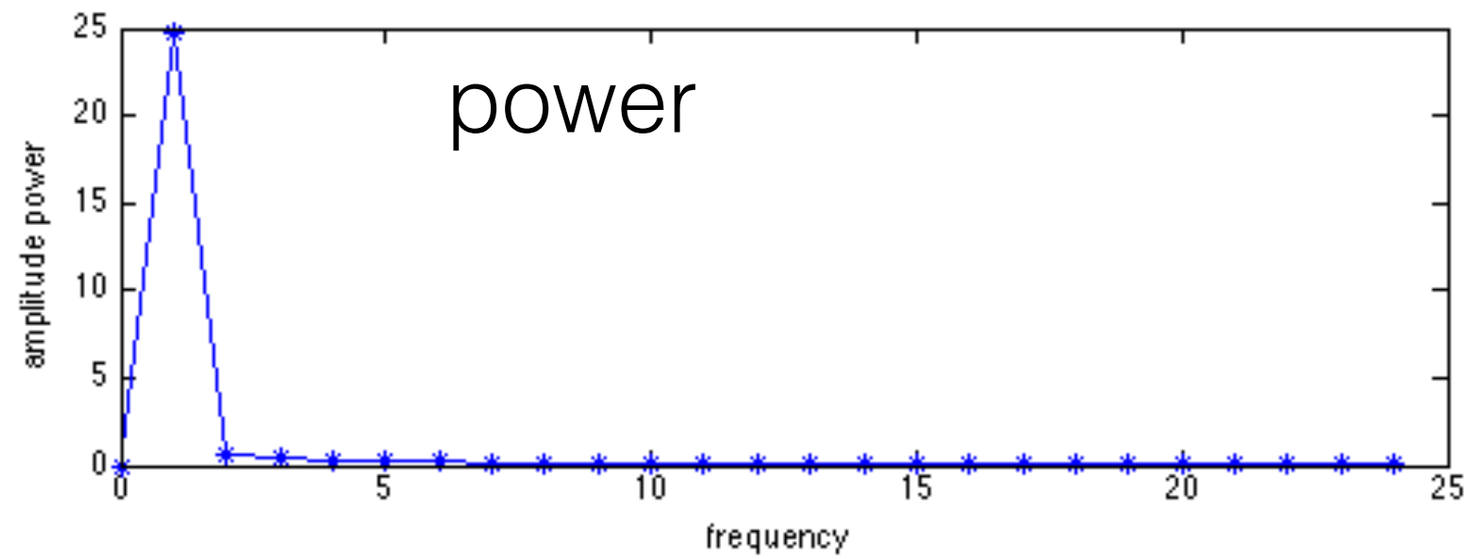


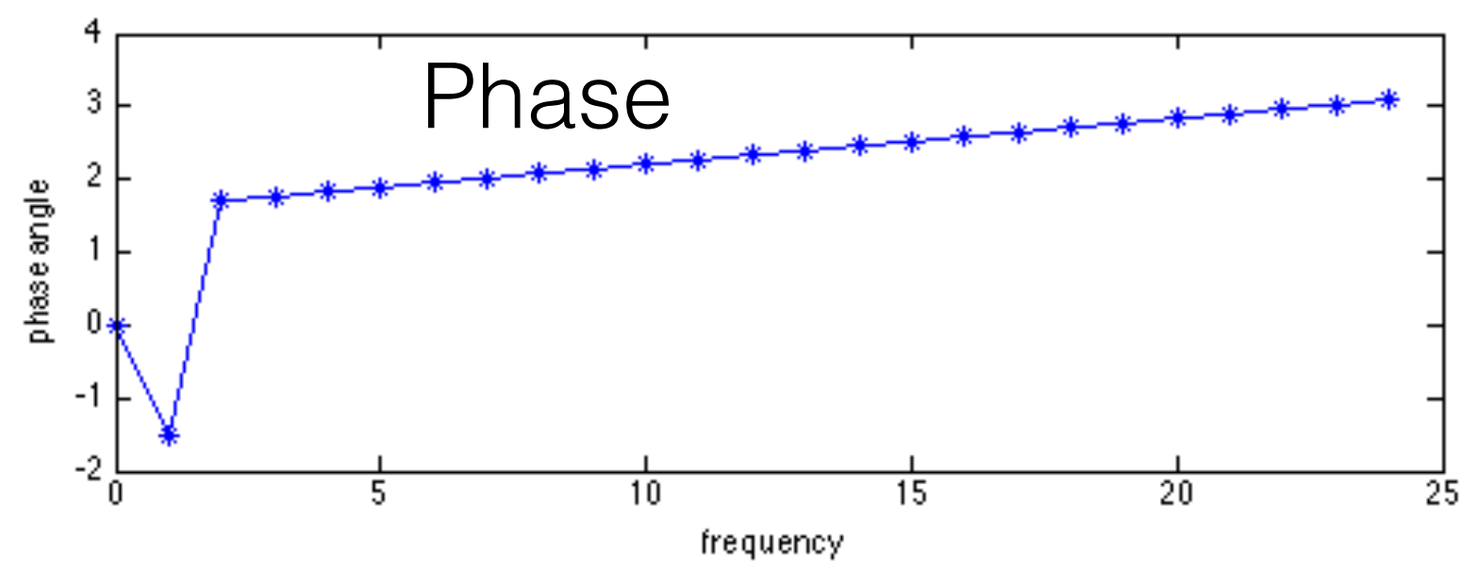
$$s(t) = 1.0 \cdot \sin(2\pi \cdot 1Hz \cdot t)$$



(Fourier_1.m)





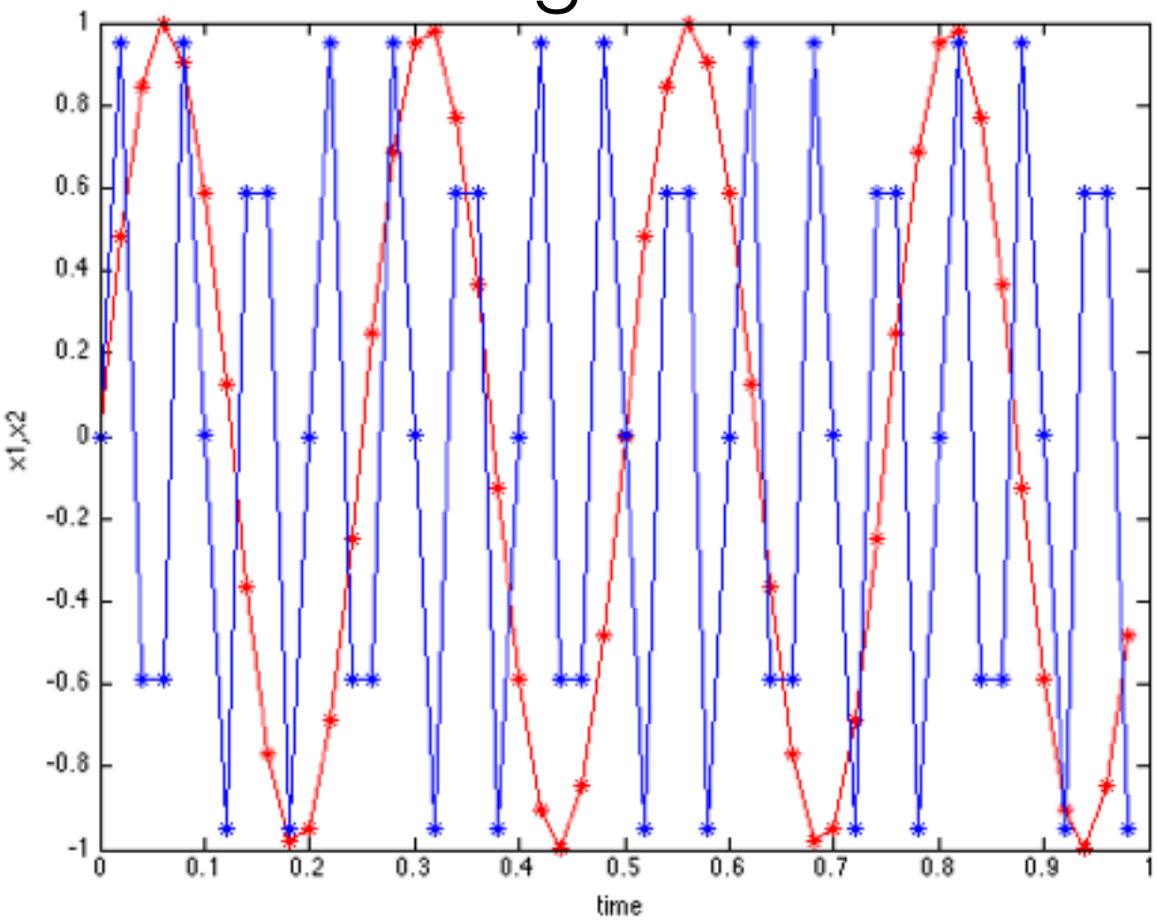


$$\phi_1 = -\pi/2$$

$$\cos(\phi - \pi/2) = \sin(\phi)$$

höhere Frequenzen

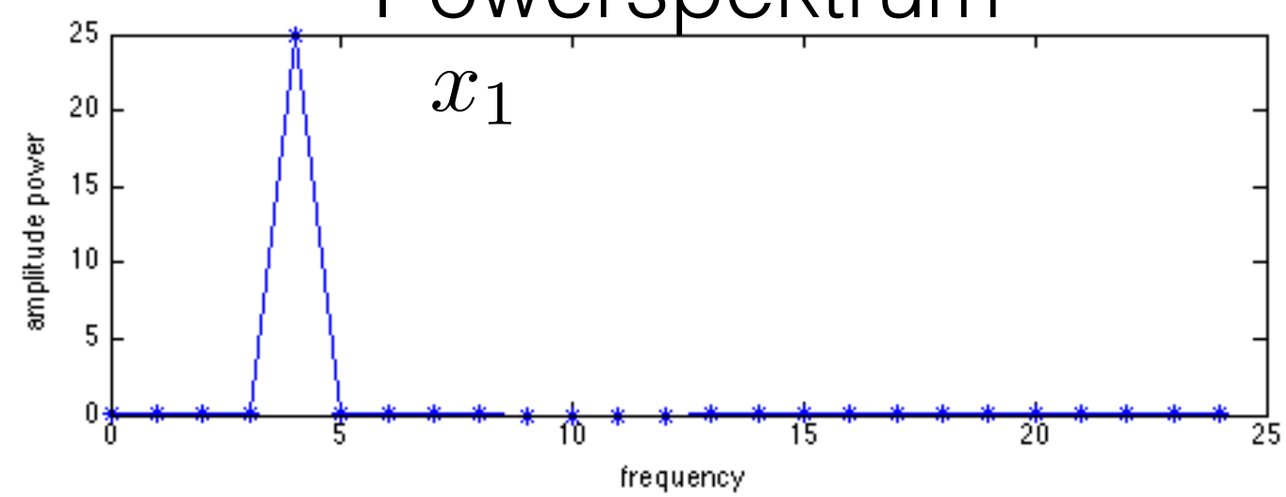
Signal

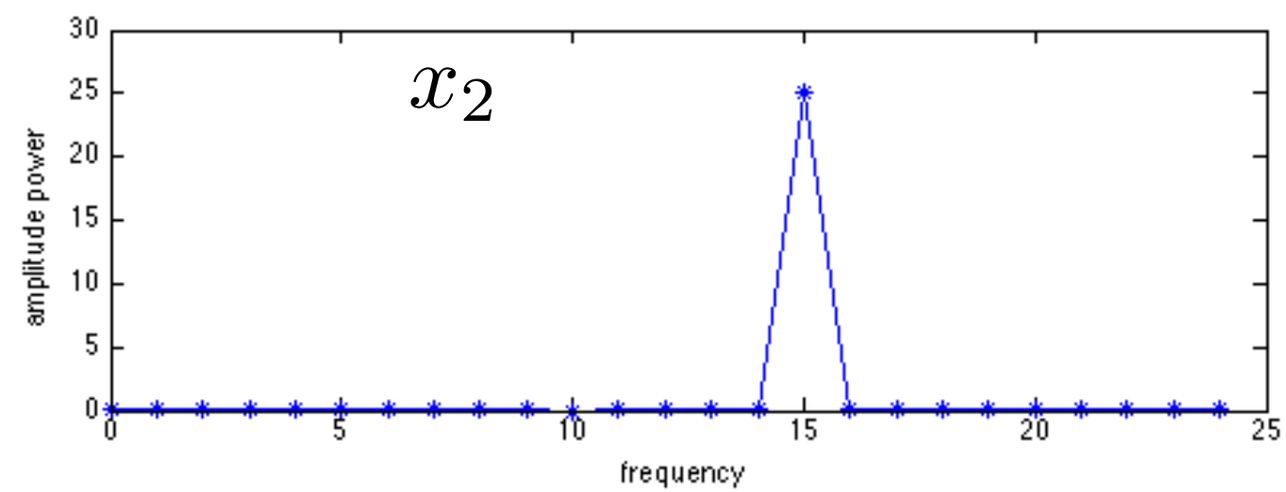


$$x_1(t) = \sin(2\pi \cdot 4 \cdot t)$$

$$x_2(t) = \sin(2\pi \cdot 15 \cdot t)$$

Powerspektrum



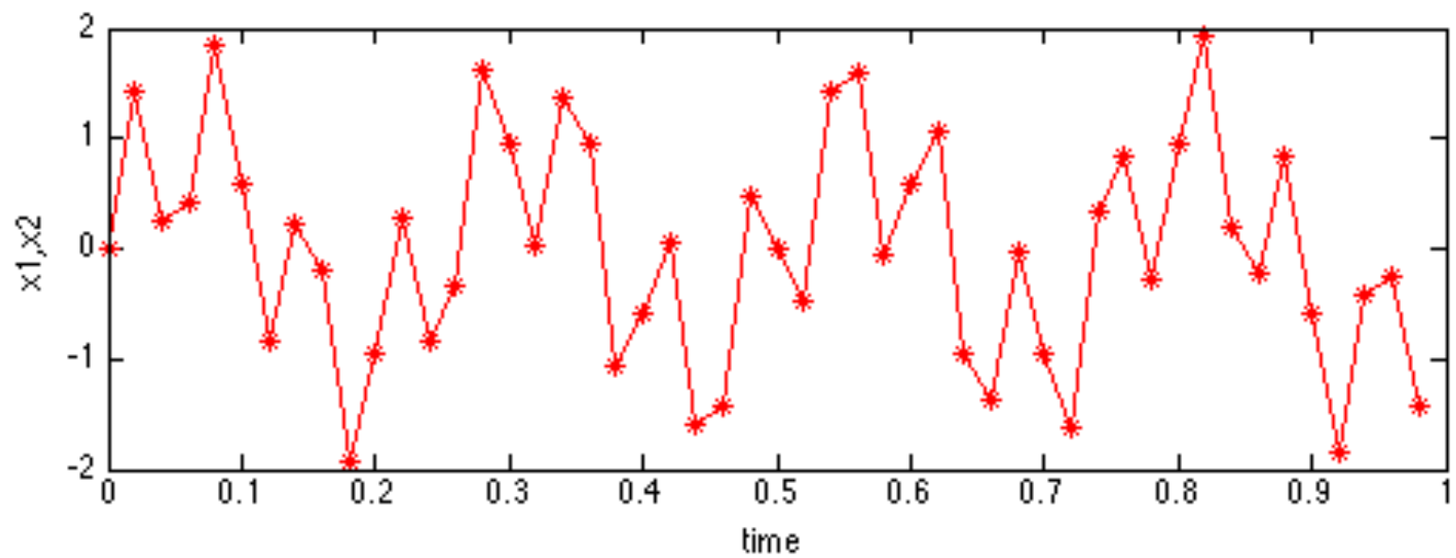


(Fourier_2.m)

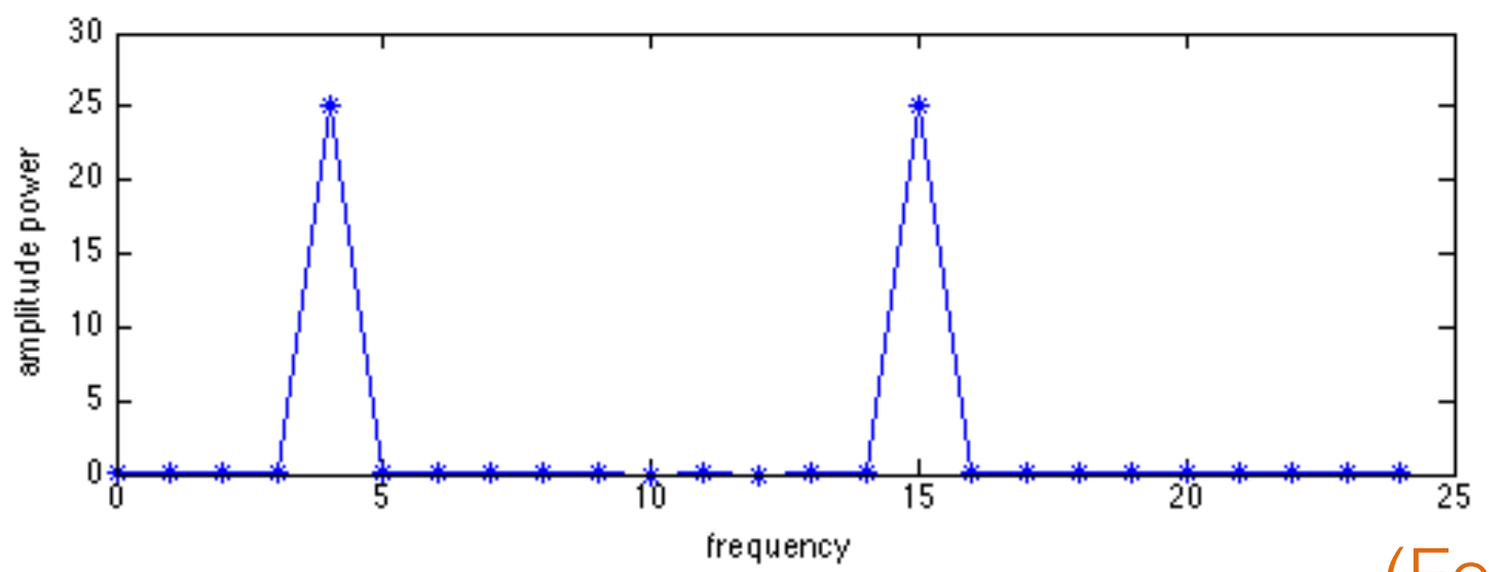
lineare Überlagerung

$$x(t) = \sin(\omega_1 t) + \sin(\omega_2 t) \ , \ \omega_1 = 2\pi \cdot 4Hz \ , \ \omega_2 = 2\pi \cdot 15Hz$$

Signal

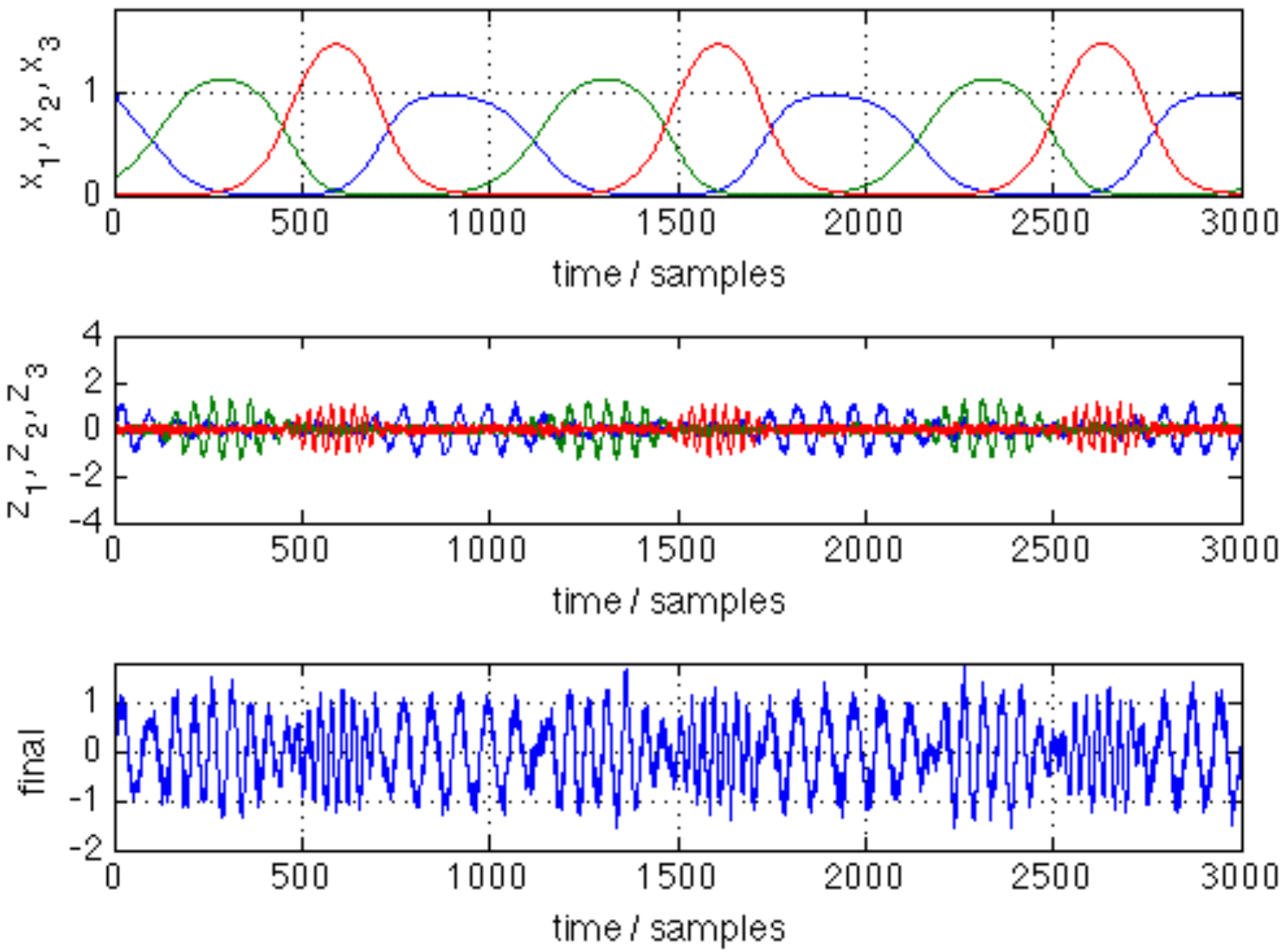


Powerspektrum



(Fourier_3.m)

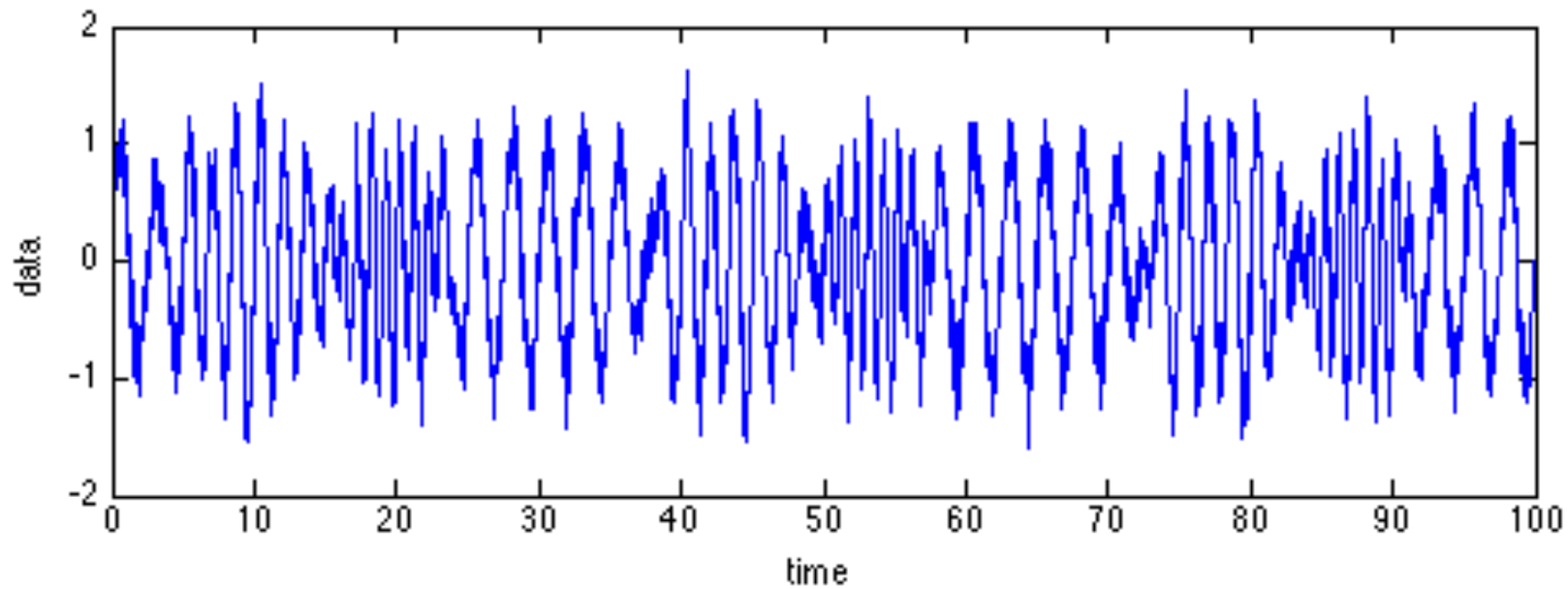
künstlicher Datensatz für spätere Untersuchungen



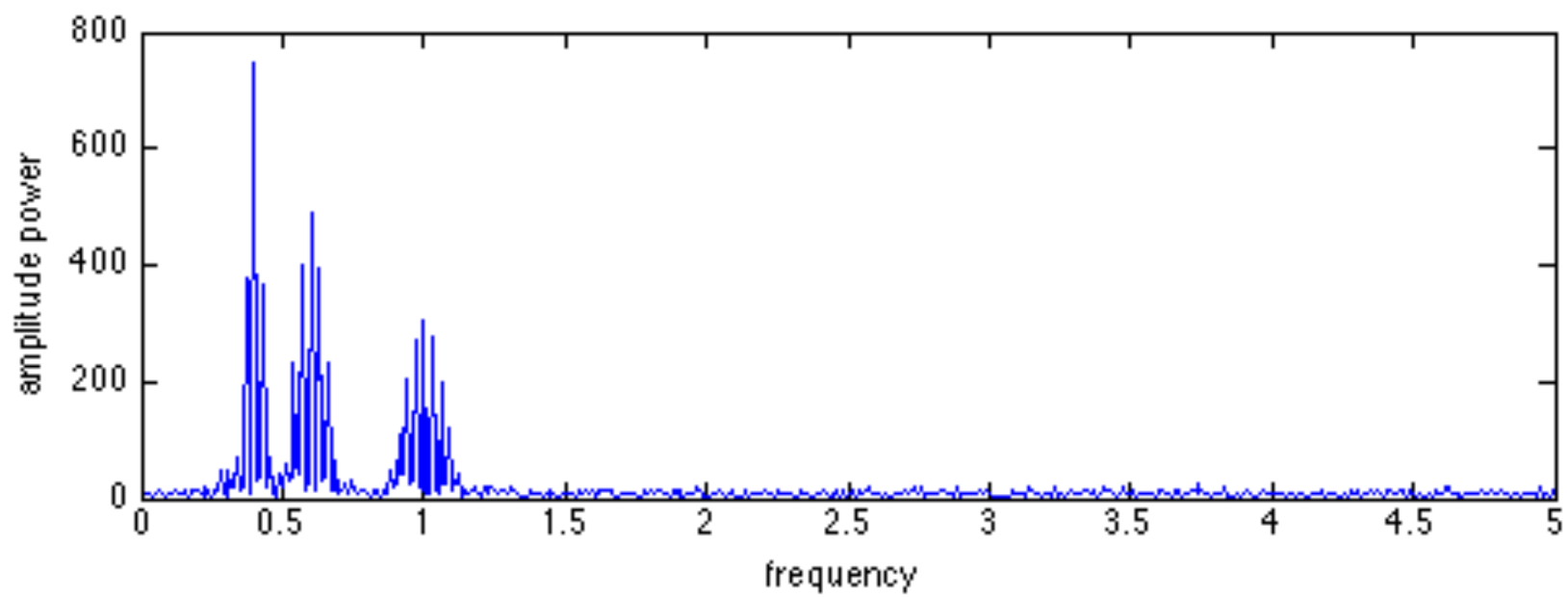
T=100s

Δt=0.033

Signal



Powerspektrum



(Fourier_5.m)

II. Fourier Analyse

II.1. Grundlagen

a) Koeffizienten

b) Fourier Theorem

II.2. Mögliche Fehler in der Fourier Analyse

Aliasing

Periodizität

Spectral leakage

II.3. Berechnung von Spektren

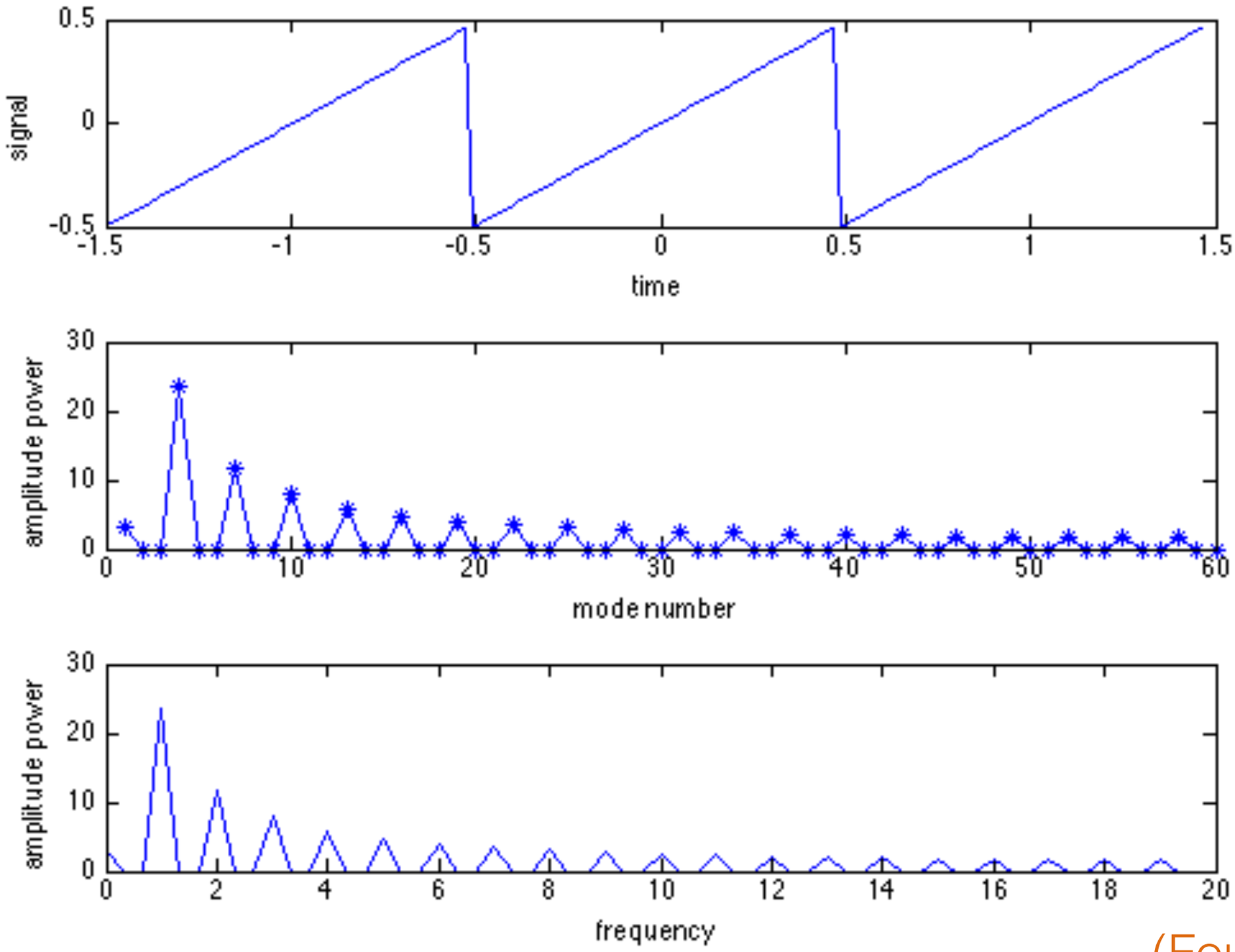
Fourier Theorem (J.B. Fourier, 1800):

Fast jedes periodische Signal

kann als eine lineare Superposition von

sinus- und cosinus Funktionen dargestellt werden.

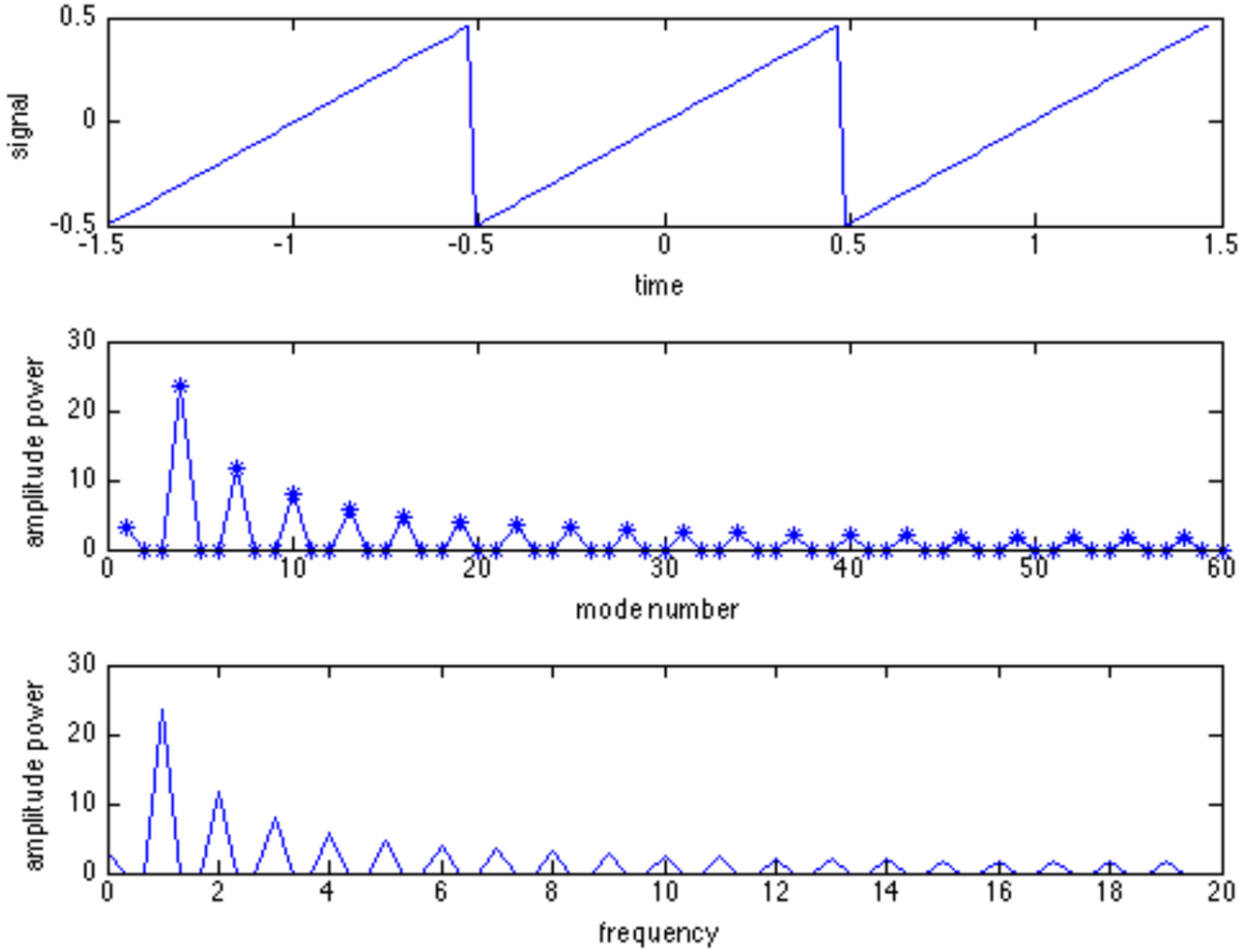
Beispiel:



M=150

Δt=0.02

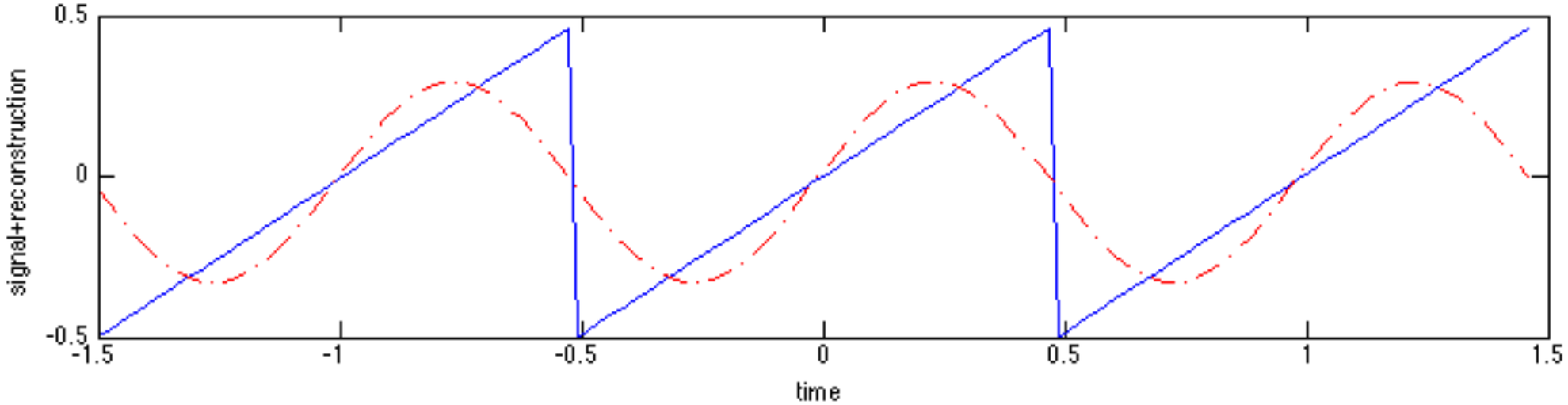
(Fourier_4.m)



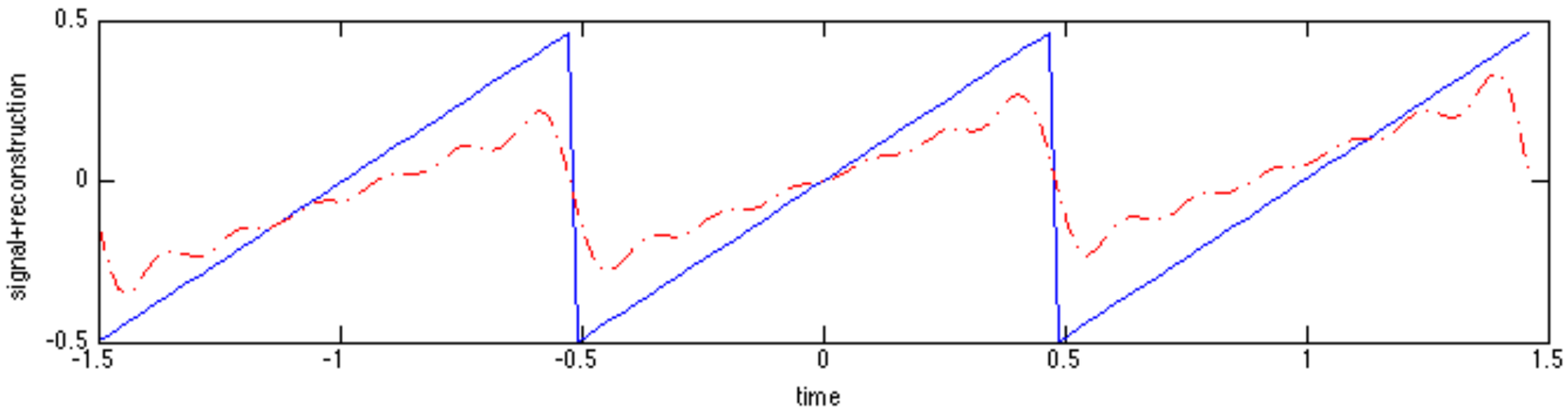
- obwohl Sägezahn Frequenz 1Hz hat
  zeigt das Powerspektrum höhere Frequenzen

- Grund: Analyse *muss alle Ecken rekonstruieren*

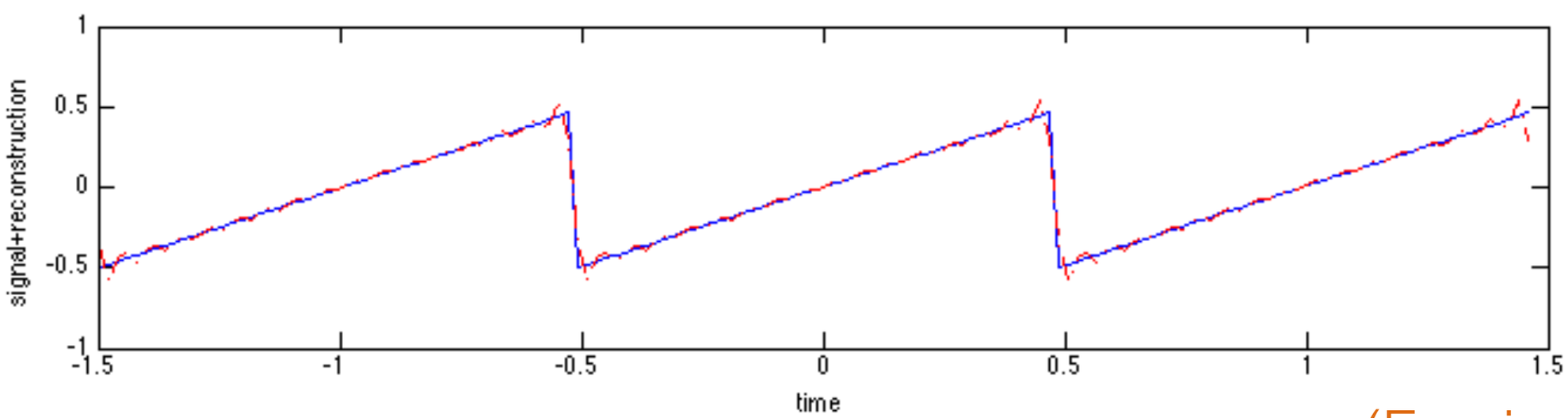
Rekonstruktion eines Signals



N=5

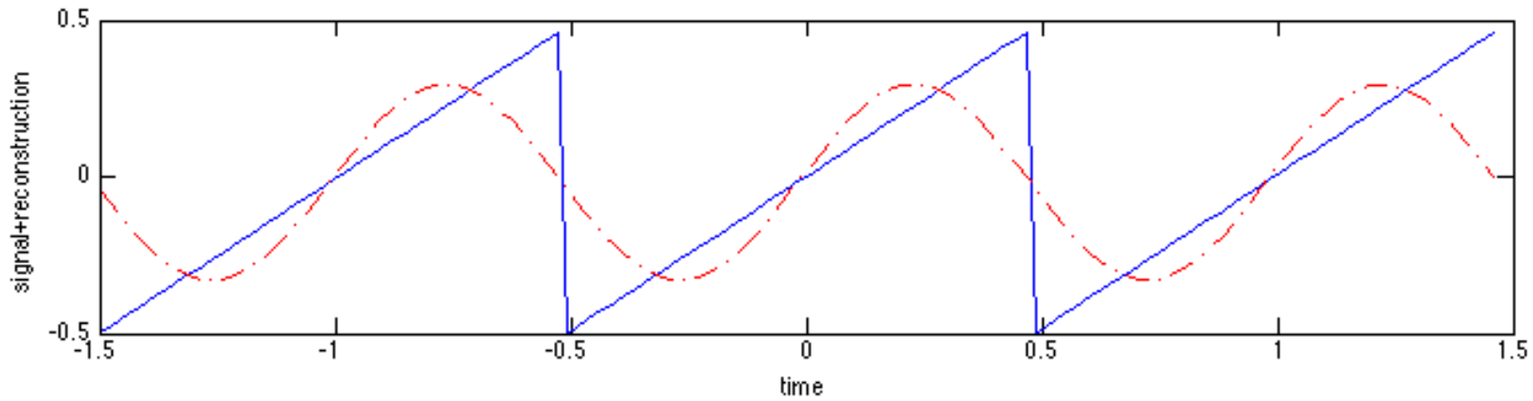


N=20

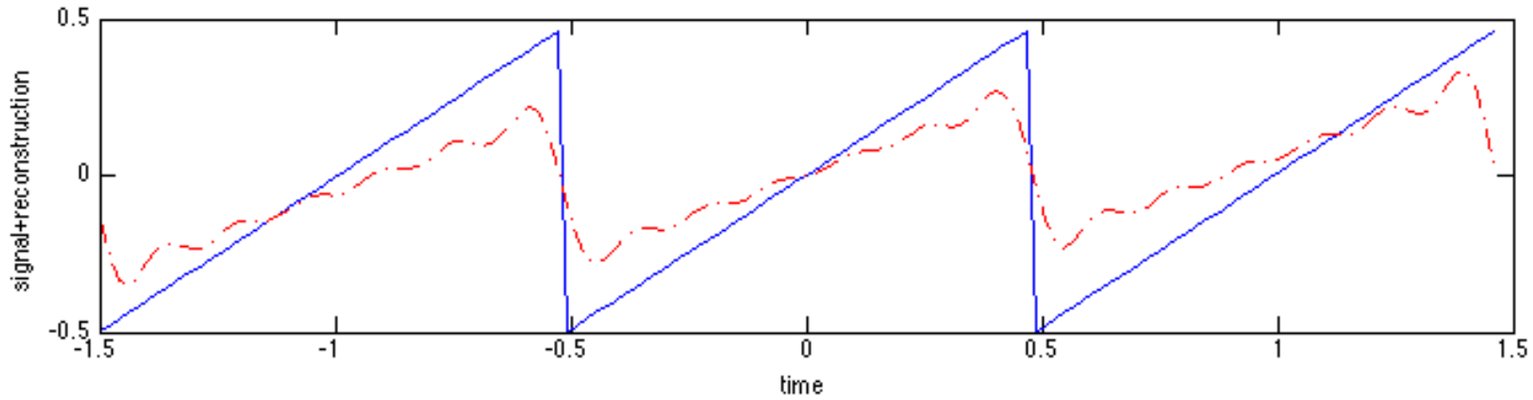


N=50

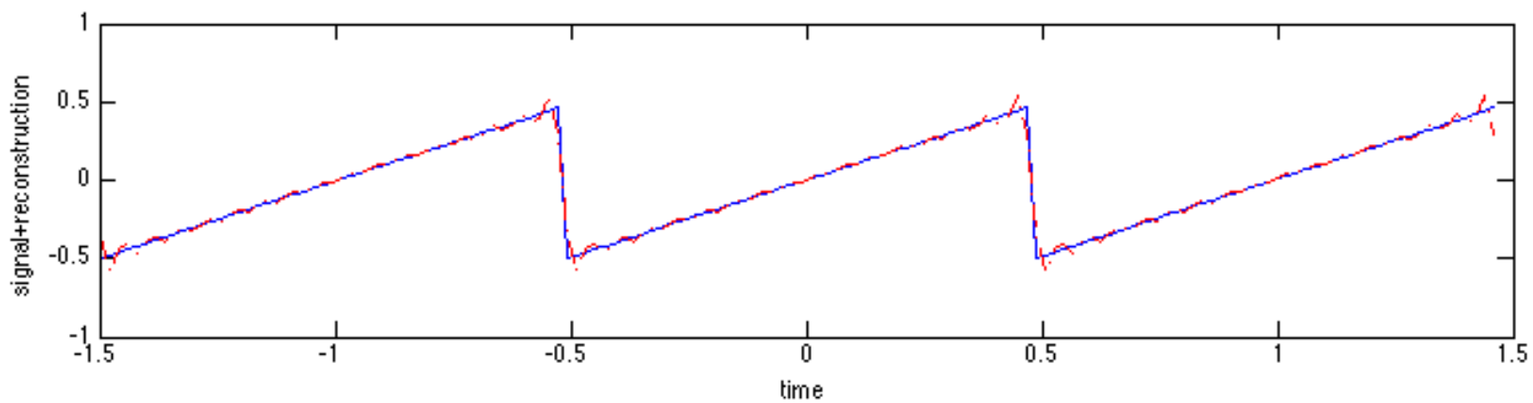
(Fourier_4.m)



N=5



N=20



N=50

um den Sägezahn gut aufzulösen muss man viele höheren Frequenzen berücksichtigen

II. Fourier Analyse

II.1. Grundlagen

a) Koeffizienten

b) Fourier Theorem

II.2. Mögliche Fehler in der Fourier Analyse

Aliasing

Periodizität

Spectral leakage

II.3. Berechnung von Spektren

Fourier transform

Fehler

a) $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

zeitlich unbegrenzt,
zeit-kontinuierlich

b) $$X_s(f) = \Delta t \sum_{n=-\infty}^{\infty} x(t_n) e^{-i2\pi f t_n}$$

zeitlich unbegrenzt,
zeit-diskret **aliasing**

c) $$X(f) = \int_{-T/2}^{T/2} x(t) e^{-i2\pi f t} dt$$

zeitlich begrenzt,
zeit-kontinuierlich

spectral leakage

d) $$X_s(f) = \Delta t \sum_{n=-M/2}^{M/2} x(t_n) e^{-i2\pi f t_n}$$

zeitlich begrenzt,
zeit-diskret

a) $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

zeitlich unbegrenzt,
zeit-kontinuierlich

Frage: was ist die inverse Fourier-Transformation ?

Ansatz: $$x(t) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} X(f') e^{i2\pi f' t} df'$$

$\longrightarrow$ bestimme $\mathcal{N}$

setze ein:

$$X(f) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(f') e^{i2\pi f' t} e^{-i2\pi f t} df' dt$$

$$= \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} X(f') \left[\int_{-\infty}^{\infty} e^{i2\pi (f' - f) t} dt \right] df'$$

.....

$$\tau = 2\pi t$$

$$= \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} X(f') \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(f'-f)\tau} d\tau \right]}_{= \delta(f'-f)} df'$$

$$= \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} X(f') \delta(f'-f) df'$$

$$= \frac{1}{\mathcal{N}} X(f) \quad \rightarrow \mathcal{N} = 1$$

inverse Fourier-Transformation: $x(t) = \int_{-\infty}^{\infty} X(f') e^{i2\pi f' t} df'$

Frage: wie transformiert sich eine
skalierte Zeit/Frequenz und eine
Translation in Zeit/Frequenz ?

$$\mathcal{F}[x(t)](\nu) = X(\nu) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi\nu t} dt$$

$$\mathcal{F}[x(at)](\nu) = \int_{-\infty}^{\infty} x(at) e^{-i2\pi\nu t} dt \ , \ a \in \mathcal{R}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-i2\pi\nu\tau/a} d\tau \ , \ a \in \mathcal{R}$$

$$\mathcal{F}[x(at)](\nu) = \frac{1}{a} \mathcal{F}[x]\left(\frac{\nu}{a}\right)$$

Frage: wie transformiert sich eine
skalierte Zeit/Frequenz und eine
Translation in Zeit/Frequenz ?

$$\mathcal{F}[x(t+b)](\nu) = \int_{-\infty}^{\infty} x(t+b)e^{-i2\pi\nu t}dt \ , \ b \in \mathcal{R}$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-i2\pi\nu\tau}e^{-i2\pi\nu b}d\tau$$

$$\mathcal{F}[x(t+b)](\nu) = \mathcal{F}[x](\nu)e^{-i2\pi\nu b}$$

Fourier transform

Fehler

a) $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

zeitlich unbegrenzt
zeit-kontinuierlich

b) $$X_s(f) = \Delta t \sum_{n=-\infty}^{\infty} x(t_n) e^{-i2\pi f t_n}$$

zeitlich unbegrenzt,
zeit-diskret **aliasing**

c) $$X(f) = \int_{-T/2}^{T/2} x(t) e^{-i2\pi f t} dt$$

zeitlich begrenzt
zeit-kontinuierlich

spectral leakage

d) $$X_s(f) = \Delta t \sum_{n=-M/2}^{M/2} x(t_n) e^{-i2\pi f t_n}$$

zeitlich begrenzt,
zeit-diskret

Sampling-Effekt auf Fourier Transformation - **aliasing**

gegeben: **unendlich lange** *abgetastete* Zeitserie

$$t_n = n\Delta t \longleftrightarrow f_s = 1/\Delta t$$

Abtastrate

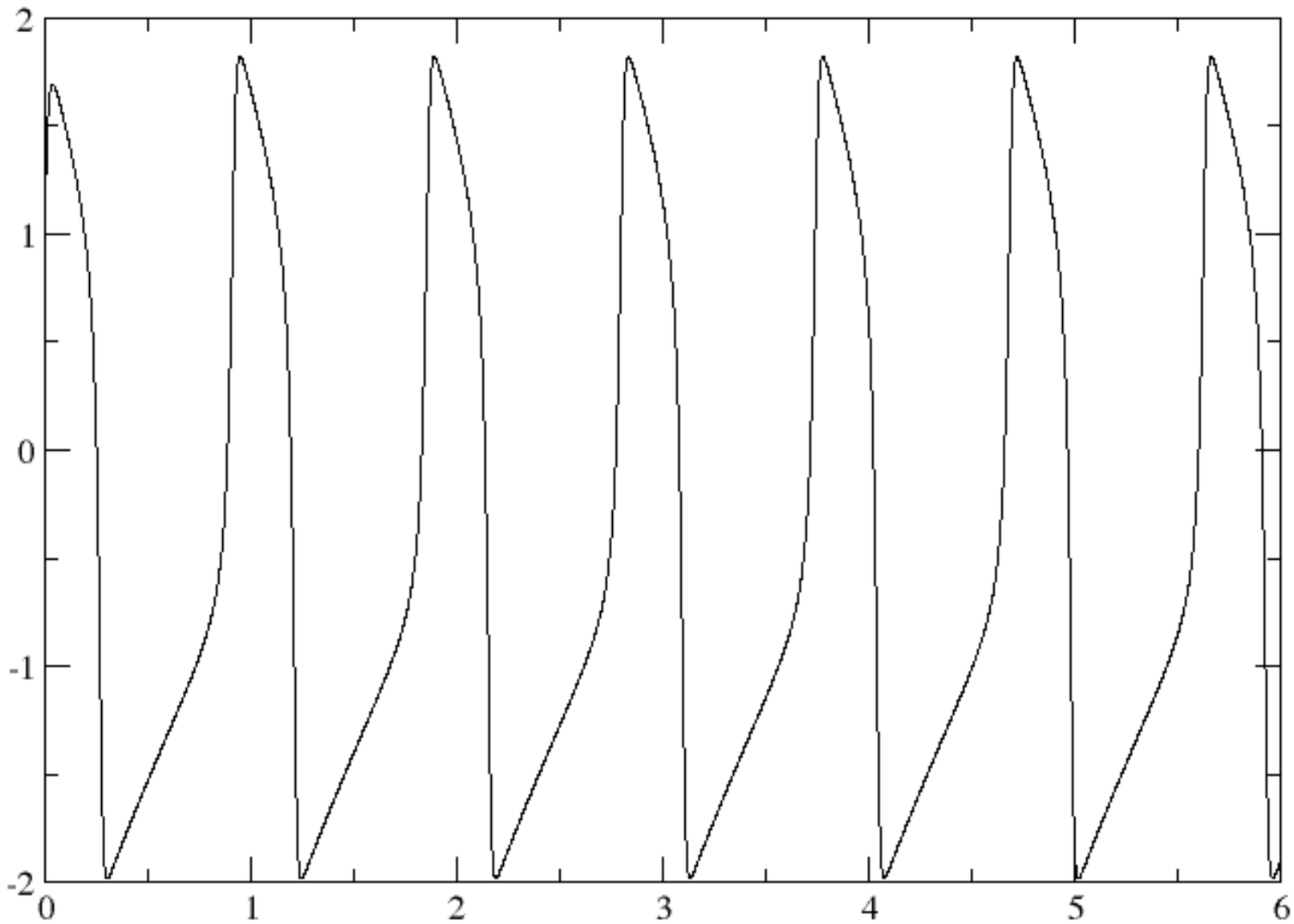
$$\longrightarrow \quad \int_{-\infty}^{\infty} dt \cdot \rightarrow \Delta t \sum_{n=-\infty}^{\infty} \cdot$$

$$\longrightarrow \quad X_s(f) = \Delta t \sum_{n=-\infty}^{\infty} x(t_n) e^{-i2\pi f t_n}$$

Discrete Time Fourier Transform (DTFT)

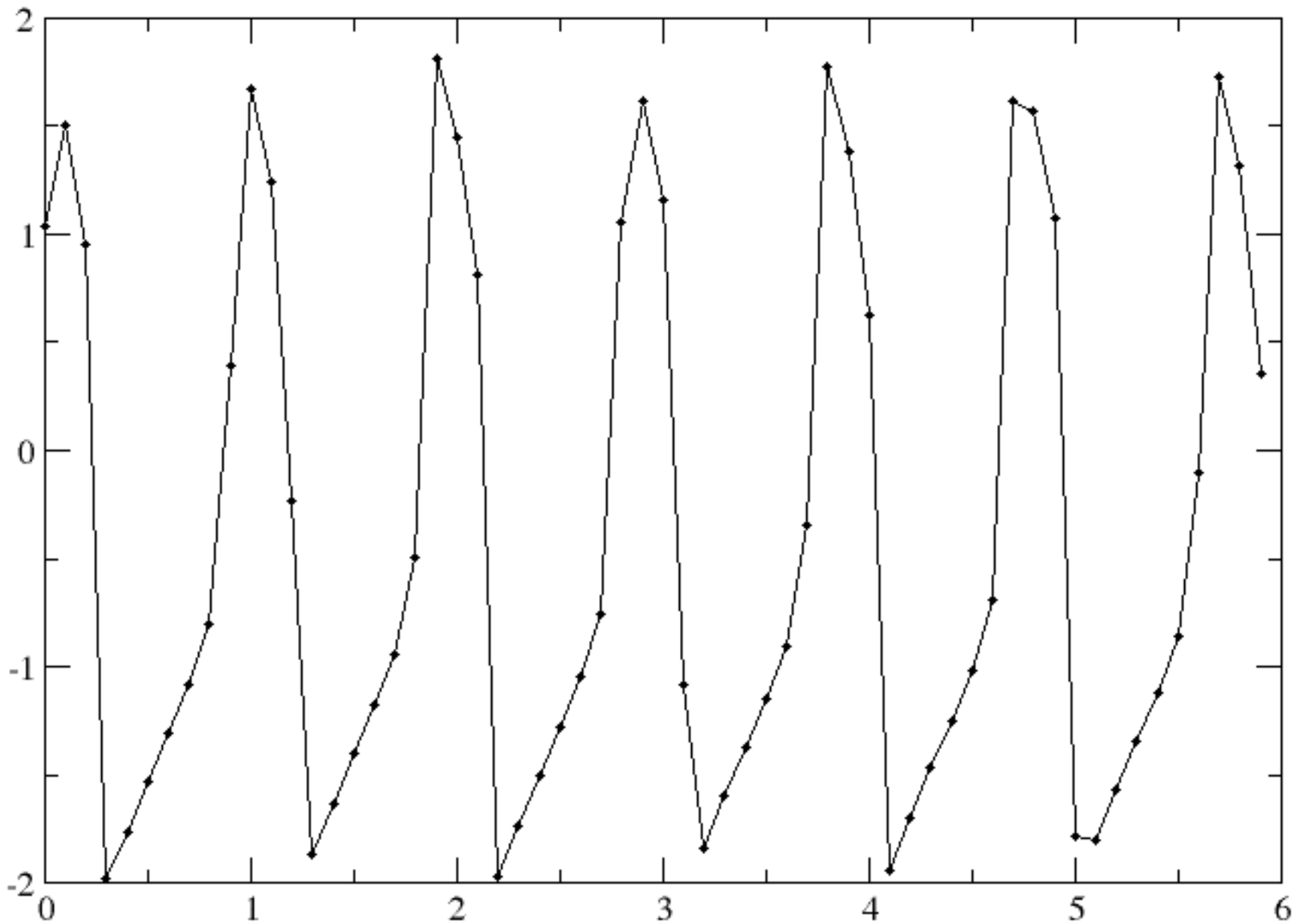
Beispiel

Originalsignal mit Abtastrate **1000Hz**



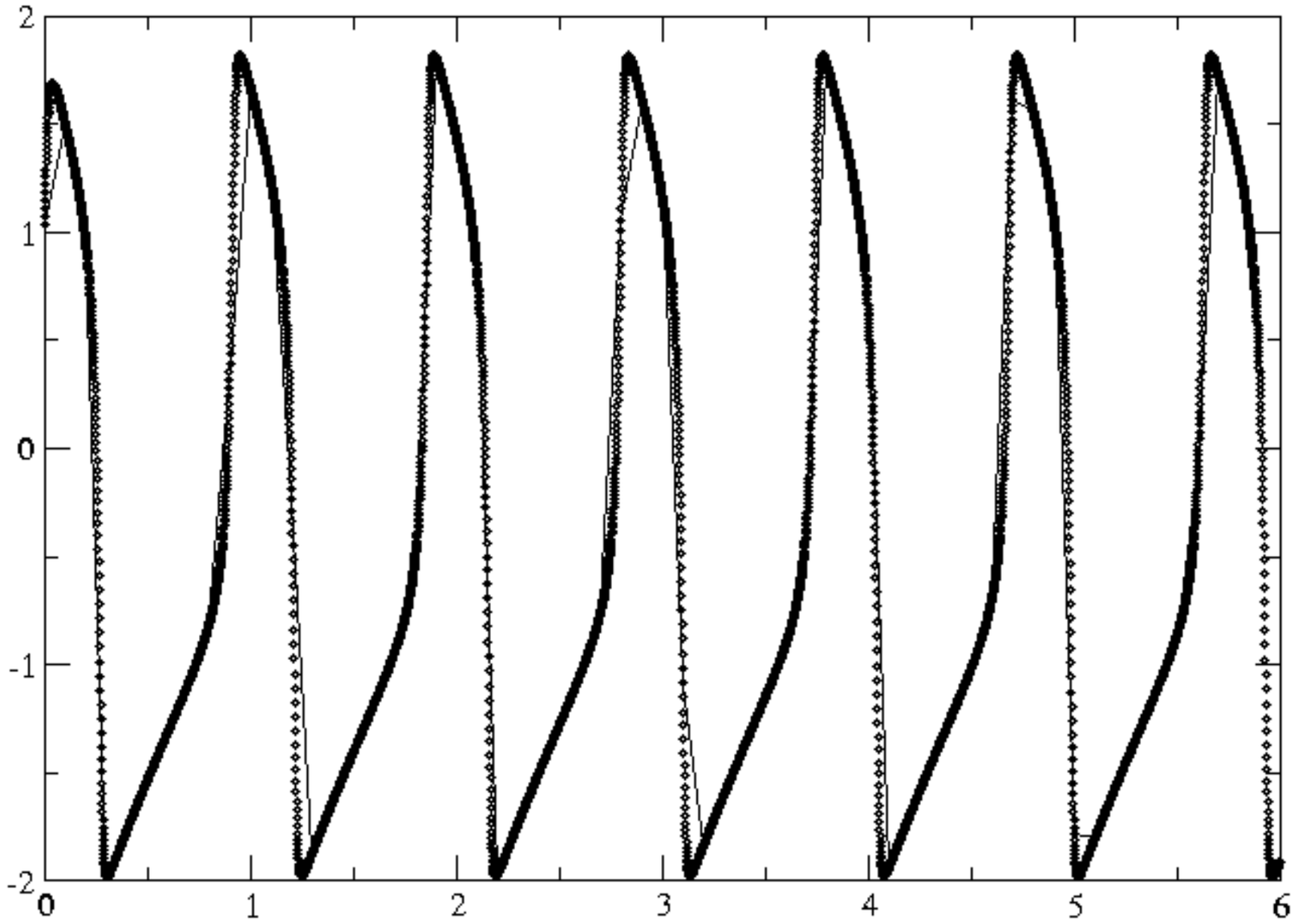
Zeit [s]

Abtastrate **10Hz**



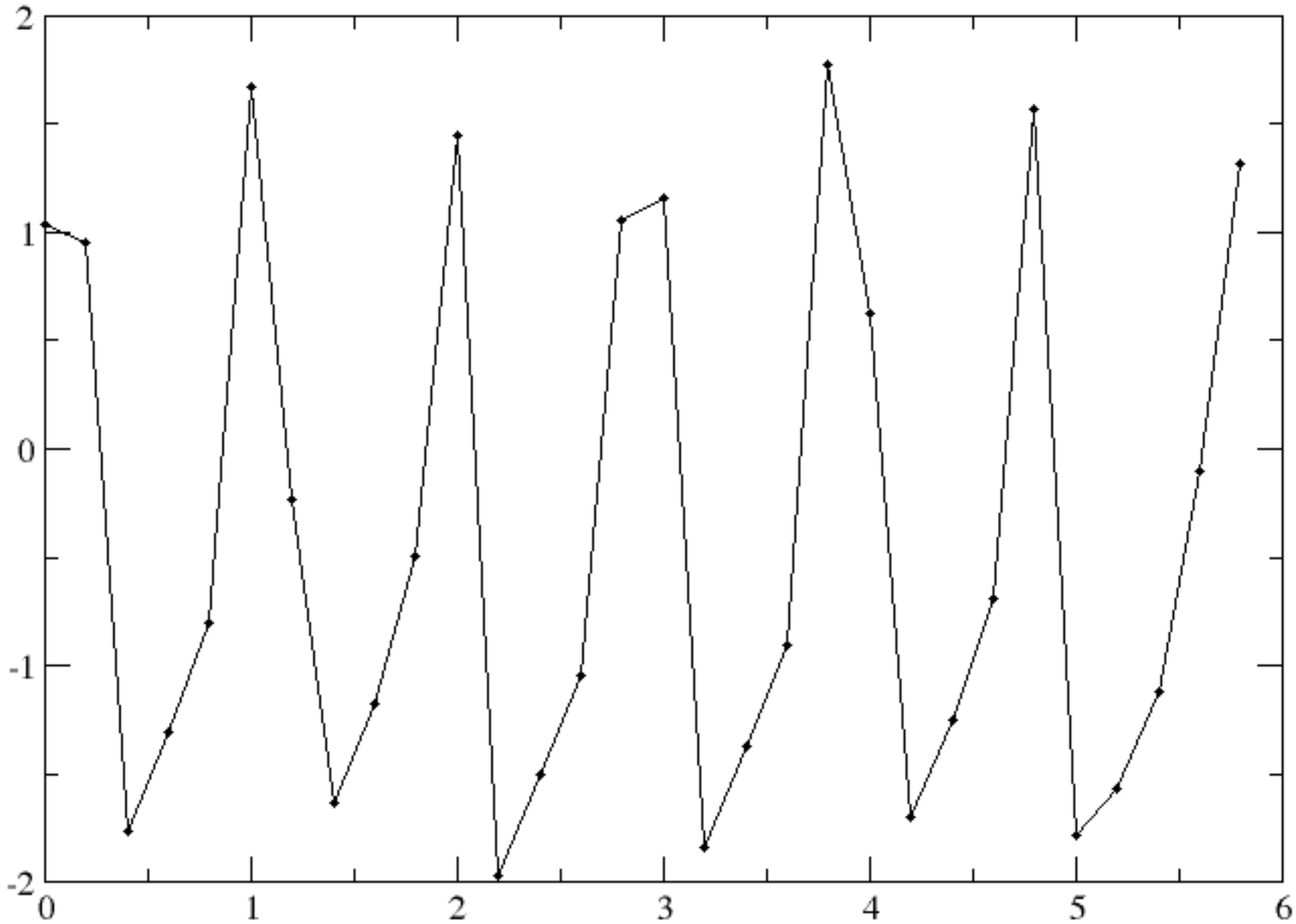
Zeit [s]

Abtastrate **1000+10Hz**



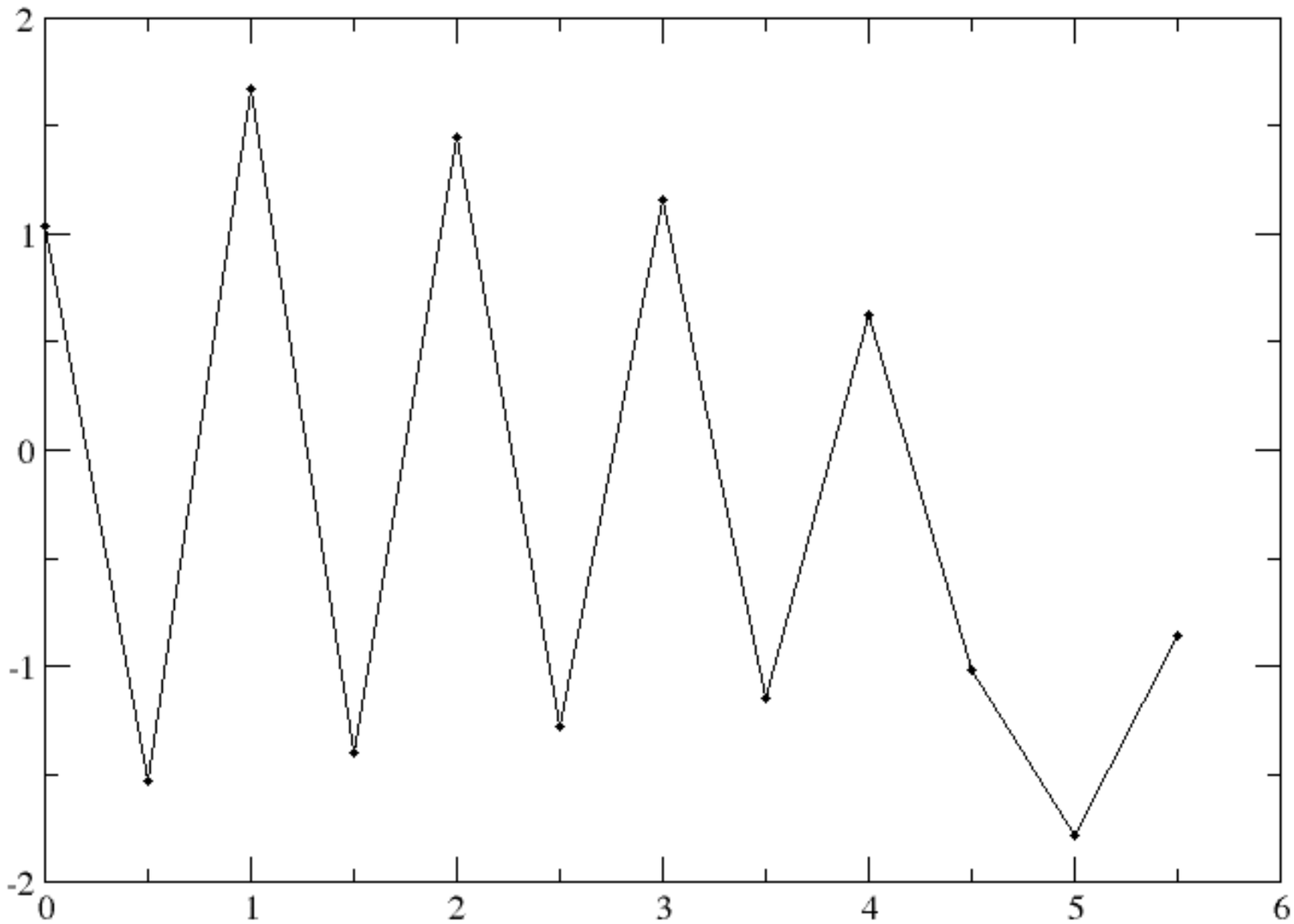
Zeit [s]

Abtastrate **5Hz**



Zeit [s]

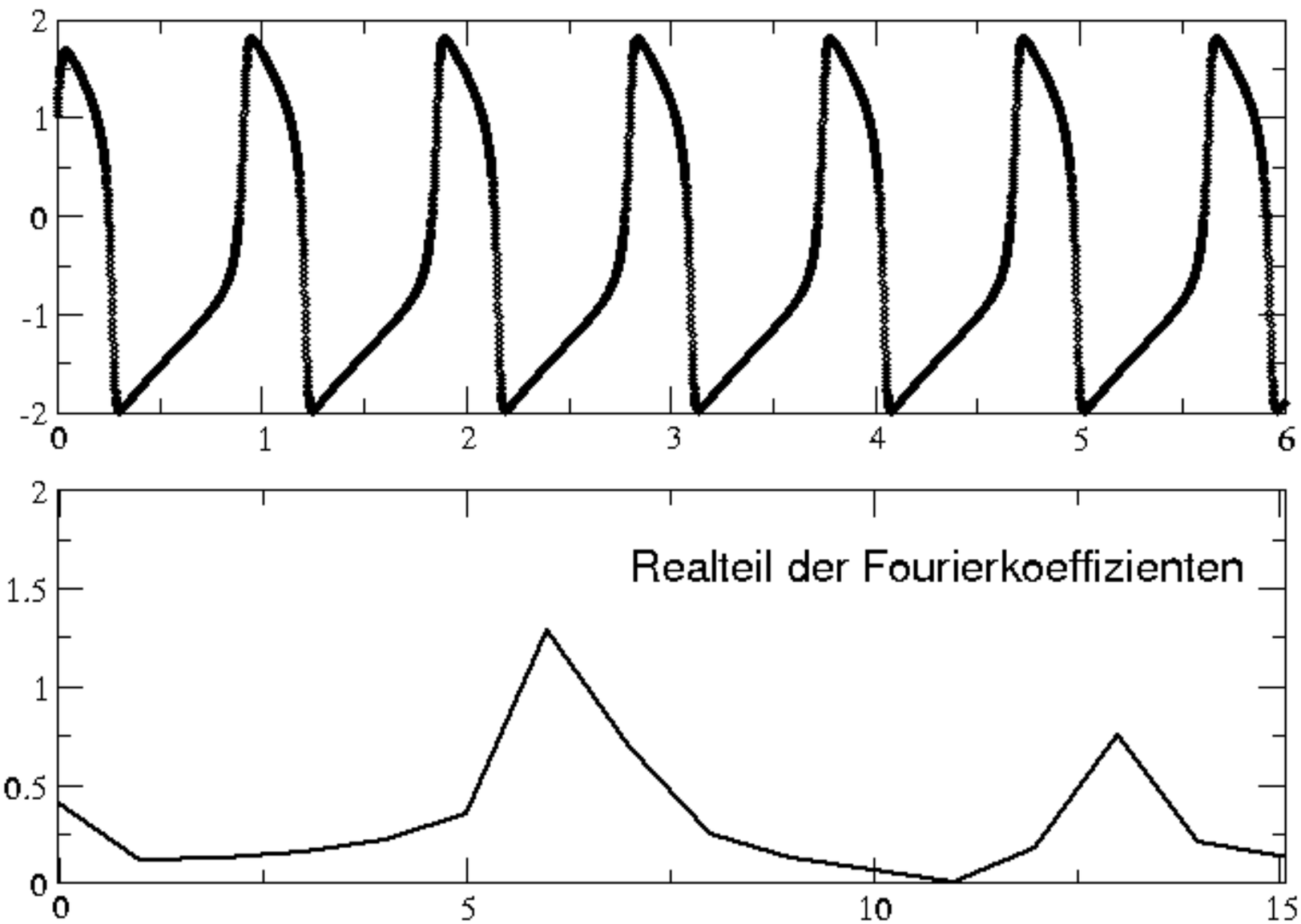
Abtastrate **2Hz**



Zeit [s]

Fourier-Transformation

Abtastrate **1000Hz**



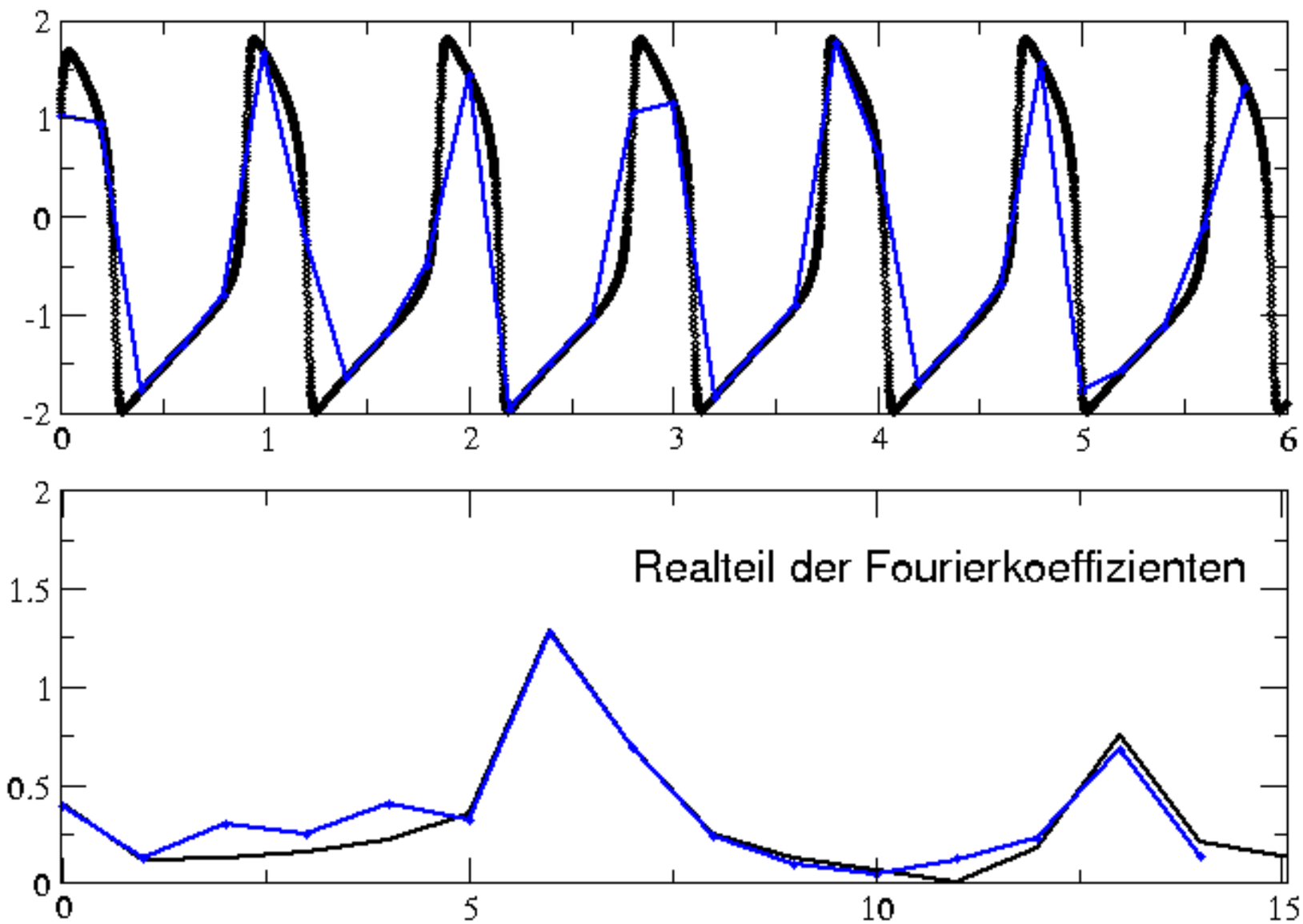
Mode

T=6s ⟶ Δf=0.166Hz

Δt=0.001s ⟶ N=6000

Fourier-Transformation

Abtastrate **5Hz**



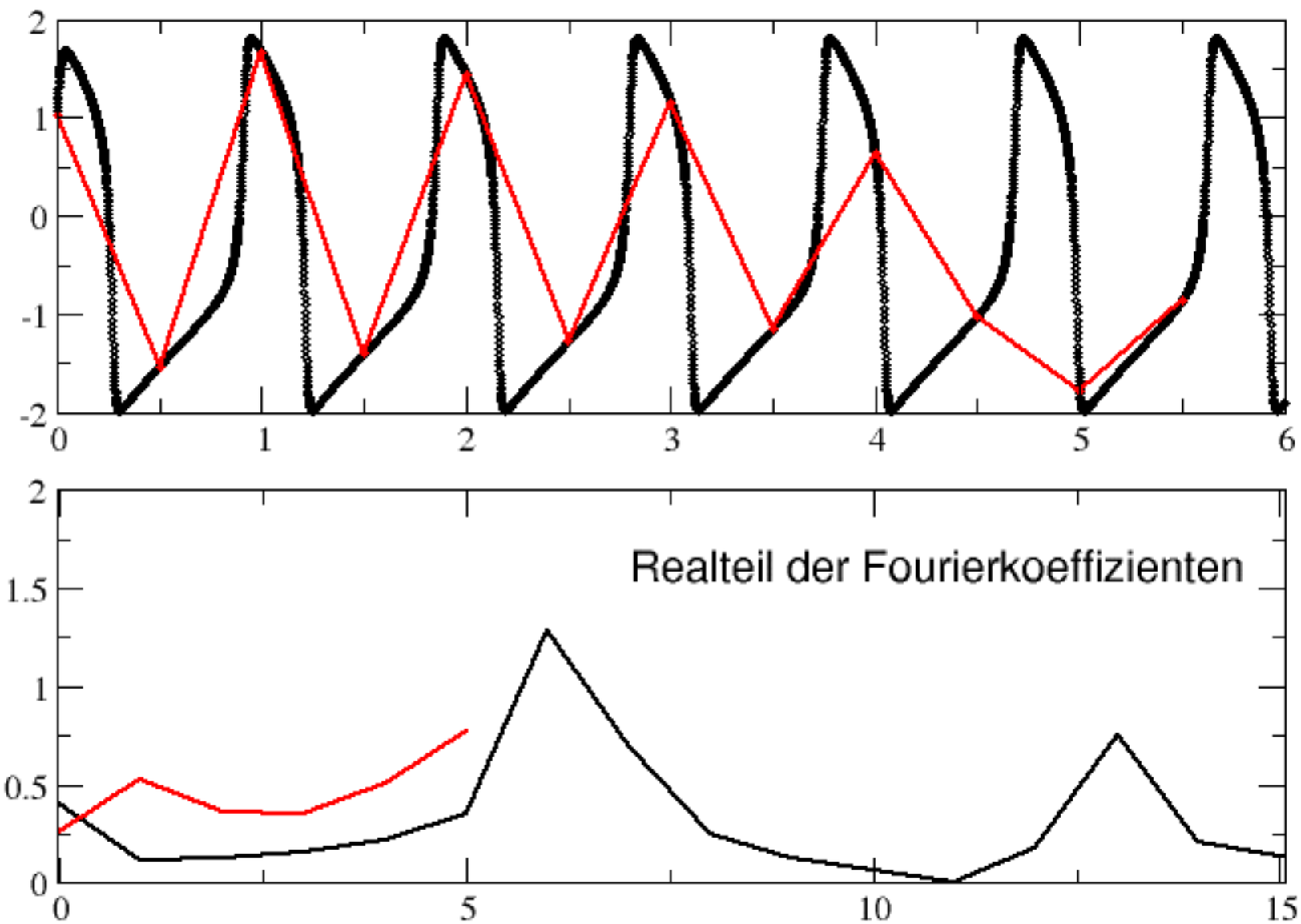
Mode

T=6s ⟶ Δf=0.166Hz

Δt=0.2s ⟶ N=30

Fourier-Transformation

Abtastrate **2Hz**



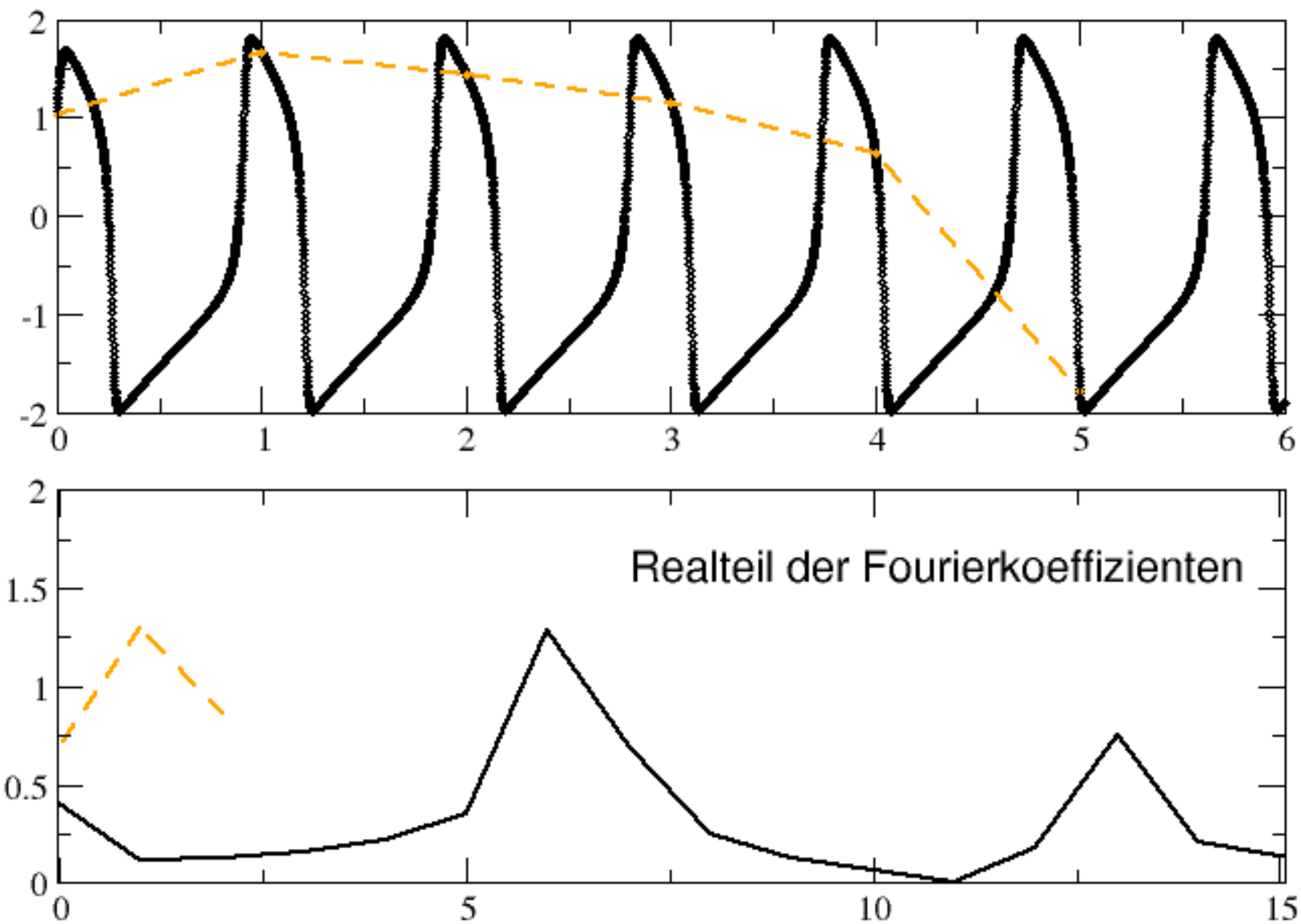
Mode

T=6s ⟶ Δf=0.166Hz

Δt=0.5s ⟶ N=12

Fourier-Transformation

Abtastrate **1Hz**



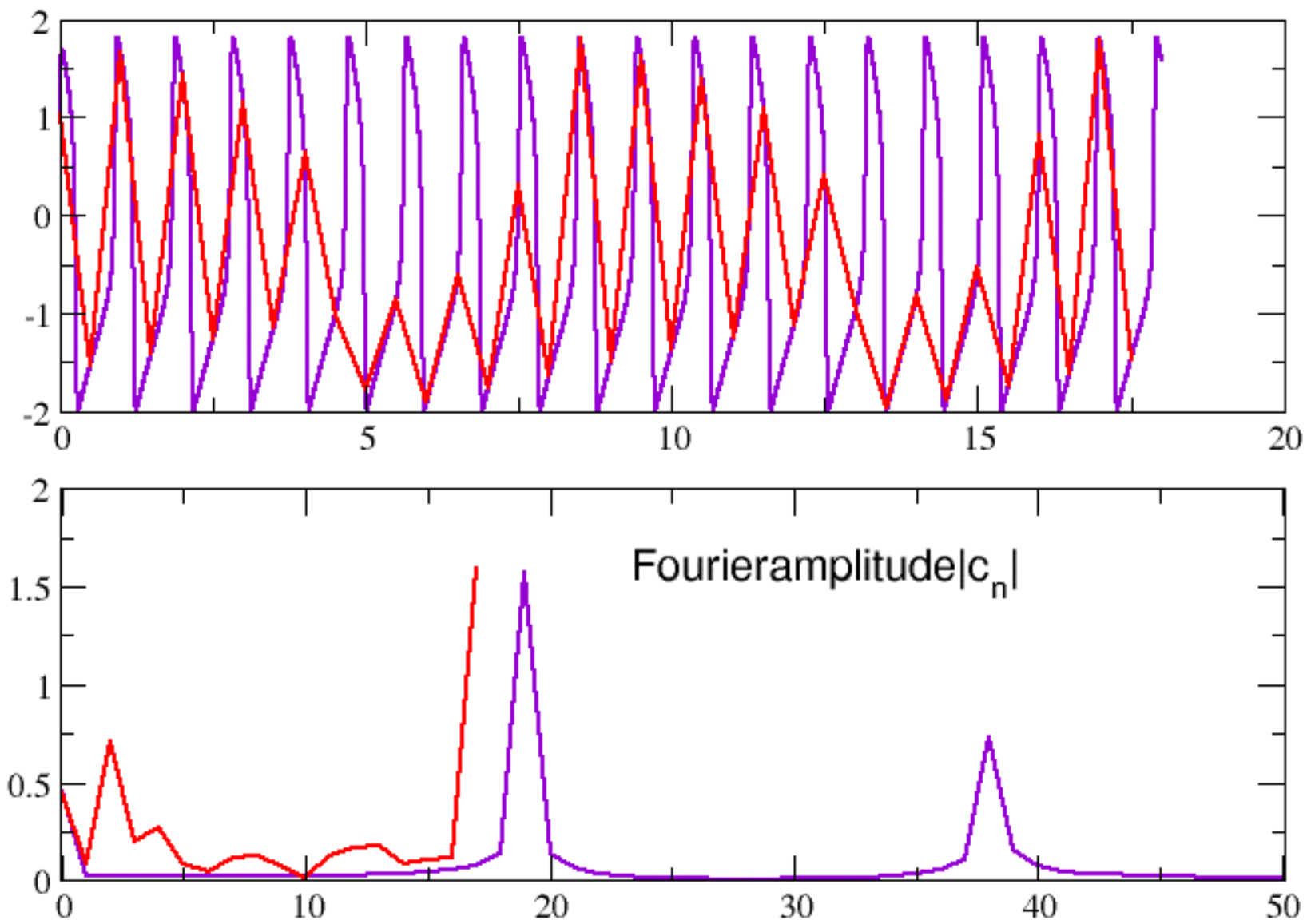
Mode

T=6s ⟶ Δf=0.166Hz

Δt=1s ⟶ N=6

longer time series

Abtastrate **2Hz**



Mode

T=18s ⟶ Δf=(1/18)Hz

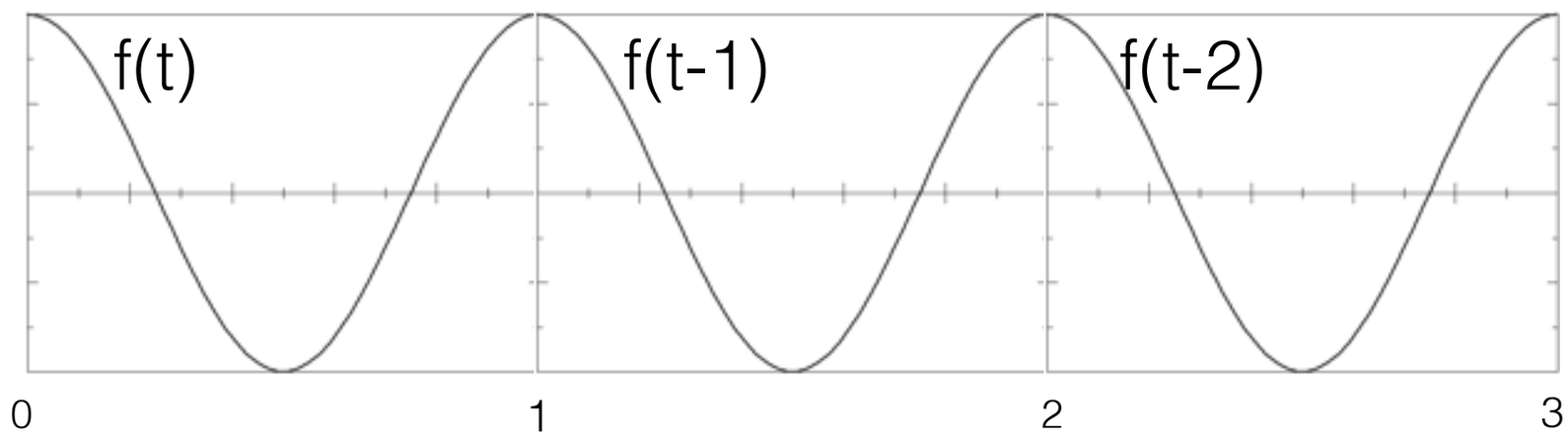
Δt=0.5s ⟶ N=36

Einschub: Poisson Summenformel

$$g(t) = \sum_{n \in \mathcal{Z}} f(t+n)$$

(f: periodisch mit Periode 1)

Beispiel:



$$g(t) = \sum_{k \in \mathcal{Z}_0} c_k e^{i2\pi kt/T} \ , \ T = 1$$

$$c_k = \int_0^T g(t) e^{-i2\pi kt/T} dt$$

$$= \int_0^T \sum_{n \in \mathcal{Z}} f(t+n) e^{-i2\pi kt/T} dt$$

$$= \int_0^1 \sum_{n \in \mathcal{Z}} f(t+n) e^{-i2\pi kt} \underbrace{e^{-i2\pi kn}}_{=1} dt$$

$$= \int_0^1 \sum_{n \in \mathcal{Z}} f(t+n) e^{-i2\pi k(t+n)} dt$$

$$s = t + n$$

$$= \sum_{n \in \mathcal{Z}} \int_n^{n+1} f(s) e^{-i2\pi ks} ds$$

$$= \int_{-\infty}^{\infty} f(s) e^{-i2\pi ks} ds$$

$$= X(k)$$

$$\rightarrow g(t) = \sum_{k \in \mathcal{Z}} X(k) e^{i2\pi kt} \quad = \sum_{n \in \mathcal{Z}} f(t+n)$$

$$t = 0: \quad \rightarrow \quad \boxed{\sum_{n \in \mathcal{Z}} f(n) = \sum_{k \in \mathcal{Z}} X(k)}$$

Poisson Summenformel

falls x(t) eine Schwartz-Funktion ist,

also $x(t) \in \mathcal{S}$ mit

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \phi \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty \right\}$$

$$= \left\{ \phi \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \alpha, \beta \in \mathbb{N}_0^n, \exists C \geq 0, \forall x \in \mathbb{R}^n : |x^\alpha D^\beta \phi(x)| \leq C \right\}.$$

es gilt (siehe Übungen für Beweis):

$$\sum_{n\in\mathcal{Z}} X(f+nf_s) = \sum_{k\in\mathcal{Z}} \frac{1}{f_s} x\left(\frac{k}{f_s}\right) e^{-i2\pi k f/f_s}$$

$$f_s = \frac{1}{\Delta t}$$

$$\rightarrow \quad = \sum_{k\in\mathcal{Z}} \Delta t \; x(k\Delta t) e^{-i2\pi f k \Delta t}$$

$$= \Delta t \sum_{k\in\mathcal{Z}} x(t_k) e^{-i2\pi f t_k} \qquad \text{(DTFT)}$$

$$= \Delta t \int_{-\infty}^{\infty} \underbrace{\left(\sum_{k\in\mathcal{Z}} \delta(t-t_k)\; x(t)\right)}_{\text{abgetastetes Signal}} e^{-i2\pi f t}\; dt$$

$$= \Delta t \; \mathcal{F}\left[\sum_{k\in\mathcal{Z}} \delta(t-t_k)\; x(t)\right]$$

$$\rightarrow \quad \boxed{\mathcal{F}\left[\sum_{k\in\mathcal{Z}} \delta(t-t_k)\; x(t)\right] = \frac{1}{\Delta t}\sum_{n\in\mathcal{Z}} X(f+nf_s)}$$

Fouriertransformierte von abgetastetem Signal

ist periodisch in Abtastfrequenz

also:

$$\sum_{n=-\infty}^{\infty} x(t_n) e^{-i2\pi f t_n} = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(f - k f_s)$$

kont. Fouriertransformation

Discrete-time Fourier Transform (DTFT)

$$t_n = n\Delta t$$

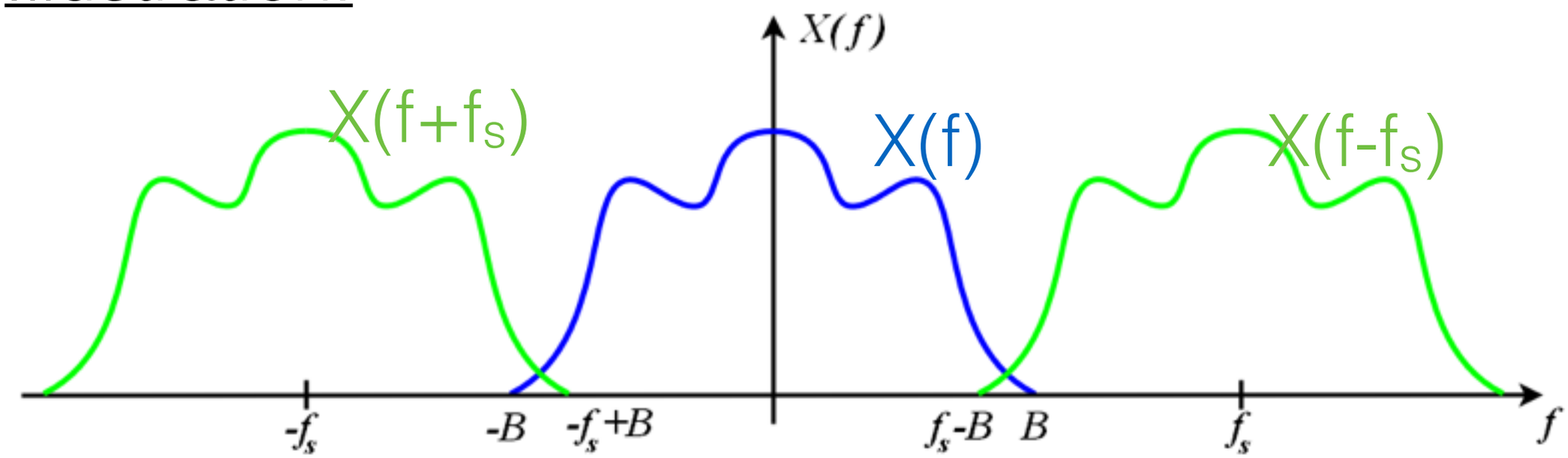
$$f_s = 1/\Delta t$$

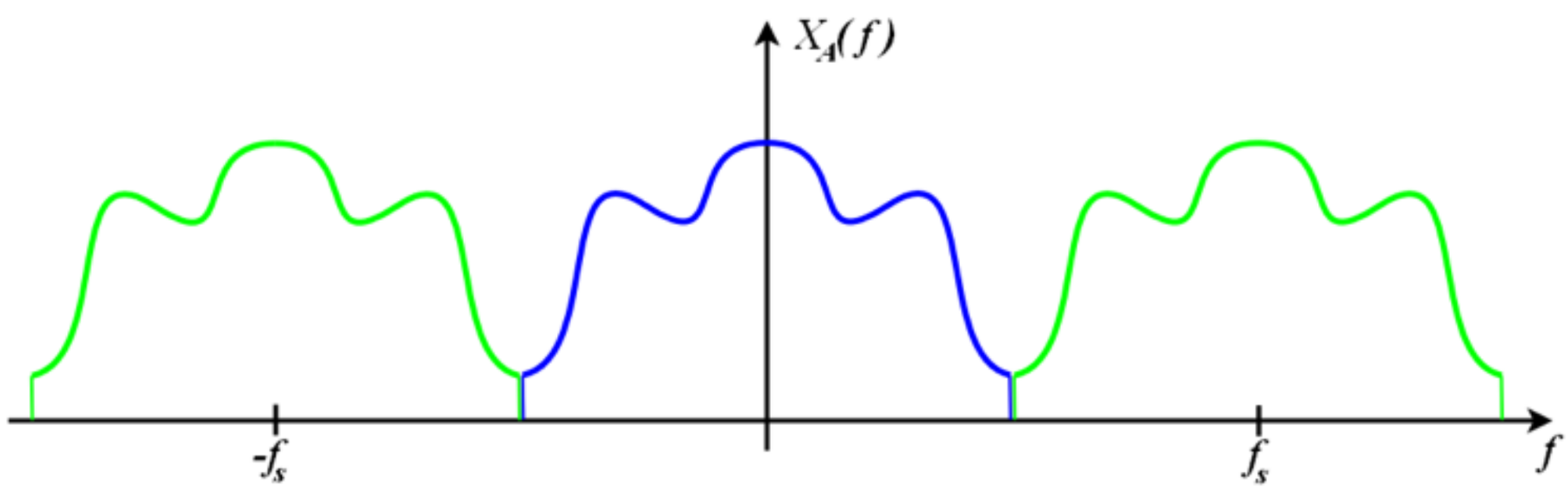
Zeitdiskretisierung erzeugt periodische Fortsetzung im Frequenzraum

kann Fehler erzeugen (aliasing) !!!

$$\sum_{n=-\infty}^{\infty} x(t_n)e^{-i2\pi f t_n} = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Illustration:





Beispiel

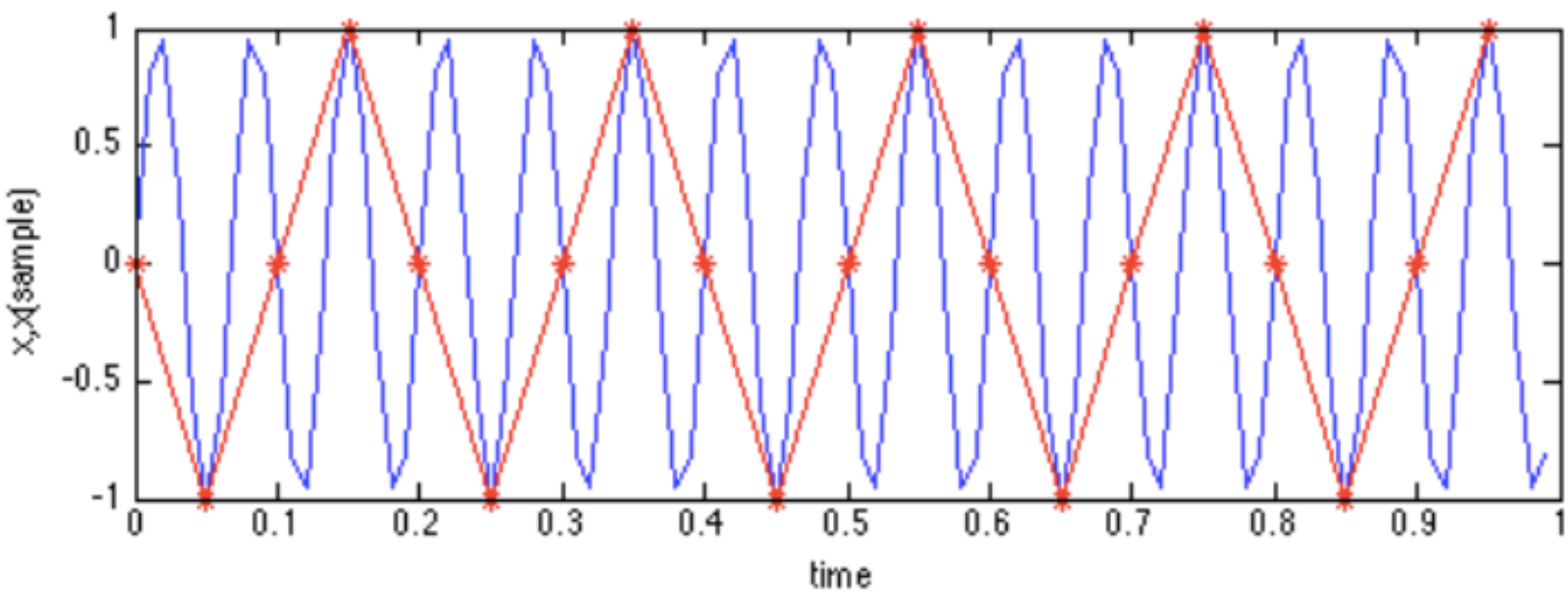
Originalsignal:

15Hz

abgetastetes Signal

sample rate: 20Hz, Nyquist: 10Hz

$\xrightarrow{?}$ 5Hz Oszillation

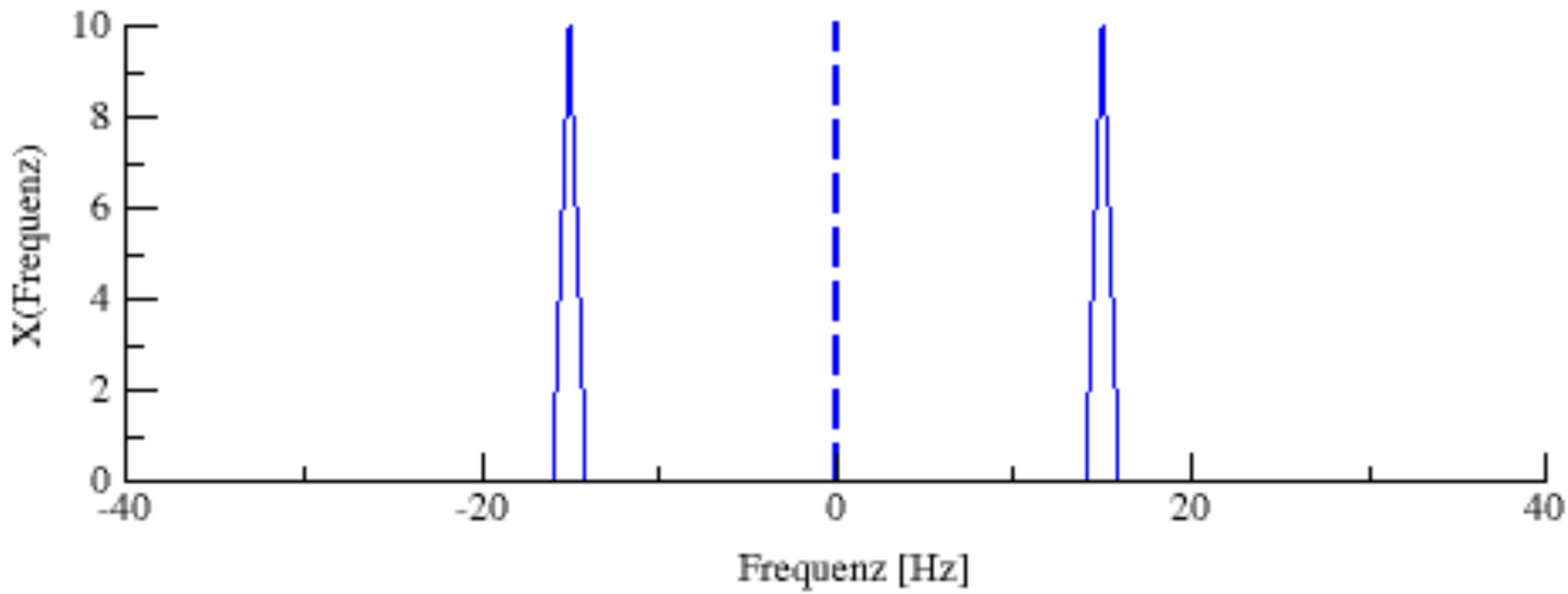


Originalsignal:

15Hz

abgetastetes Signal

sample rate: 20Hz, Nyquist: 10Hz



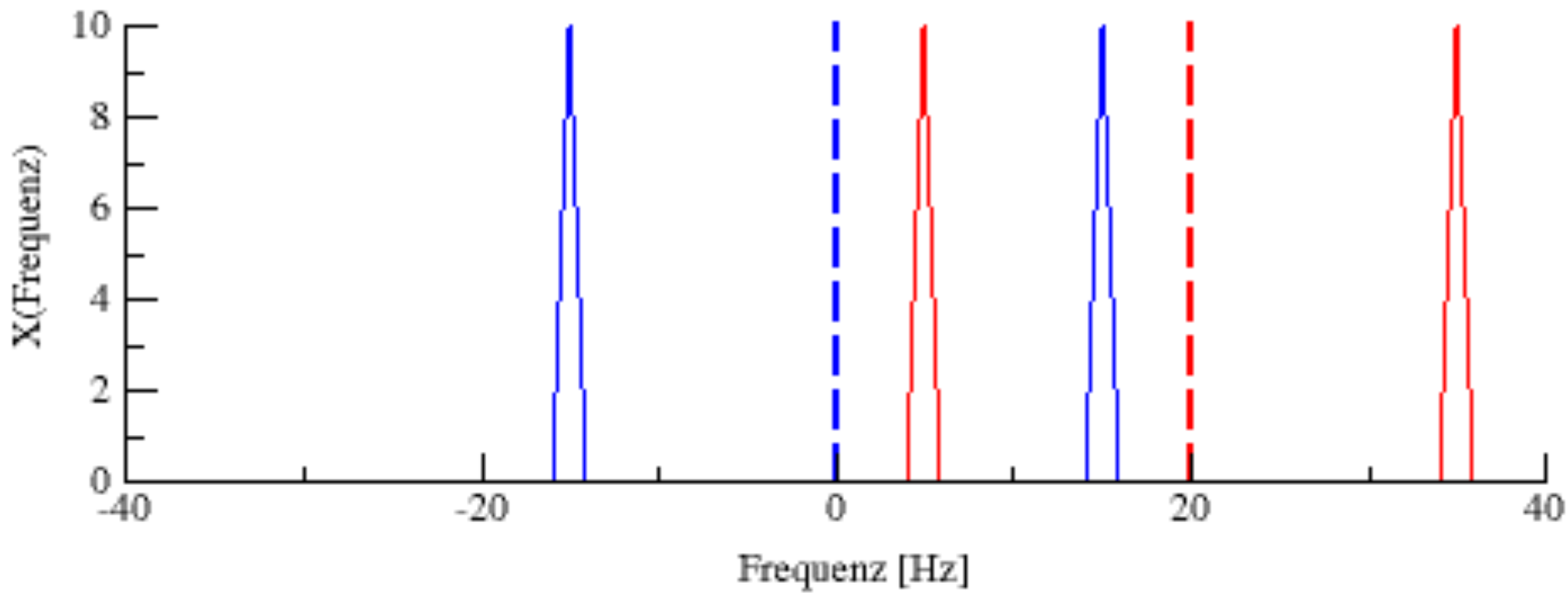
$$\sum_{n=-\infty}^{\infty} x(t_n)e^{-i2\pi f t_n} = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Originalsignal:

abgetastetes Signal

15Hz

sample rate: 20Hz, Nyquist: 10Hz



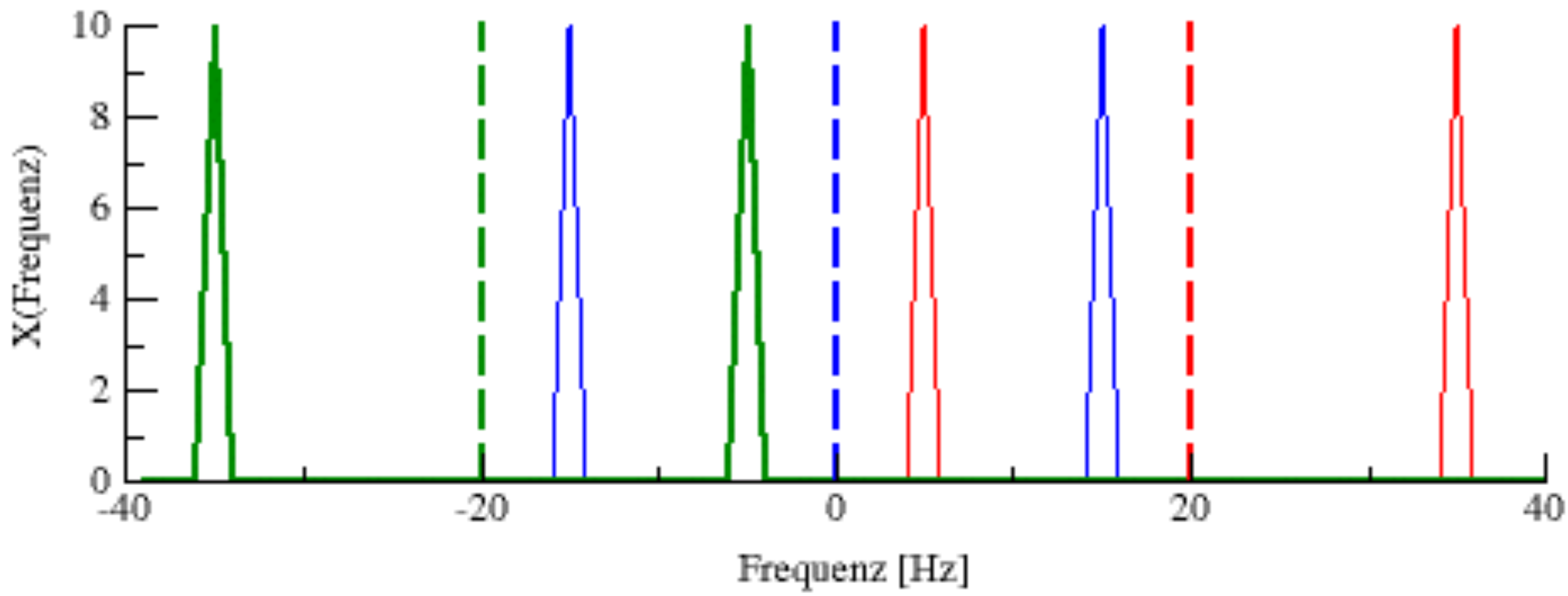
$$\sum_{n=-\infty}^{\infty} x(t_n)e^{-i2\pi f t_n} = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Originalsignal:

abgetastetes Signal

15Hz

sample rate: 20Hz, Nyquist: 10Hz

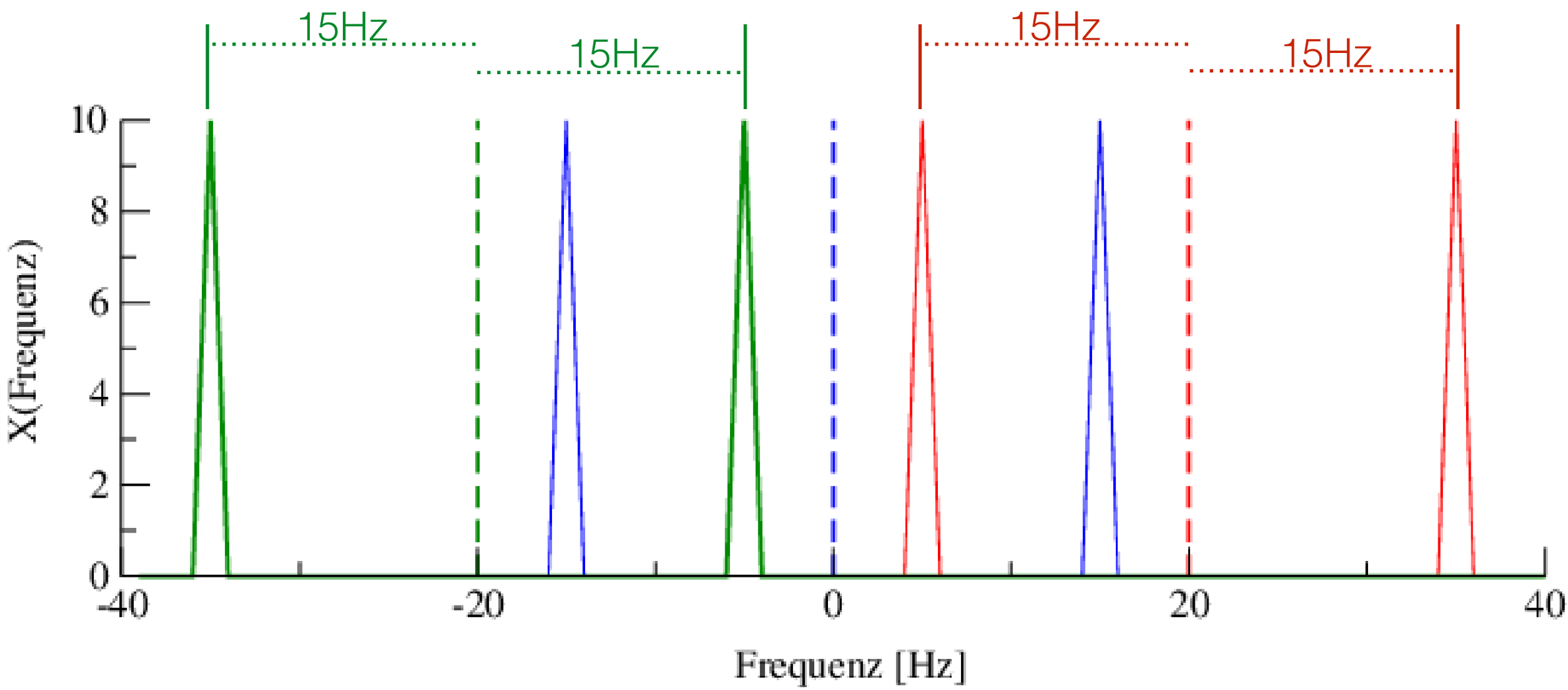


$$\sum_{n=-\infty}^{\infty} x(t_n)e^{-i2\pi f t_n} = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

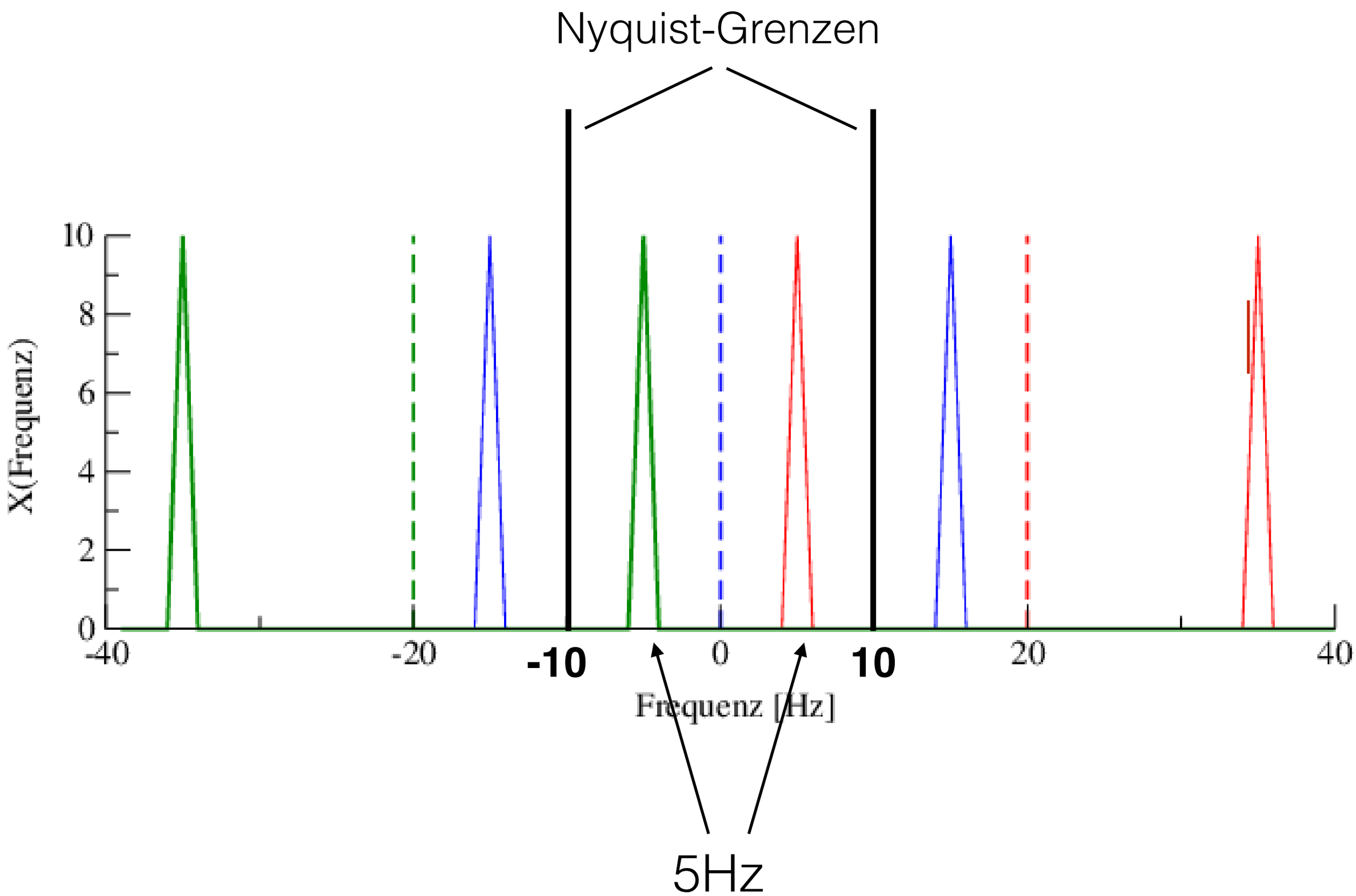
Originalsignal: abgetastetes Signal

15Hz

sample rate: 20Hz, Nyquist: 10Hz ⟶ 5Hz

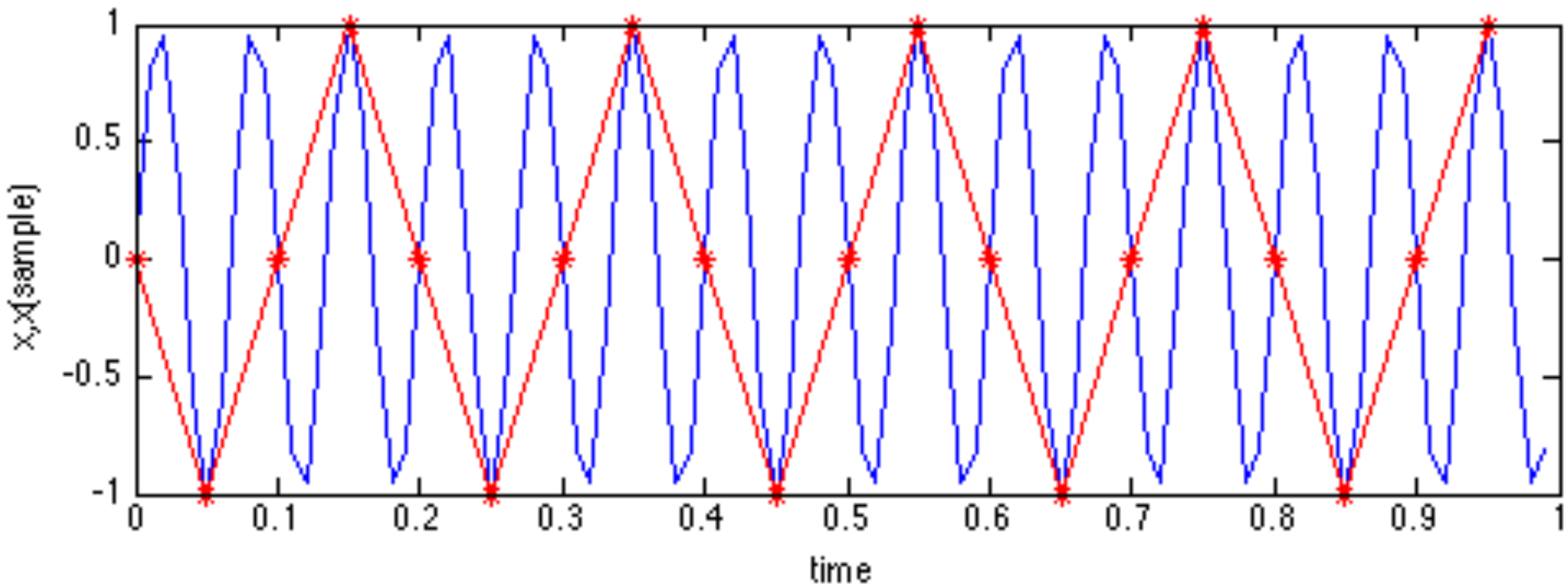


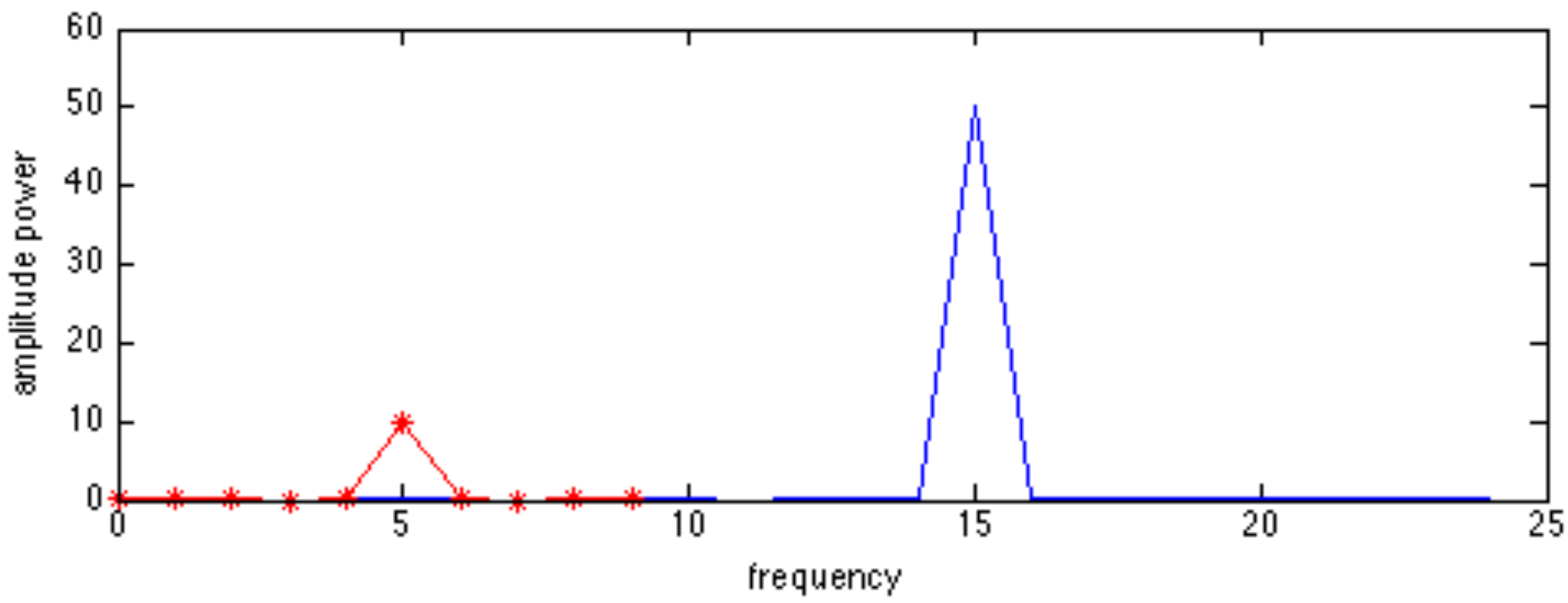
$$\sum_{n=-\infty}^{\infty} x(t_n)e^{-i2\pi f t_n} = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} X(f - kf_s)$$


Nyquist-Grenzen
10
8
6
4
2
0
X(Frequenz)
-40
-20
-10
0
10
20
40
Frequenz [Hz]
5Hz

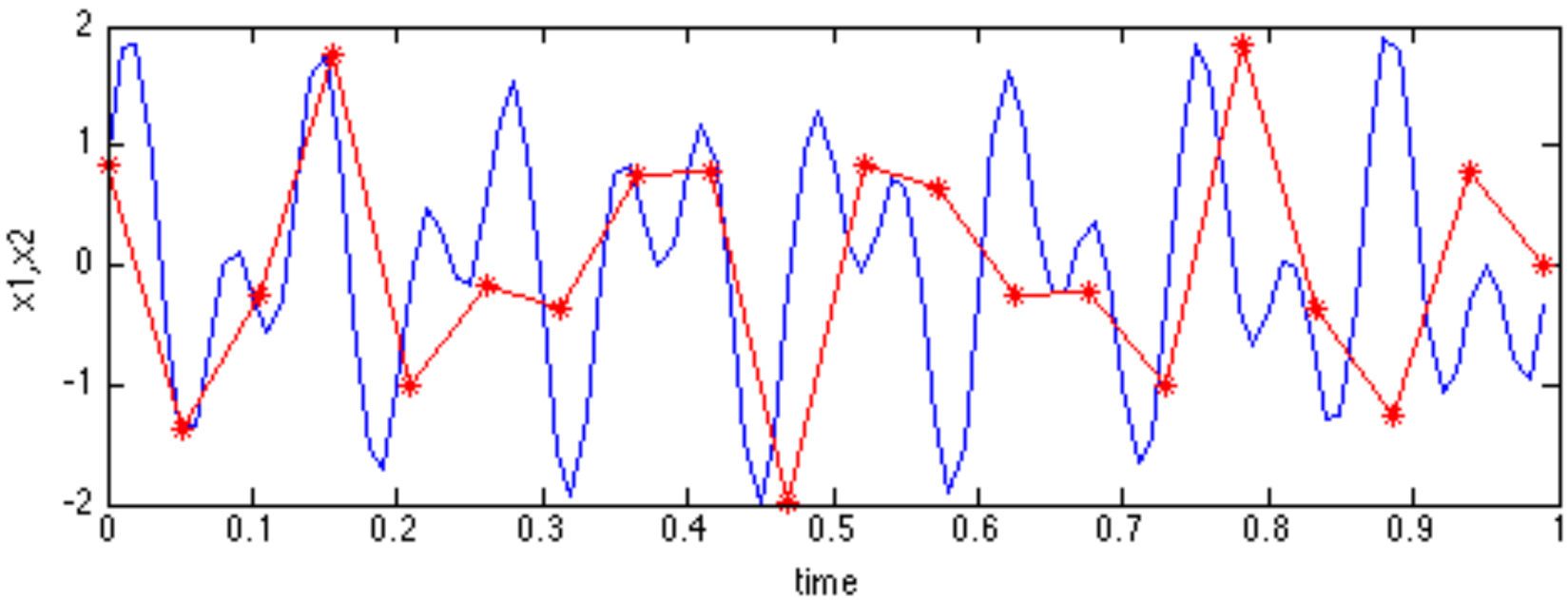
(Fourier_7.m)

Resultat:





weiteres Beispiel:



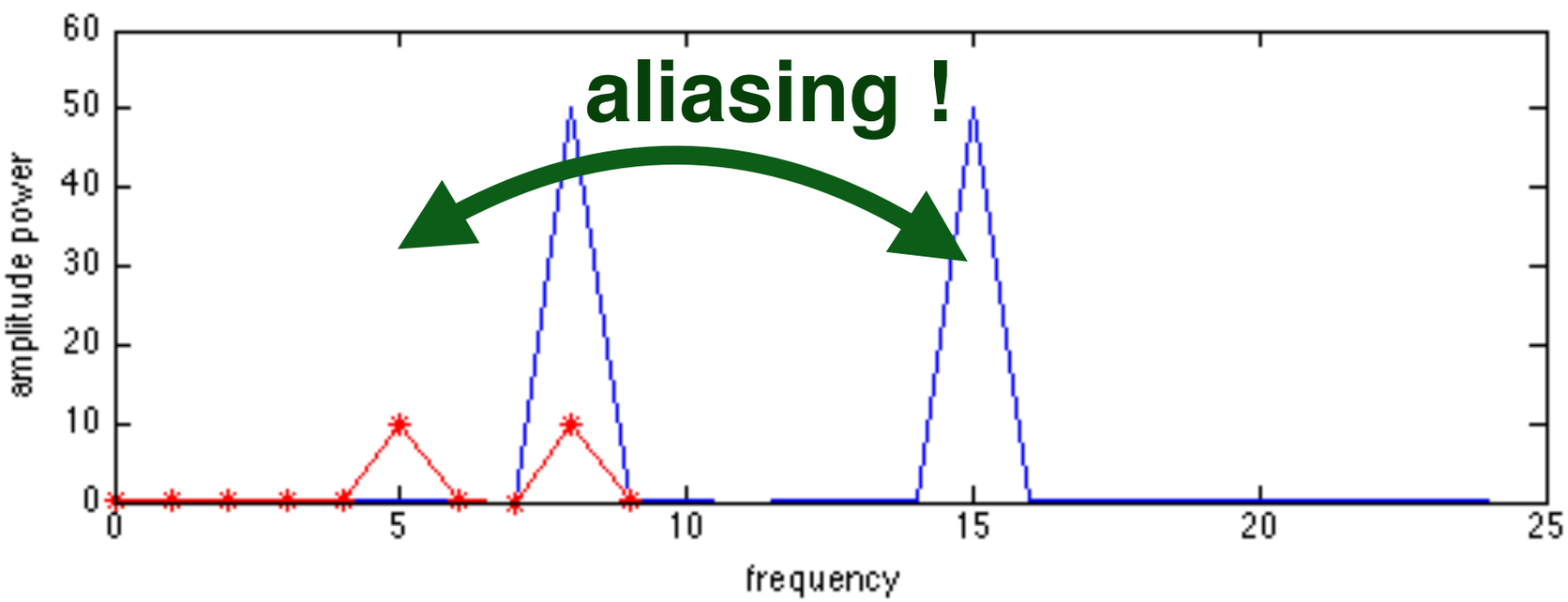
Originalsignal:

8Hz and 15Hz

abgetastetes Signal:

Abtastrate: 20Hz

Nyquist: 10Hz

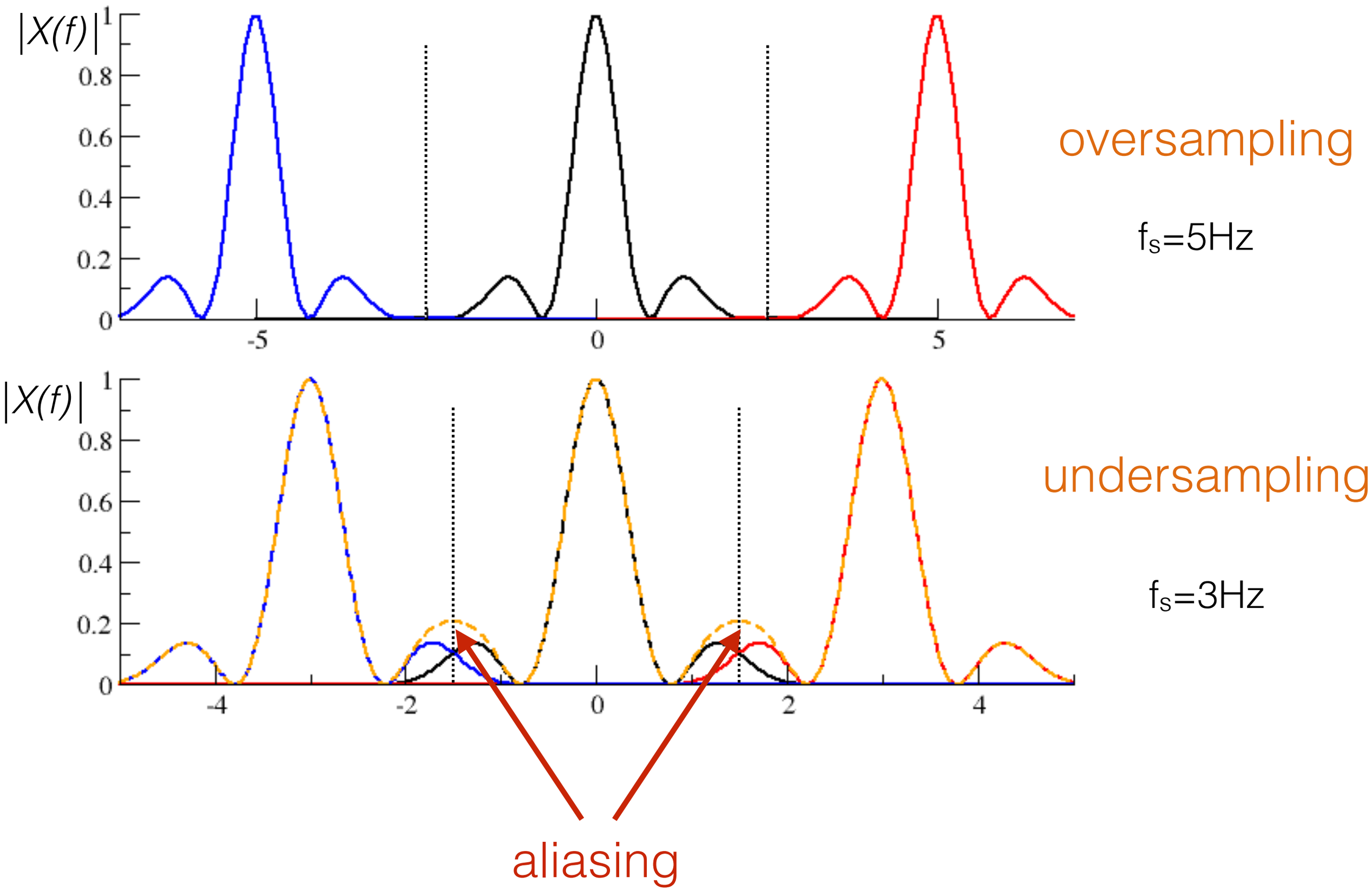


new alias frequency: 5Hz = 15Hz-Nyquist

(Fourier_6.m)

anderes Beispiel für aliasing: Stroboskop-Effekt

illustratives Beispiel



Fourier transform

Fehler

a) $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

zeitlich unbegrenzt
zeit-kontinuierlich

b) $$X_s(f) = \Delta t \sum_{n=-\infty}^{\infty} x(t_n) e^{-i2\pi f t_n}$$

zeitlich unbegrenzt,
zeit-diskret **aliasing**

c) $$X(f) = \int_{-T/2}^{T/2} x(t) e^{-i2\pi f t} dt$$

zeitlich begrenzt,
zeit-kontinuierlich

spectral leakage

d) $$X_s(f) = \Delta t \sum_{n=-M/2}^{M/2} x(t_n) e^{-i2\pi f t_n}$$

zeitlich begrenzt,
zeit-diskret

$$X_T(f) = \int_{-T/2}^{T/2} x(t) e^{-i2\pi f t} dt$$

<u>Spezialfall</u>: Signal periodisch mit Periode P

$$x(t) = x(t+P) \qquad\qquad P=T/M$$

$$= \int_{-T/2}^{T/2} x(t+P) e^{-i2\pi f t} dt$$

$$= \int_{-T/2+P}^{T/2+P} x(\tau) e^{-i2\pi f \tau} e^{i2\pi f P} d\tau$$

$$= e^{i2\pi f P} \int_{-T/2}^{T/2} x(\tau) e^{-i2\pi f \tau} d\tau$$

$$e^{i2\pi f P} = 1 \;\rightarrow\; 2\pi f P = 2\pi n, n \in \mathcal{Z}_0$$

$$\boxed{f_n = \frac{1}{P} n}$$

$$X_T(f) = X_T\left(\frac{n}{P}\right)$$

Periode im Signal diskretisiert Frequenzen

$$X_T(f) = \int_{-T/2}^{T/2} x(t)e^{-i2\pi ft}dt$$

$$= \int_{-T/2}^{-T/2+P} x(t)e^{-i2\pi ft}dt + \int_{-T/2+P}^{-T/2+2P} x(t)e^{-i2\pi ft}dt + \cdots$$

$$\cdots + \int_{T/2-P}^{T/2} x(t)e^{-i2\pi ft}dt$$

$$= \sum_{m=1}^{M} \int_{-T/2+(m-1)P}^{-T/2+mP} x(t)e^{-i2\pi ft}dt$$

$$= \sum_{m=1}^{M} \int_{-T/2+(m-1)P}^{-T/2+mP} x(t) e^{-i2\pi f t} dt$$

Periodizität

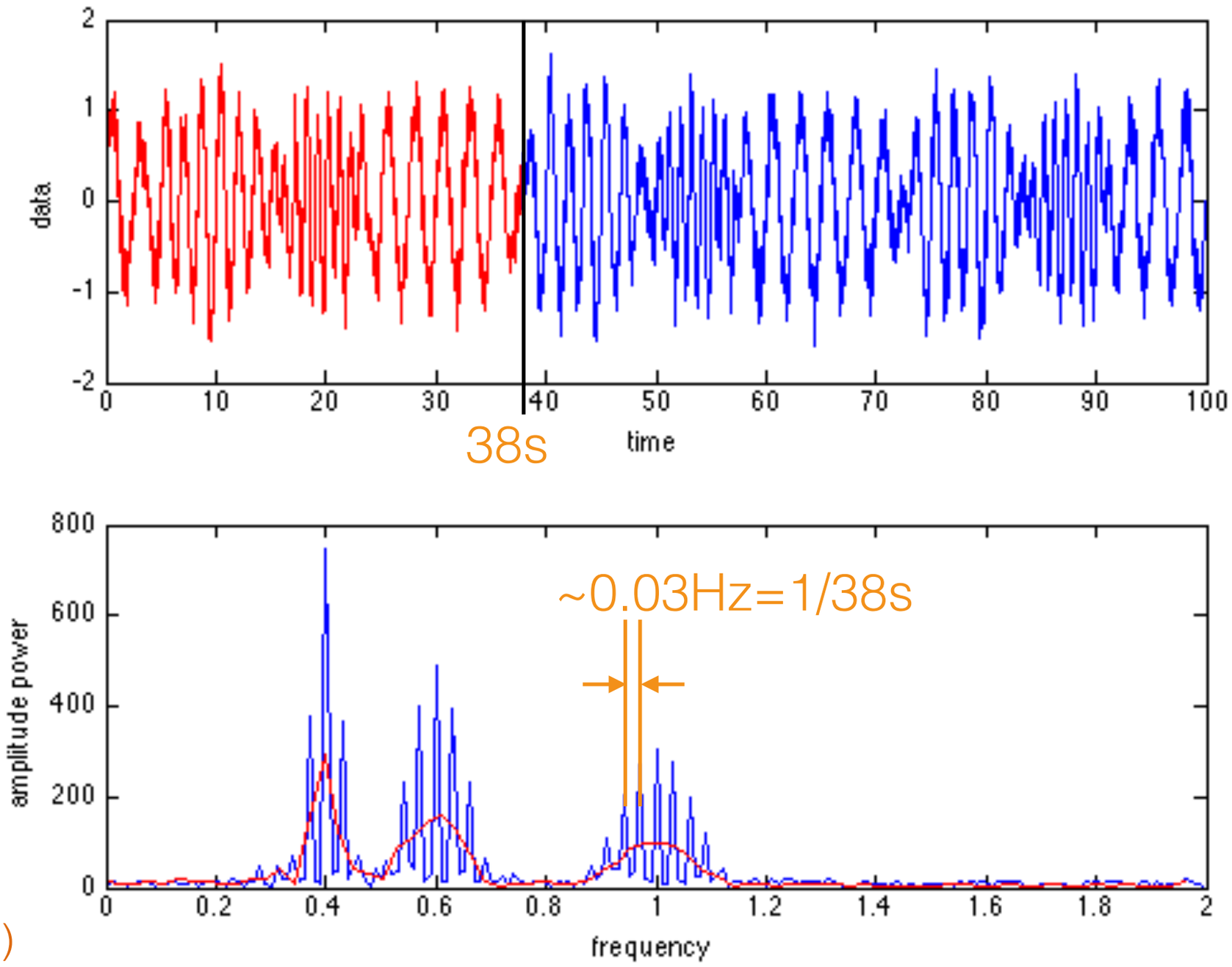
$$X_T(f) = M \int_{-T/2}^{-T/2+P} x(t) e^{-i\frac{2\pi n t}{P}} dt$$

$$= MPc_n = Tc_n$$

Fourier-Koeffizient

falls man annimmt, dass Zeitserie periodisch ist
und man aber **nur eine Periode** betrachtet,
dann kann man die Fourieranalyse anwenden.

Beispiel:



(Fourier_8.m)

wenn Signal periodisch ist mit Periode P, dann zeigt das Spektrum eine Kamm-Struktur mit Diskretisierung *1/P*

Fourierreihe nimmt Periodizität an

wenn Signal selbst **keine** periodische Struktur hat:

Ergebnis ist wie **erwartet** **:-)**

wenn Signal selbst **eine** periodische Struktur hat:

zusätzliche **unbeabsichtigte** Struktur **:-(**

aus der Praxis:

- wenn Powerspektrum regelmäßige Unterstruktur hat:

  die Daten sind periodisch !

- falls Daten *a priori* periodisch sind:

  versuche dies zu vermeiden !!

Frage: wie lautet die inverse Fourier Transformation ?

gegeben: Zeitserie mit Länge T

$$\rightarrow f = f_n = \frac{n}{T}$$

$\longrightarrow$ Ansatz: $$x(t) = \frac{1}{\mathcal{N}} \sum_{n=-\infty}^{\infty} X_T(f_n) e^{i2\pi f_n t}$$

eingesetzt:

$$X_T(f_m) = \frac{1}{\mathcal{N}} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} X_T(f_n) e^{i2\pi f_n t} e^{-i2\pi f_m t} dt$$

$$= \frac{1}{\mathcal{N}} \sum_{n=-\infty}^{\infty} X_T(f_n) \int_{-T/2}^{T/2} e^{i2\pi (f_n - f_m) t} dt$$

N.R.:

$$\int_{-T/2}^{T/2} e^{i2\pi(f_n-f_m)t}dt = \frac{e^{i2\pi(f_n-f_m)T/2} - e^{-i2\pi(f_n-f_m)T/2}}{i2\pi(f_n-f_m)}$$

$$= T\frac{\sin(\pi(n-m))}{\pi(n-m)} = \begin{cases} T \ \forall n = m \\ 0 \ \forall n \neq m \end{cases}$$

$$= T\delta_{n,m}$$

....

$$= \frac{1}{\mathcal{N}} \sum_{n=-\infty}^{\infty} X_T(f_n) T\delta_{n,m}$$

$$X_T(f_m) = \frac{T}{\mathcal{N}} X_T(f_m)$$

$$x(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_T(f_n) e^{i2\pi f_n t}$$