Calculus I AP Workshop Problems

May 5, 2007

Problems 1–14 No calculator

1. What is $\lim_{h\to 0}\dfrac{\cos\left(\frac{3\pi}{2}+h\right)-\cos\left(\frac{3\pi}{2}\right)}{h}$? $= \frac{0}{0} \Rightarrow$ indeterminant

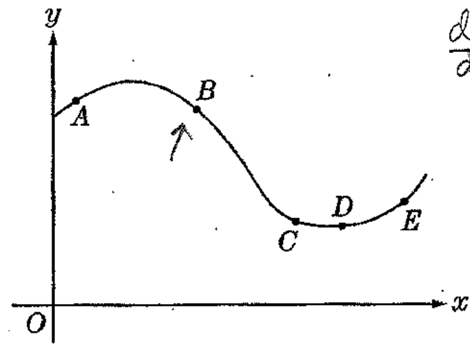
(A) 1
(B) $\dfrac{\sqrt{2}}{2}$
(C) 0
(D) -1
(E) The limit does not exist.

using l'hopital's rule

$$\equiv \lim_{h\to 0}\frac{-\sin\left(\frac{3\pi}{2}+h\right)-0}{1} = -\sin\frac{3\pi}{2} = -(-1) = \boxed{1}$$

2. At which of the five points on the graph in the figure at the right are $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ both negative?

(A) A
(B) B
(C) C
(D) D
(E) E



$\frac{dy}{dx} < 0$ decreasing

$\frac{d^2y}{dx^2} < 0$ concave down

$\therefore$ B

3. The slope of the tangent to the curve $y^3x + y^2x^2 = 6$ at $(2, 1)$ is

(A) $-\dfrac{3}{2}$
(B) -1
(C) $-\dfrac{5}{14}$
(D) $-\dfrac{3}{14}$
(E) 0

$$y^3\cdot 1 + x\cdot 3y^2\frac{dy}{dx} + y^2\cdot 2x + x^2\cdot 2y\frac{dy}{dx} = 0$$

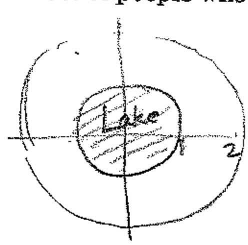
$$(3xy^2 + 2x^2y)\frac{dy}{dx} = -y^3 - y^2\cdot 2x$$

$$\frac{dy}{dx} = \frac{-y^2(y+2x)}{xy(3y+2x)}$$

at $(2,1)$, $\frac{dy}{dx} = \frac{-1(5)}{2(3+4)} = \boxed{\frac{-5}{14}}$

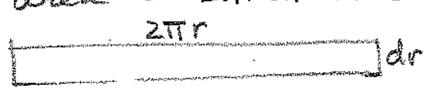
4. A city is built around a circular lake that has a radius of 1 mile. The population density of the city is $f(r)$ people per square mile, where r is the distance from the center of the lake, in miles. Which of the following expressions gives the number of people who live within 1 mile of the lake?

(A) $2\pi\int_0^1 r f(r)\,dr$
(B) $2\pi\int_0^1 r(1+f(r))\,dr$
(C) $2\pi\int_0^2 r(1+f(r))\,dr$
(D) $2\pi\int_1^2 r f(r)\,dr$
(E) $2\pi\int_1^2 r(1+f(r))\,dr$



$f(r)$ = density in $\frac{\text{people}}{\text{mile}^2}$

each ring with thickness dr has area $= 2\pi r\,dr$ miles^2



Multiply area by density

miles$^2 \cdot \frac{\text{people}}{\text{miles}^2}$ = people

integrate $1 < r < 2$

$$\int_1^2 2\pi r\, f(r)\,dr = \boxed{2\pi\int_1^2 r f(r)\,dr}$$

Calculus I AP Workshop Problems
May 5, 2007

5. Which of the following statements about the function given by $f(x) = x^4 - 2x^3$ is true?

(A) The function has no relative extremum. False
(B) The graph of the function has one point of inflection and the function has two relative extrema. True
(C) The graph of the function has two points of inflection and the function has one relative extremum. False
(D) The graph of the function has two points of inflection and the function has two relative extrema. False
(E) The graph of the function has two points of inflection and the function has three relative extrema. False

$f'(x) = 4x^4 - 6x^2 = 0$
$2x^2(2x^2-3)=0$
$x=0 \quad x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$

$f'(x)$: $+ \; 0 \; - \; 0 \; - \; 0 \; +$
x: $-\frac{\sqrt{6}}{2}$, 0, $\frac{\sqrt{6}}{2}$
rel max, rel min

$f''(x) = 16x^3 - 12x = 0$
$4x(4x^2-3)=0$
$x=0 \quad x = \pm\frac{\sqrt{3}}{2}$

$f''(x)$: $- \; 0 \; - \; 0 \; + \; 0 \; +$
x: $-\frac{\sqrt{3}}{2}$, 0, $\frac{\sqrt{3}}{2}$
inf

6. If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

(A) $-2\cos 3$
(B) $-2\sin 3\cos 3$
(C) $6\cos 3$
(D) $2\sin 3\cos 3$
(E) $6\sin 3\cos 3$

$f(x) = (\sin(3-x))^2$
$f'(x) = 2\sin(3-x)\cdot\cos(3-x)\cdot -1$
$= -2\sin(3-x)\cos(3-x)$
$f'(0) = \boxed{-2\sin 3\cos 3}$

7. The solution to the differential equation $\frac{dy}{dx} = \frac{x^3}{y^2}$, where $y(2) = 3$, is

(A) $y = \sqrt[3]{\frac{3}{4}x^4}$
(B) $y = \sqrt[3]{\frac{3}{4}x^4} + \sqrt[3]{15}$
(C) $y = \sqrt[3]{\frac{3}{4}x^4} + 15$
(D) $y = \sqrt[3]{\frac{3}{4}x^4 + 5}$
(E) $y = \sqrt[3]{\frac{3}{4}x^4 + 15}$

$y^2\,dy = x^3\,dx$
$\int y^2\,dy = \int x^3\,dx$
$\frac{y^3}{3} = \frac{x^4}{4} + c$
$\frac{27}{3} = \frac{16}{4} + c$
$5 = c$

$\frac{y^3}{3} = \frac{x^4}{4} + 5$
$y^3 = \frac{3}{4}x^4 + 15$
$\boxed{y = \sqrt[3]{\frac{3}{4}x^4 + 15}}$

8. What is the average rate of change of the function f given by $f(x) = x^4 - 5x$ on the closed interval [0, 3]?

(A) 8.5
(B) 8.7
(C) 22
(D) 33
(E) 66

Average of $f'(x) = \frac{1}{3-0}\int_0^3 f'(x)\,dx = \frac{1}{3}[f(3) - f(0)]$
$= \frac{1}{3}[66 - 0]$
$= \boxed{22}$

$f(3) = 3^4 - 5(3) = 81 - 15 = 66$

9. The position of a particle moving along a line is given by $s(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$. For what values of t is the speed of the particle increasing?

(A) $3 < t < 4$ only
(B) $t > 4$ only
(C) $t > 5$ only
(D) $0 < t < 3$ and $t > 5$
(E) $3 < t < 4$ and $t > 5$

Speed is increasing when the signs on $V(t)$ and $a(t)$ are the same

$V(t) = 6t^2 - 48t + 90 = 0$
$t^2 - 8t + 15 = 0$
$(t-3)(t-5) = 0$
$t=3 \quad t=5$

$V(t)$: $+ \; 0 \; - \; 0 \; +$
t: 3, 5

$a(t) = 12t - 48 = 0$
$t = 4$

$a(t)$: $- \; 0 \; +$
t: 4

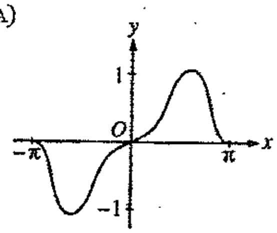
$\therefore$ both negative $(3,4)$
both positive $(5,\infty)$

Calculus I AP Workshop Problems

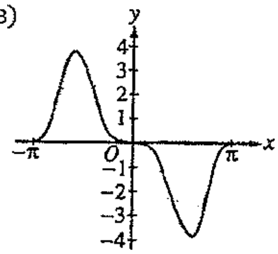
May 5, 2007

21. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $[-\pi, \pi]$?

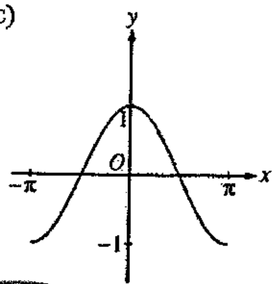
(A)



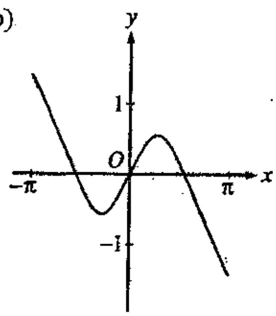
(B)



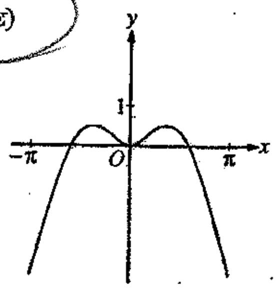
(C)



(D)



(E)



$\int_{-\pi}^{\pi} f(x)\,dx$ = area under the curve on $[-\pi, \pi]$

A, B + D are odd functions, so

$$\int_{-\pi}^{0} f(x)\,dx = -\int_{0}^{\pi} f(x)\,dx$$

C appears to be $f(x) = \cos x$, so

$$\int_{-\pi}^{-\frac{\pi}{2}} f(x)\,dx = -\int_{-\frac{\pi}{2}}^{0} f(x)\,dx$$

and

$$\int_{0}^{\frac{\pi}{2}} f(x)\,dx = -\int_{\frac{\pi}{2}}^{\pi} f(x)\,dx$$

22. The base of a solid S is the semicircular region enclosed by the graph of $y = \sqrt{4 - x^2}$ and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, then the volume of S is

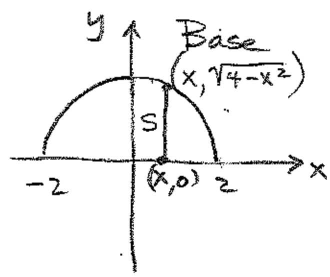
(A) $\frac{32\pi}{3}$

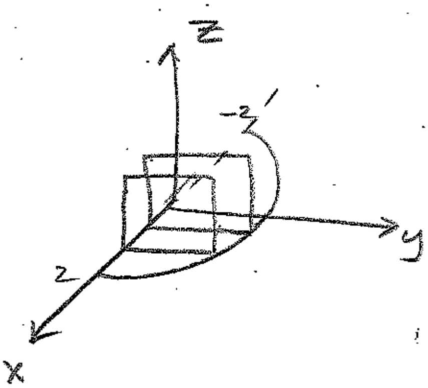
(B) $\frac{16\pi}{3}$

(C) $\frac{40}{3}$

(D) $\frac{32}{3}$

(E) $\frac{16}{3}$





$$S = \sqrt{4-x^2}$$

$$A_{sq} = S^2 = 4 - x^2$$

$$V = \int_{-2}^{2} (4 - x^2)\,dx = 2\int_{0}^{2} (4 - x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3} + c\right]_0^2$$

$$= 2\left[\left(8 - \frac{8}{3}\right) - 0\right] = \boxed{\frac{32}{3}} \text{ units}^3$$

Calculus I AP Workshop Problems
May 5, 2007

10. $\int (x-1)\sqrt{x}\,dx =$ $\int (x\sqrt{x} - \sqrt{x})\,dx = \int (x^{3/2} - x^{1/2})\,dx$

$= \boxed{\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c}$

(A) $\frac{3}{2}\sqrt{x} - \frac{1}{\sqrt{x}} + C$

(B) $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

(C) $\frac{1}{2}x^2 - x + C$

(D) $\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C$

(E) $\frac{1}{2}x^2 + 2x^{\frac{3}{2}} - x + C$

11. What is $\lim_{x\to\infty} \frac{x^2-4}{2+x-4x^2}$?

(A) -2

(B) $-\frac{1}{4}$

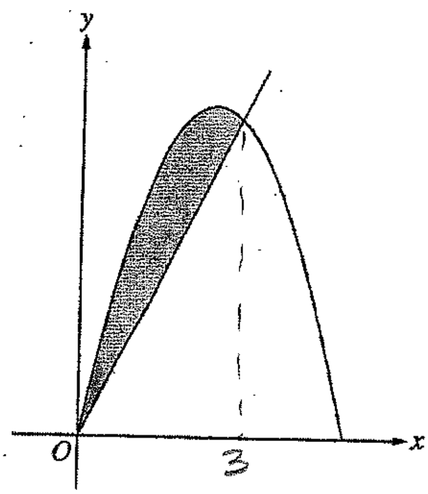
(C) $\frac{1}{2}$

(D) 1

(E) The limit does not exist.

At ∞, a function approaches its horizontal asymptote

$\lim_{x\to\infty} \frac{1x^4-4}{2+x-4x^2} = \frac{1}{-4}$



$2x = 5x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \quad x = 3$

$\text{area} = \int_0^3 [5x - x^2 - 2x]\,dx$

$= \int_0^3 (3x - x^2)\,dx$

$= \frac{3}{2}x^2 - \frac{x^3}{3} + c \Big]_0^3$

$= \frac{27}{2} - \frac{27}{3}$

$= \frac{81}{6} - \frac{54}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$

12. The figure above shows the graph of $y = 5x - x^2$ and the graph of the line $y = 2x$. What is the area of the shaded region?

(A) $\frac{25}{6}$

(B) $\frac{9}{2}$

(C) 9

(D) $\frac{27}{2}$

(E) $\frac{45}{2}$

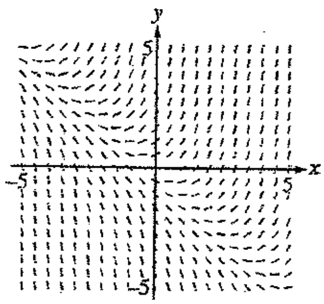
13. If f is a function that is continuous for all real numbers, then

$\frac{d}{dx}\int_0^{x^2} f(t)\,dt =$ $f(x^2) \cdot 2x$

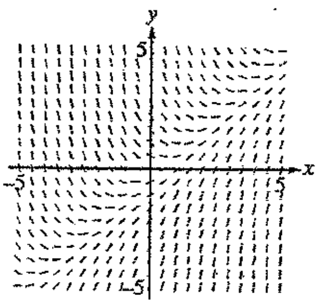
(A) $2x\, f(x^2)$
(B) $2x\, f(2x)$
(C) $f(2x)$
(D) $f(x^2)$
(E) $f'(x^2)$

14. Which of the following is a slope field for the differential equation $\frac{dy}{dx} = \frac{x}{y}$?

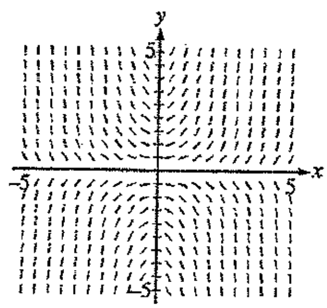
(A)



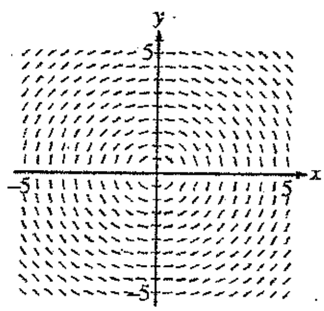
(B)



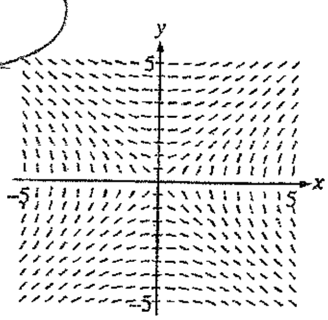
(C)



(D)



(E)



$y\,dy = x\,dx$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c$

$\frac{y^2}{2} - \frac{x^2}{2} = c$

Hyperbola w/ vertices on y-axis

OR

Calculate the slope for several points

i.e. $\frac{dy}{dx} = 1$ (1, 1) (2, 2) (3, 3) Eliminates A, C, D

$\frac{dy}{dx} = 0$ (0, 1) (0, 2) (0, 3) Eliminates B

Calculus I AP Workshop Problems

May 5, 2007

Problems 1–15 With a Calculator

15. A particle travels along a straight line with a velocity of $v(t) = 3e^{(-t/2)}\sin(2t)$ meters per second. What is the total distance, in meters, traveled by the particle during the time interval $0 \le t \le 2$ seconds?

(A) 0.835
(B) 1.850
(C) 2.055
(D) 2.261
(E) 7.025

Total Distance $= \int_0^2 |v(t)|\,dt$

$= \int_0^2 \left|3e^{-\frac{t}{2}}\sin(2t)\right|\,dt = \boxed{2.261}$

16. Let S be the region enclosed by the graphs of $y = 2x$ and $y = 2x^2$ for $0 \le x \le 1$. What is the volume of the solid generated when S is revolved about the line $y = 3$?

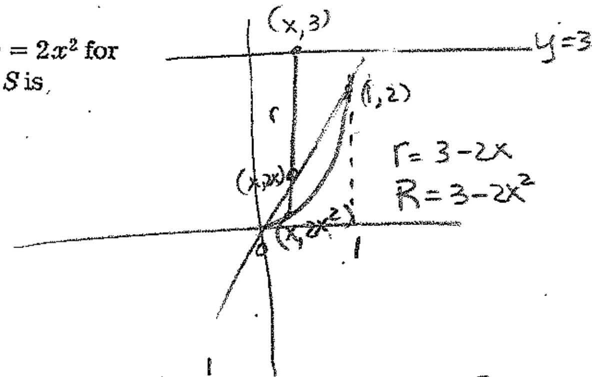
(A) $\pi\int_0^1 \left(\left(3-2x^2\right)^2 - \left(3-2x\right)^2\right)dx$

(B) $\pi\int_0^1 \left(\left(3-2x\right)^2 - \left(3-2x^2\right)^2\right)dx$

(C) $\pi\int_0^1 \left(4x^2 - 4x^4\right)dx$

(D) $\pi\int_0^2 \left(\left(3-\frac{y}{2}\right)^2 - \left(3-\sqrt{\frac{y}{2}}\right)^2\right)dy$

(E) $\pi\int_0^2 \left(\left(3-\sqrt{\frac{y}{2}}\right)^2 - \left(3-\frac{y}{2}\right)^2\right)dy$



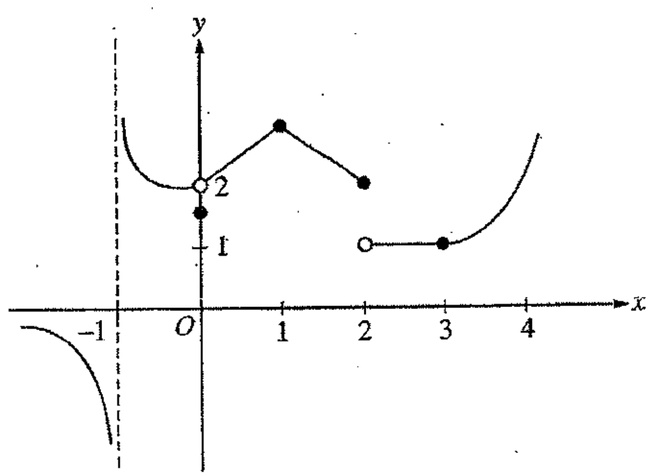
$r = 3-2x$

$R = 3-2x^2$

$V = \pi\int_0^1 \left[(3-2x^2)^2 - (3-2x)^2\right]dx$

$\therefore$ A

17.



The graph of a function f is shown above. If $\lim_{x \to b} f(x)$ exists and f is not continuous at b, then $b =$

(A) -1
(B) 0
(C) 1
(D) 2
(E) 3

$\boxed{b=0}$

because $\lim_{x\to 0} f(x) = 2$

but $f(0) \ne \lim_{x \to 0} f(x)$

Calculus I AP Workshop Problems
May 5, 2007

18.

x	1.1	1.2	1.3	1.4
$f(x)$	4.18	4.38	4.56	4.73

Let f be a function such that $f''(x) < 0$ for all x in the closed interval $[1, 2]$. Selected values of f are shown in the table above. Which of the following must be true about $f'(1.2)$?

(A) $f'(1.2) < 0$
(B) $0 < f'(1.2) < 1.6$
(C) $1.6 < f'(1.2) < 1.8$
(D) $1.8 < f'(1.2) < 2.0$
(E) $f'(1.2) > 2.0$

$f'(1.2) \approx \dfrac{f(1.3)-f(1.1)}{1.3-1.1}$

$\approx \dfrac{4.56-4.18}{1.2} = 1.9$

$f'(1.2) \approx \dfrac{f(1.2)-f(1.1)}{1.2-1.1} = \dfrac{.20}{.1} = 2.0$

$f'(1.2) \approx \dfrac{f(1.3)-f(1.2)}{1.3-1.2} = \dfrac{.18}{.1} = 1.8$

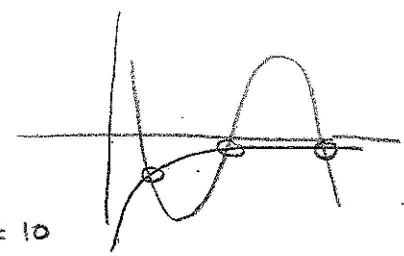
19. Two particles start at the origin and move along the x-axis. For $0 \leq t \leq 10$, their respective position functions are given by $x_1 = \sin t$ and $x_2 = e^{-2t} - 1$. For how many values of t do the particles have the same velocity?

(A) None
(B) One
(C) Two
(D) Three
(E) Four

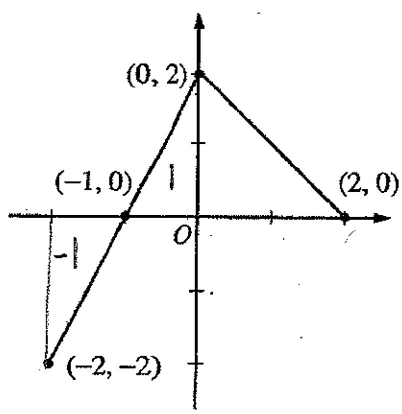
$x_1 = \sin t \qquad x_2 = e^{-2t} - 1$

$v_1 = \cos t \qquad v_2 = -2e^{-2t}$

graph v_1 and v_2 $\quad 0 \leq x \leq 10$



20.



Graph of f

The graph of the function f shown above consists of two line segments. If g is the function defined by $g(x) = \int_0^x f(t)\,dt$, then $g(-1) =$

(A) -2
(B) -1
(C) 0
(D) 1
(E) 2

$g(-1) = \int_0^{-1} f(t)\,dt$

$= -\int_{-1}^{0} f(t)\,dt$

$= -[-1+1]$

$= 0$

Calculus I AP Workshop Problems
May 5, 2007

23. Oil is leaking from a tanker at the rate of $R(t) = 2{,}000e^{-0.2t}$ gallons per hour, where t is measured in hours. How much oil leaks out of the tanker from time $t = 0$ to $t = 10$?

(A) 54 gallons
(B) 271 gallons
(C) 865 gallons
(D) 8,647 gallons
(E) 14,778 gallons

$$\int_0^{10} \left(2000e^{-.2t}\right) dt = \boxed{8646.647} \text{ gallons}$$

24. If $f'(x) = \sin\left(\frac{\pi e^x}{2}\right)$ and $f(0) = 1$, then $f(2) =$

(A) -1.819
(B) -0.843
(C) -0.819
(D) 0.157
(E) 1.157

$$f(2) = f(0) + \int_0^2 f'(x)\,dx$$
$$= 1 + \int_0^2 \sin\left(\frac{\pi e^x}{2}\right) dx = \boxed{1.157}$$

Answers to Calculus AB Multiple-Choice Questions

Part A

1. A	6. B	11. B
2. B	7. E	12. B
3. C	8. C	13. A
4. D	9. E	14. E
5. C	10. D	

Part B

15.* D	19.* D	22. D
16. A	20. B	23.* D
17. B	21. E	24.* E
18. D		

* Indicates a graphing calculator-active question.