Solutions For PUMaC 2006

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These are the official solutions for the 2006 Princeton University Mathematics Competition tests.

Contents

Chapter 1

Individual Tests

1.1 Rules

1. Five different Individual Tests will be offered in the following topics: Algebra, Geometry, Number Theory, Calculus, and Advanced Topics. The Advanced Topics Test will include topics such as inequalities, sequences and series, combinatorics, probability, and game theory.

2. Each student will take only one Individual Test.

3. No more than two students from each team may take the same Individual Test.

4. Half-teams (teams of 5) must have one student take each individual test.

5. The Individual Test that each student is taking must be declared on the registration form by the Late Registration Deadline (November 22, 2006), and may be changed up to the beginning of the Team Test on the day of the competition.

6. If a student takes an Individual Test that he or she is not signed up for, his or her results will be disqualified for that test.

7. All Individual Tests will consist of 10 short-answer questions, and will last 60 minutes.

8. The first five problems will be worth 2 points each and the last five problems will be worth 3 points each.

9. During the Individual Test, there will be absolutely no communication between the students from the time the questions are handed out to the time the answers are collected.

10. Proctors will give 5-minute and 1-minute warnings, and no other information.

11. At the end of the 60 minutes, students must submit completed official answer sheets to the Proctor. Copying of answers from other sheets of paper to the official answer sheet after the time limit will not be allowed.

1.2 Problems

1.2.1 Algebra

1. *Given that $x^2+5x+6=20$, find the value of $3x^2+15x+17$.*

   Solution: Note that

   $$3x^2+15x+17 = 3(x^2+5x+6)-1 = 3(20)-1 = \boxed{59}$$

2. *Express $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$ in the simplest possible form.*

   Solution 1: Let $x = \left(\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}\right)$. Then

   $$x^2 = \left(\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}\right)2 = \left(7+4\sqrt{3}\right)+2\left(\sqrt{49-48}\right)+\left(7-4\sqrt{3}\right) = 16$$

   whence $\boxed{x=4}$.

   Solution 2: Let $(a+b\sqrt{3})2 = 7+4\sqrt{3}$ (and then also $7-4\sqrt{3}$), and solve the resulting system of equations in a and b. We find that

   $$\sqrt{7+4\sqrt{3}} = 2+\sqrt{3}$$
   $$\sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$$
   $$(2+\sqrt{3})+(2-\sqrt{3}) = \boxed{4}$$

3. *Let $r_1,\dots,r_5$ be the roots of the polynomial $x^5+5x^4-79x^3+64x^2+60x+144$. What is $r_1^2+\ldots+r_5^2$?*

   Solution: Using Vieta's Formulas, $r_1+\cdots+r_5 = 5$ and $r_1r_2+r_1r_3+r_2r_3+\cdots+r_4r_5 = -79$. Then $r_12+\cdots+r_52 = (r_1+\cdots+r_5)^2-2(r_1r_2+r_1r_3+\cdots+r_4r_5) = 25+2(79) = \boxed{183}$.

4. *Find all pairs of real numbers (a,b) so that there exists a polynomial $P(x)$ with real cofficients and $P(P(x)) = x^4-8x^3+ax^2+bx+40$.*

   Solution: Suppose $P(x) = x^2+rx+s$. Then $P(P(x)) = x^4+2rx^3+(r^2+r+2s)x^2+(2rs+r^2)x+(rs+s+s^2)$. Equating coefficients, we find

   $$2r = -8$$
   $$r^2+r+2s = a$$
   $$2rs+r^2 = b$$
   $$rs+s+s^2 = 40$$

   From the first equation we get $r=-4$. The last becomes $-3s+s^2 = 40 \implies (s-8)(s+5) = 0$. If $s=-5$ we get $\boxed{a=2, b=56}$. If $s=8$ we get $\boxed{a=28, b=-48}$. These are the only possible values of a and b.

5. *Find the greatest integer less than the number*

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{1000000}}$$

Solution: Let S be the sum in question. We just note that

$$2(\sqrt{n+1}-\sqrt{n})=\frac{2}{\sqrt{n+1}+\sqrt{n}}$$

and clearly

$$\frac{2}{\sqrt{n}+\sqrt{n-1}}>\frac{1}{\sqrt{n}}>\frac{2}{\sqrt{n+1}+\sqrt{n}}$$

Adding these inequalities from $n=2$ to 10^6, we obtain $1998>S-1>2(\sqrt{10^6+1}-\sqrt{2})>1997$. Thus $\boxed{\lfloor S\rfloor=1998}$.

6. *Suppose that $P(x)$ is a polynomial with the property that there exists another polynomial $Q(x)$ to satisfy $P(x)Q(x)=P(x^2)$. $P(x)$ and $Q(x)$ may have complex coefficients. If $P(x)$ is a quintic with distinct complex roots $r_1,\ldots,r_5$, find all possible values of $|r_1|+\ldots+|r_5|$.*

Solution: Suppose that $P(x_0)=0$. Then $P(x^2)$ has roots at $\sqrt{x_0}$ and $-\sqrt{x_0}$. Since $P(x)\mid P(x^2)$, we must also have $P(\pm\sqrt{x_0})=0$. Proceeding in this fashion, we find that all of the numbers $x_0,\pm\sqrt{x_0},\pm\sqrt{\pm\sqrt{x_0}},\ldots$ must be roots of $P(x)$. Since $P(x)$ is a quintic, that sequence must be finite. Since either $1<|\sqrt{x_0}|<|x_0|$ or $0<|x_0|<|\sqrt{x_0}|<1$ must be true if $|x_0|\notin\{0,1\}$, clearly we must have $x_0=0$ or $|x_0|=1$. No roots are repeated, so the set of 2^{nth} roots of $x_0(\neq 0)$ must contain either 4 or 5 elements (depending on whether $x\mid P(x)$). So $|r_1|+\ldots+|r_5|\in\{4,5\}$. To show that both are obtained, consider the polynomials

$$P_1(x)=x^5+x^4-x^3+x^2-x+1=(x-1)(x^2+x+1)(x^2-x+1)$$
$$P_2(x)=x^5+x^4+x^3+x^2+x=x(x^4+x^3+x^2+x+1)$$

Thus, the sum in question may obtain the values $\boxed{\{4,5\}}$.

7. *Find one complex value of x that satisfies the equation $\sqrt{3}x^7+x^4+2=0$.*

Solution: Think geometrically in the complex plane. Note that $1,\sqrt{3}$ and 2 are the sides of a right triange. This suggests that x^4 and x^7 are separated by 90 degrees, and that the magnitude of x is 1. This suggests that x makes an angle of 30 with the real number line, or $x=\cos(30)+\sin(30)\cdot i=$ $\boxed{\frac{\sqrt{3}}{2}+\frac{i}{2}}$. It can be easily checked that this is a solution. Since the polynomial has real coefficients, $\boxed{x=\frac{\sqrt{3}}{2}-\frac{i}{2}}$, the complex conjugate, will also be a solution.

8. *The Lucas numbers L_n are defined recursively as follows: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $r = 0.21347\ldots$, whose digits form the pattern of the Lucas numbers. When the numbers have multiple digits, they will "overlap," so $r = 0.2134830\ldots$,* ***not*** *$0.213471118\ldots$. Express r as a rational number $\frac{p}{q}$, where p and q are relatively prime.*

   Solution: Let L_n be the n^{th} Lucas number. We want to find $S = \sum_{n=0}^{\infty} \frac{L_n}{10^{n+1}}$.

   Let $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio. Note that $1+\phi = \phi^2$, so $\phi^{n-2} + \phi^{n-1} = \phi^n$ and $(-\phi)^{-n+2} + (-\phi)^{-n+1} = (-\phi)^{-n}$, and as $\phi^0 + (-\phi)^0 = 2, \phi^1 + (-\phi)^{-1} = 1$, we conclude that $L_n = \phi^n + (-\phi)^{-n}$. Then we are looking for

$$\begin{aligned} S &= \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{L_n}{10}\right)^n \\ &= \frac{1}{10} \sum_{n=0}^{\infty} \left[\left(\frac{\phi}{10}\right)^n + \left(\frac{-1}{10\phi}\right)^n\right] \\ &= \frac{1}{10} \left(\frac{1}{1-\phi/10} + \frac{1}{1+1/10\phi}\right) \\ &= \frac{1}{10} \left(\frac{20}{19-\sqrt{5}} + \frac{20}{19+\sqrt{5}}\right) \\ &= \frac{2(19+\sqrt{5}+19-\sqrt{5})}{19^2-5} \\ &= \boxed{\frac{19}{89}} \end{aligned}$$

9. *The curve $y = x^4 + 2x^3 - 11x^2 - 13x + 35$ has a bitangent (a line tangent to the curve at two points). What is the equation of the bitangent?*

   Solution: A polynomial with two double roots has $y = 0$ for a bitangent. Then let's subtract a linear function from this polynomial to form one with two double roots. We want

$$\begin{aligned} (x+a)^2(x+b)^2 &= (x^2+2ax+a^2)(x^2+2bx+b^2) \\ &= x^4 + (2a+2b)x^3 + (a^2+4ab+b^2)x^2 + 2ab(a+b)x + (a^2+b^2) \end{aligned}$$

   So

$$\begin{gathered} (2a+2b) = 2 \implies (a+b) = 1 \\ a^2+4ab+b^2 = -11 \text{ we subtract } (a+b)^2 = 1 \text{ to get } 2ab = -12 \end{gathered}$$

   By inspection we see a solution is $a = 3, b = -2$. So the x-coefficient should be -12 and the constant should be $a^2b^2 = 36$. Then the difference is $x+1$, so our bitangent is $\boxed{y = -x-1}$.

10. *If x, y, z are real numbers and*

$$\begin{aligned} 2x+y+z &\le 66 \\ x+2y+z &\le 60 \\ x+y+2z &\le 70 \\ x+2y+3z &\le 110 \\ 3x+y+2z &\le 98 \\ 2x+3y+z &\le 89 \end{aligned}$$

What is the maximum possible value of $x+y+z$?

Solution: This is an example of the linear programming problem, and there are a number of ways to go about this. Here is one solution:

We are given six inequalities. Visualize this in 3 dimensions. Each inequality represents a plane, where our point must be on one side of the plane.

Imagine that we start at a point (x, y, z) that satisfies all of the inequalities and then pick a direction to travel in. If this decreases $x+y+z$, travel in the opposite direction. In this, way, we will be either increasing x+y+z or keeping it constant. When we hit a plane, choose a direction parallel to that plane. When we hit another plane, the intersection will form a line, and we can travel along this line. The only places where we cannot do this are when we hit a point where three planes intersect, or when 3 inequalities are equalities. At least one maximum will be found at one of these intersection points or at infinity.

A quick look at the inequalities shows that if x, y, or z is infinity, $x+y+z$ must be -infinity. This is clearly not a maximum. Thus, there must be at least one intersection point giving the maximum value for $x+y+z$.

Because of the symmetry, brute force is feasible here. However, we will use a slightly more efficient method.

Pick one intersection point to start with. The first three inequalities are a natural place to start. Solving gives $x = 17$, $y = 11$, and $z = 21$. However, this violates inequality 5, as

$$\begin{aligned} x+2y+3z &= 102 \\ 3x+y+2z &= 104 \\ 2x+3y+z &= 88. \end{aligned}$$

Let's try to find the maximum value of $x+y+z$ given just inequalities 1,2,3, and 5, moving to an adjacent intersection point.

If we take $x = 17-3p$, $y = 11+p$, $z = 21+p$, this preserves the 2nd and 3rd equalities, decreases $3x+y+2z$ by $6p$, and decreases $x+y+z$ by p. If we take $x = 17+p$, $y = 11-3p$, $z = 21+p$, this preserves the 1st and 3rd equalities, increases $3x+y+2z$ by $2p$, and decreases $x+y+z$ by p. If we take $x = 17+p$, $y = 11+p$, $z = 21-3p$, this preserves the 1st and 2nd equalities, decreases $3x+y+2z$ by $2p$, and decreases $x+y+z$ by p.

Note that $p > 0$, as otherwise one of the first three inequalities will be violated.

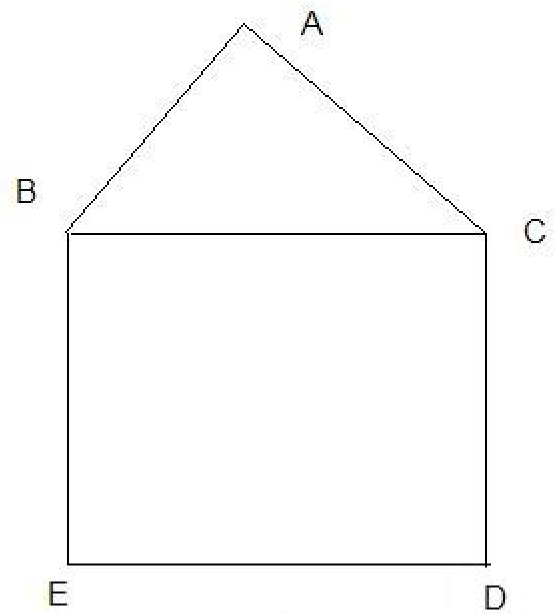
We want to decrease $x+y+z$ as little as possible while decreasing $3x+y+z$, so we use the first path. Setting $p = 1$ reduces $3x+y+2z$ to 98, as needed and gives $x = 14$, $y = 12$, and $z = 22$. A quick check shows that this satisfies all of the inequalities. Thus, the answer is $\boxed{48}$.

1.2.2 Geometry

1. *A, B, C, D, E, and F are points of a convex hexagon, and there is a circle such that A,B,C,D,E, and F are all on the circle. If $\measuredangle ABC = 72$, $\measuredangle BCD = 96$, $\measuredangle CDE = 118$, and $\measuredangle DEF = 104$, what is $\measuredangle EFA$?*

   Solution: $CDEF$ is cyclic, meaning that $\measuredangle CFE = 62°$, and $ABCF$ is cyclic, meaning that $\measuredangle AFC = 108°$. So $\boxed{\measuredangle EFA = 170}$.

2. *ABC is an equilateral triangle with side length 1. BCDE is a square. Some point F is equidistant from A, D, and E. Find the length of AF.*



   Solution: Imagine shifting triangle BAC down by 1. Now B is on E and C is on D. Call the point where A would be P. Clearly, $EP = 1$ and $DP = 1$. Also, $AP = 1$ because the triangle was shifted downwards by 1. Therefore $F = P$, and the answer is $\boxed{1}$.

3. *Find the exact value of* $\sin 36°$.

   Solution: Take a regular pentagon $ABCDE$ of side length 1. Segment AC has length $2\cos(36)$. Draw segments perpenicular to AC that meet D and E. It is now clear that segment AC has length $2\cos(72)+1$. By the double angle formula for cosine, $\cos(72) = 2\cos^2(36) - 1$. Solving the resulting quadratic equation gives $\cos(36) = \frac{1+\sqrt{5}}{4}$, whence $\boxed{\sin(36) = \sqrt{\frac{5-\sqrt{5}}{8}}}$.

4. *There is a circle c centered about the origin of radius 1. There are circles $c_1, \ldots, c_6$, each of radius r_1, such that each circle is completely inside c and is tangent to it, and c_2 is tangent to c_1, c_3 is tangent to $c_2, \ldots$, and c_1 is tangent to c_6. There is a circle d which is tangent to c, c_1, and c_2, but does not intersect any of these circles. What is the radius of circle d? Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive and as small as possible, and c is squarefree.*

Solution: The easiest way to solve this is to use inverse geometry. Let C be the center of the big circle and r be its radius. Invert the plane about C, keeping the big circle in place, the six little circles become six circles of the same size, forming a hexagon. Pick two adjacent circles plus the original circle. The circle we are looking for is now in the center of this triangle. The distance from C to its center is $\frac{2r\sqrt{3}}{3}$. Its radius is $\frac{2r\sqrt{3}-3r}{3}$. The distance from the other side of the circle to C is $r+2\frac{2r\sqrt{3}-3r}{3}=\frac{4r\sqrt{3}-3r}{3}$. Invert the plane about C to restore everything to the way it started. The distance from C to the near side of the circle is

$$\frac{r^2}{\frac{4r\sqrt{3}-3r}{3}}=\frac{3(4\sqrt{3}+3)}{48-9}r$$

Taking r minus this and dividing by 2 to get the radius (and recalling that $r=1$) gives $\boxed{\frac{14-6\sqrt{3}}{37}}$

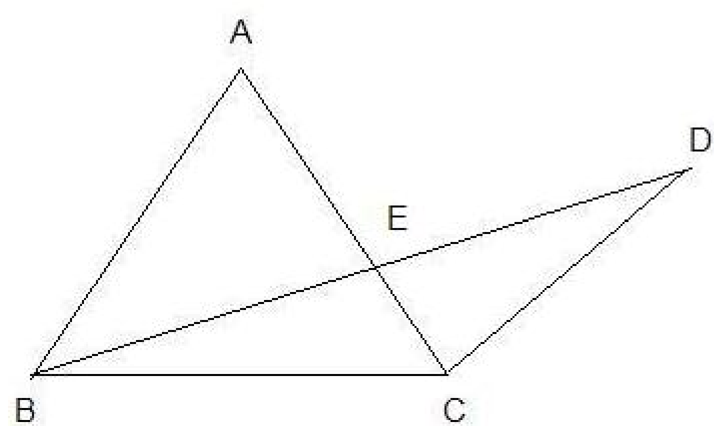
5. *A,B, and C are vertices of a triangle, and P is a point within the triangle. If angles $\measuredangle BAP, \measuredangle BCP$, and $\measuredangle ABP$ are all 30° and angle $\measuredangle ACP$ is 45°, what is $\sin(\measuredangle CBP)$?*

   Solution: Let $\angle CBP=x, BP=a$. Since $\angle BAP=\angle ABP$, then $AP=BP=a$. The length of the altitude from P to BC is then $a\sin x$, making $CP=2a\sin x$. Since $\angle ACP=45^\circ$, the length of the altitude from P to AC is $\sqrt{2}a\sin x$. (1)

   Now the sum of the angles in a triangle is 180°, so $\angle CAP=45^\circ-x$. Since$AP=a$, then the length of the altitude from P to AC is $a\sin(45^\circ-x)$. (2)

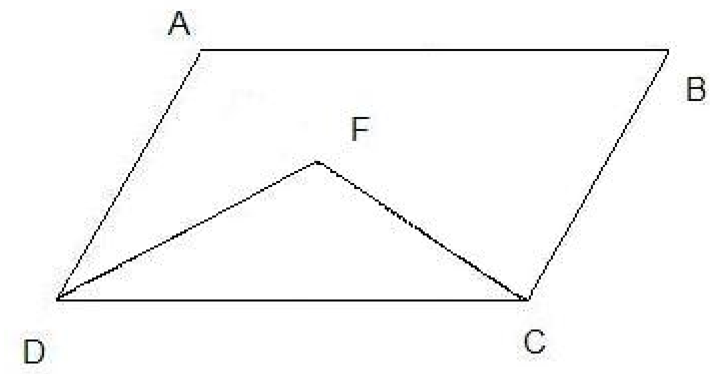
   Equating (1) and (2), $\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x=\sqrt{2}\sin x$, whichsimplifies to $\cos x=3\sin x$. And $\cos^2 x+\sin^2 x=1$, so $10\sin^2 x=1$. x is acute, so $\sin x>0$. Therefore, $\boxed{\sin x=\frac{\sqrt{10}}{10}}$.

6. *Given that in the diagram shown, $\measuredangle ACB=65^\circ$, $\measuredangle BAC=50^\circ$, $\measuredangle BDC=25^\circ$, $AB=5$, and $AE=1$, determine the value of $BE\cdot DE$.*



   Solution: You see that $\measuredangle ABC=180^\circ-\measuredangle BAC-\measuredangle ACB=65^\circ$. So $ABC=ACB$, and triangle ABC is isoceles, with $AB=AC=5$. Notice that $\measuredangle BDC=25^\circ=\frac{1}{2}\measuredangle BAC$. This reminds me of a circle. Construct a circle with A as the center, and AB and AC as radii. We see from a well-known theorem that D must lie on the circle. Now extend AC to meet the circle again at F, and we have Power of the Point about E. So $BE \cdot DE = CE \cdot EF = 4 \cdot 6 = \boxed{24}$.

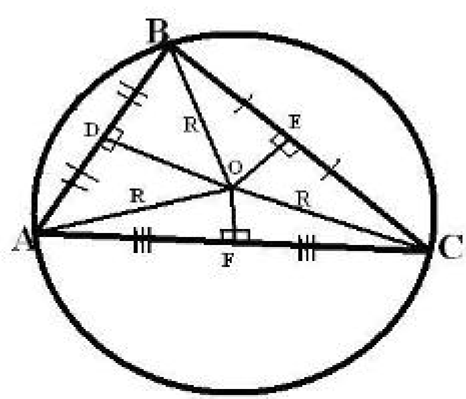
7. *Given parallelogram ABCD, construct point F so that $CF \perp BC$, as shown. Also F is placed so that $\measuredangle DFC = 120^\circ$. If $DF = 4$ and $BC = CF = 2$, what is the area of the parallelogram?*



Solution: Extend CE to meet the extension of AD at G. Then $CE \perp BC$ and $BC \parallel AD$, so $CG \perp AG$. $\angle DFG = 180^\circ - \angle DFC = 120^\circ$, $\angle DGF = 90^\circ$, and $DF = 4$, so $FG = 4\cos 60^\circ = 2$. Then $CG = 4$, so $[ABCD] = AD \cdot CG = \boxed{8}$.

8. *Given that triangle ABC has side lengths $a = 7, b = 8, c = 5$, find* $(\sin(A) + \sin(B) + \sin(C)) \cdot \left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)$.

Solution: To evaluate the left-hand side, consider the circumcircle (whose radius is the junction of the perpendicular bisectors):

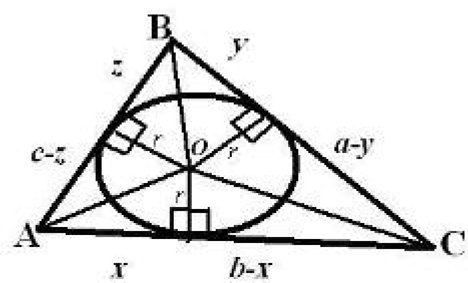


where R is the circumradius. Now $m\angle BCA = \frac{1}{2} m\angle BOA$, and $\triangle BOD \cong \triangle AOD$ by SSS (or SAS).

Thus, $m\angle BOD = \frac{1}{2}m\angle BOA = \angle C$. So $\sin C = \sin(\angle BOD) = \frac{c/2}{R}$. And simlarly,

$$\sin A = \sin(\angle BOE) = \frac{a/2}{R}$$
$$\sin B = \sin(\angle AOF) = \frac{b/2}{R}$$
$$\sin A + \sin B + \sin C = \frac{1}{2R}(a+b+c)$$

To evaluate the left-hand side, consider the incircle (whose radius is the junction of the angle bisectors):



where r is the radius. Now because BO, AO, and CO are angle bisectors, we have:

$$\begin{aligned}\cot(\frac{A}{2})+\cot(\frac{B}{2})+\cot(\frac{C}{2}) &= \frac{1}{2}\left(\left(\cot(\frac{A}{2})+\cot(\frac{B}{2})\right)+\left(\cot(\frac{A}{2})+\cot(\frac{C}{2})\right)+\left(\cot(\frac{B}{2})+\cot(\frac{C}{2})\right)\right)\\ &= \frac{1}{2}\left(\frac{(c-z)+z}{r}+\frac{(b-z)+z}{r}+\frac{(a-z)+z}{r}\right)\\ &= \frac{1}{2r}(a+b+c)\end{aligned}$$

So

$$\begin{aligned}(\sin A+\sin B+\sin C)\cdot(\cot(\frac{A}{2})+\cot(\frac{B}{2})+\cot(\frac{C}{2})) &= \frac{1}{2R}(a+b+c)\cdot\frac{1}{2r}(a+b+c)\\ &= \frac{1}{4rR}(a+b+c)^2\end{aligned}$$

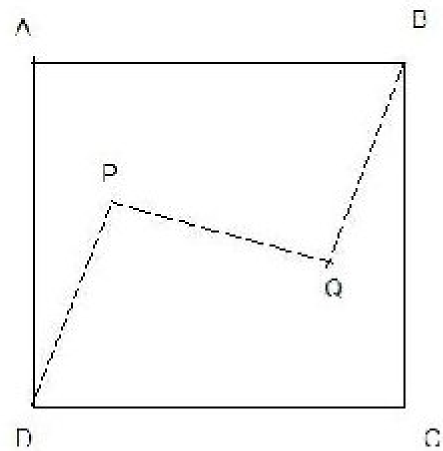
Now since $R = \frac{abc}{4rs}$, where s is the semiperimeter, $Rr = \frac{abc}{4s}$. And so we have

$$\frac{1}{4rR}(a+b+c)^2 = \frac{1}{4}\cdot\frac{4s}{abc}\cdot(2s)^2 = \frac{4s^3}{abc} = \boxed{\frac{100}{7}}.$$

9. *Consider all line segments of length 4 with one end-point on the line $y = x$ and the other end-point on the line $y = 2x$. Find the equation of the locus of the midpoints of these line segments.*

Solution: Consider a point (a,a) on the line $y = x$ and a point $(b,2b)$ on the line $y = 2x$. If these two points are the endpoints of a line segment of length 4, then $(a-b)2+(a-2b)2 = 42$ (*). The midpoint of this line segment is $(x,y) = (\frac{a+b}{2}, \frac{a+2b}{2})$. Solving the equations $x = \frac{a+b}{2}$ and $y = \frac{a+2b}{2}$ for a and b in terms of x and y, we get $a = 4x - 2y$ and $b = 2y - 2x$. Substituting into (*) and simplifying, the equation of the locus is $\boxed{25x^2 - 36xy - 13y^2 = 4}$.

10. *Points P and Q are located inside square ABCD such that DP is parallel to QB and DP = QB = PQ. Determine the minimum possible value of ∡ADP.*



Solution: Let $\theta = \measuredangle ADP$ and $x = DP$. For a given θ, let's consider the minimum possible value of $PQ - x$ (Ignoring the restiction that $PQ = x$) If we increase x by dx, we increase PQ by $2 \cdot dx \cdot (cos(\measuredangle DPQ))$. Thus, this is minimum when $\measuredangle DPQ = \frac{\pi}{3}$.

For a given value of θ, let's say we construct this minimum. Now reduce θ by da. This increases PQ by $\frac{\sqrt{3}}{2} \cdot x \cdot da$ (unless $\theta > \frac{\pi}{4}$ which clearly is not the answer.

Thus, for the minimum angle, $PQ - x$ is minimum when $PQ = x$. If $PQ - x$ has a negative minimum value, the angle can be reduced until $PQ = x$. If $PQ - x$ has a positive minimum value, then the conditions of the problem cannot be satisfied for that angle.

Now it is clear that $\measuredangle PDQ = \measuredangle PQD = \frac{\pi}{3}$ when θ is minimum. By symmetry, it follows quickly that $\boxed{\theta = \frac{\pi}{12}}$.

1.2.3 Number Theory

1. *Find the smallest positive integer that is a multiple of* 18 *and whose digits can only be* 4 *or* 7.

   Solution: Let the number be n. Certainly $18 \mid n \implies 9 \mid n$, so the sum of the digits of n is also a multiple of 9. We can do that with three digits in exactly one way (can't do it with less than three), that is with one 4 and two 7s. But since $18 \mid n$, n is even, hence $\boxed{n = 774}$.

2. *Professor Conway collects a total of* 58 *midterms from the two sections of his introductory linear algebra course. He notices that the number of midterms from the smaller section is equal to the product of the digits of the number of midterms from his larger section. Assuming that every student handed in a midterm, how many students are there in the smaller section?*

Solution: The number of students in the larger section must be a two-digit number, say $10a+b$, where $a,b<10, a>0$, and b is non-negative. We know that $ab+10a+b=58$. Use Simon's Favorite Factoring Trick: $(a+1)(b+10)=ab+10a+b+10=68$. Given the conditions on a and b, the only solution is $a+1=4$, $b+10=17$, yielding $a=3, b=7$, and $\boxed{ab=21}$.

3. *Find the fifth root of 14348907.*

   Solution: Let $y=14348907$. Consider $(y/10)^5=143.48907$. Obviously $2<y/10<3$, so the leading digit is 2. What gives us a 7 in the last place for 5th powers? Only 7. So the answer is $\boxed{27}$.

4. *What are the last two digits of* $2003^{2005^{2007^{2009}}}$*, where* a^{b^c} *means* $a^{(b^c)}$*?*

   Solution We write $2003=100x+3$, since only the last two digits can affect the last two digits of the power: $(100x+3)^n=100(\text{irrelevant})+3^n$. The last digits of powers of 3 repeat: 1 3 9 27 81 43 29 87 61 83 49 47 41 23 69 7 21 63 89 67 1. The cycle repeats every 20 terms. Now, for the same reason 2005^n is like $5^n \pmod{20}$, and other than $n=0, 5^n \pmod{20}=5$. The 5th one in the sequence (the 1 is really the 0th because we started at $n=0$) is $\boxed{43}$.

5. *Find the largest integer k such that* $12^k \mid 66!$.

   Solution: We need to count both 2s and 3s – there are $\left[\frac{66}{2}\right]+\left[\frac{66}{4}\right]+\left[\frac{66}{8}\right]+\left[\frac{66}{16}\right]+\left[\frac{66}{32}\right]+\left[\frac{66}{64}\right]=33+16+8+4+2+1=64$ 2s, so 32 4s, and $\left[\frac{66}{3}\right]+\left[\frac{66}{9}\right]+\left[\frac{66}{27}\right]=22+7+2=31$ 3s, so that makes $\boxed{31}$ total 12s.

6. *I have a set A containing n distinct integers. This set has the property that if* $a,b\in A$*, then* $12 \nmid |a+b|$ *and* $12 \nmid |a-b|$*. What is the largest possible value of n?*

   Solution: There are 12 possible equivalences $\pmod{12}$. Obviously we can't have two that are the same. If we take 0 thru 6, we have 7 different ones and no problem, but if we try to add an 8th it won't work. This is because if we have something that's $k \pmod{12}$, we exclude k and $12-k$, so if we take $\boxed{7}$ distinct ones we necessarily excluded all the rest.

7. *Find the largest possible value of the expression* $x+y+z$*,* $x,y,z\in\mathbb{Z}$*, given that the equation* $10x^3+20y^3+2006xyz=2007z^3$ *holds.*

   Solution: $10x^3+20y^3+2006xyz=2007z^3$. Take this equation mod 2. Clearly, $z\equiv 0 \pmod 2$. Now take this equation mod 4. $z\equiv 0 \pmod 4$, so this leaves $10x^3\equiv 0 \pmod 4$. $x\equiv 0 \pmod 2$. Now take this equation mod 8. $z\equiv 0 \pmod 2$ and $x\equiv 0 \pmod 2$, so this leaves $20y^3\equiv 0 \pmod 8$. $y\equiv 0 \pmod 2$. However, if x, y, and z are a solution, we can get another solution by dividing x, y, and z by 2, and it must still be integral because $x\equiv y\equiv z\equiv 0 \pmod 2$. Thus, we can continue this process forever.

   This only makes sense if $x=y=z=0$, and this is the only integral solution. Therefore, the largest possible value of $x+y+z$ is $\boxed{0}$.

8. *Find all integers n (not necessarily positive) such that* $7n^3-3n^2-3n-1$ *is a perfect cube.*

   Solution: Write the given expression as $(2n)^3-(n+1)^3=m^3 \implies (2n)^3=m^3+(n+1)^3$. By Fermat's Last Theorem, this equation can hold in integers only if at least one of the numbers is 0. So, to make $2n=0$ we have $n=0$, to make $m=0$ we have $2n=n+1 \implies n=1$, and to make $n+1=0$ we have $n=-1$. Our solutions are $\boxed{\{-1,0,1\}.}$

9. *Consider the set of sequences $\{S_i\}$ that start with $S_0 = 12$, $S_1 = 21$, $S_2 = 28$, and for $n > 2$ S_n is the sum of two (not necessarily distinct) S_{k_n} and S_{j_n} with $k_n, j_n < n$. Find the largest integer that cannot be found in any sequence S_i.*

   Solution: The language in the problem boils down to: what is the largest integer that we cannot express as $12x+21y+28z$ with $x,y,z \geq 0$ and integer. If we consider this question in mod 12, notice that the $12x$ term doesn't matter. Once we find a linear combination of 21 and 28 that is in a paricular residue class of mod 12, we can add multiples of 12 to get all integers greater than this in the same equivalence class. So we are really just interested in the smallest linear combination required for each equivalence class. Taking the maximum of these and subtracting 12 gives the largest number not attainable. So consider any constant c that we are trying to reach in mod 12 with linear combinations of 21 and 28. We have that $21y+28z \equiv c \pmod{12}$ To do this we just need to make them equal in mod 3 and in mod 4 because of the Chinese Remainder Theorem. In mod 3, this equation is very simple: $z \equiv c \pmod 3$ and in mod 4 it is also simple $y \equiv c \pmod 4$. We want to choose c so that z and y are both maximized, so we have $z = 2, y = 3$ Thus our maximum is $2 \cdot 28 + 3 \cdot 21 - 12 = \boxed{107}$. This exact same approach generalizes so that the maximum for ab, bc, ac where a, b, c are pairwise relatively prime positive integers, is equal to $2abc - ab - bc - ac$.

10. *If $a_1, \ldots, a_{12}$ are twelve nonzero integers such that $a_1^6 + \cdots + a_{12}^6 = 450697$, what is the value of $a_1^2 + \cdots + a_{12}^2$?*

    Solution: Since there are 6th powers, we might want to consider Fermat's Little Theorem. 450697 is 2 mod 7; from FLT and that we deduce that either 3 or 10 of the a_i equal 7. $7^6 = 117649$, so there can't be 10 (too large). Then $a_1 = a_2 = a_3 = 7$. Now we have $a_4^6 + ... + a_{12}^6 = 97750$. The RHS is equivalent to 6 mod 8. Sixth powers are either 1 or 0 mod 8. So we deduce there are 6 odds and 3 evens among the remaining a_i. The RHS is equivalent to 1 mod 9. Sixth powers are either 1 or 0 mod 9. From this we deduce that there are 8 multiples of 3 and 1 that isn't. Since $6^6 = 46656$, not all three evens can be 6, so we find that $a_4 = a_5 = 6$, $a_6 = ... = a_{11} = 3$. Subtracting, we find $a_{12} = 2$. The answer is then $4 + 6(9) + 2(36) + 3(49) = \boxed{277}$.

1.2.4 Advanced Topics

1. *What is the greatest possible number of edges in a planar graph with 12 vertices? A planar graph is one that can be drawn in a plane with none of the edges crossing (they intersect only at vertices).*

   Solution: Euler's formula tells us that $V - E + F = 2$. Suppose we have some vertices and some edges. Consider any region (including the exterior) created by the graph. If this region touches more than 3 vertices, then there are two vertices that can be connected without making the graph nonplanar. Suppose we go through every region and connect them all in this way. Then all regions have exactly three vertices and three edges associated. Any particular edge must be associated with exactly two regions, and each region is associated with three edges. So the number of regions is 2/3 the number of edges. Then we have $V - E + \frac{2}{3}E = 2$, so $3(V-2) = E$. If $V = 12$ then $\boxed{E = 30}$.

2. 3 *green,* 4 *yellow, and* 5 *red balls are placed in a bag. (Large piles of balls of each colour are outside the bag.) Two balls of different colours are selected at random, and* replaced *by two balls of the third colour. If, at some point, there are* 5 *green balls left in the bag, and there are at least as many yellow balls as red balls left in the bag, how many balls of each colour are left in the bag? Write your answer in the form (g, y, r), where g is the number of green balls and so on.*

Solution: Consider this question modulo 3. At each step, we remove two balls of different colours, and add two balls of the third colour, effectively adding -1 mod 3 to the number of balls of each colour. Letting the number of green balls be G, the number of yellow balls be Y, and the number of red balls be R, $R \pmod 3 \equiv (Y+1) \pmod 3 \equiv (G+2) \pmod 3$. So if $G \pmod 3 \equiv 2$, then Y $\pmod 3 \equiv 0$, and $R \pmod 3 \equiv 1$. Also notice that the total number of balls in the bag never changes. So if $G=5$, then $Y+R=7$. So $(Y,R)=(6,1)$, $(Y,R)=(3,4)$, or $(Y,R)=(0,7)$. We are told that $Y \gneq R$, so $(G,Y,R)=(5,6,1)$. We see that this can be obtained as follows: Take a green and a red. Then we have (2, 6, 4). Take a green and a red. Then we have (1, 8, 3). Then we take yellow+red twice and we have $\boxed{(5,6,1)}$ as desired.

3. *Find the minimum value of* $x^2+2x+\frac{24}{x}$ *for* $x>0$.

Solution: Write the function as $x^2+2x+\frac{8}{x}+\frac{8}{x}+\frac{8}{x}$. We apply the AM-GM inequality to obtain that the function is greater than or equal to $5\sqrt[5]{2\cdot 8^3}=5\sqrt[5]{2\cdot 2^9}=\boxed{20}$. This is obtained if we can have equality among all the terms, which we see works when $x=2$.

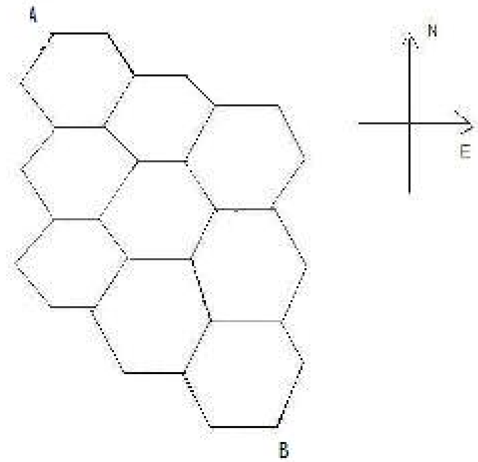
4. *A modern artist paints all of his paintings by dividing his 3 ft by 5 ft canvas into 21 random regions. He then colours some of the regions, and leaves some of them white. If the smallest region has area* $a=10$ *square inches, and the probability that any given region with area* a_i *is left white is* $\frac{a}{a_i}$*, then what is the probability that any given point on the canvas is left white? (1 ft = 12 in)*

Solution: Let the total area of the canvas be $A=3ft\cdot 5ft=2160in^2$. Suppose the regions of area a_i are called A_i. The probability that a random point x is white is equal to the probability that the region it is in is white. Therefore,

$$\begin{aligned} P\{x \text{ is white}\} &= P\{x \in A_i\}\cdot P\{A_i \text{ is white}\} \\ &= \frac{a_i}{A}\cdot\frac{a}{a_i} \\ &= \frac{a_i}{2160}\cdot\frac{10}{a_i} \\ &= \frac{1}{216} \end{aligned}$$

After multiplying by the number of regions, the desired probability is $\boxed{\frac{21}{216}=\frac{7}{72}}$.

5. *In the diagram shown, how many pathways are there from point A to point B if you are only allowed to travel due East, Southeast, or Southwest?*



Solution: Use the following algorithm: at point A write the number 1. At every other vertex V, if all of the vertices from which one could travel to V have numbers, write the sum of these numbers on V. Continue until all vertices are numbered. The number on B is the number of paths from A to B. It is easy to verify that this method counts all paths from A to B. We obtain the answer $\boxed{20}$.

6. *Evaluate the sum:*

$$\sum_{k=0}^{r} \binom{r}{6} \binom{12-r}{6-k}$$

Solution: You have a group of 12 objects, which you then split up into a group of r objects and a group of $12-r$ objects. To pick 6 objects, you must pick k objects from the group with r objects for some k. Since we are summing over k, each possibility is counted once and only once, and the answer is $\boxed{\binom{12}{6} = 924}$.

7. *Aaron has a coin that is slightly unbalanced. The odds of getting heads are* 60%. *What are the odds that if he flips it endlessly, at some point during his flipping he has a total of three more tails than heads?*

Solution: Let $P(n)$ be the probability that we get n more tails than heads at some point, so we're looking for $P(3)$. We have $P(k)P(j) = P(k+j)$ because there is no distinction between starting points when the sequence is infinite. We also have $P(n) = \frac{3}{5}P(n+1) + \frac{2}{5}P(n-1)$ from the result of one flip. It is trivially obvious that $P(0) = 1$, since we start there. Then $P(1) = \frac{3}{5}P(1)^2 + \frac{2}{5}P(0)$, and so $3P(1)^2 - 5P(1) + 2 = 0$. So $P(1) = \frac{5\pm\sqrt{25-24}}{6} = \frac{5\pm1}{6}$. Obviously we want the nontrivial solution $P(1) = \frac{2}{3}$. Then $P(3) = P(1)^3 = \boxed{\frac{8}{27}}$.

8. *Evaluate the sum:*

$$\sum_{n=0}^{\infty} \frac{5n+7}{6^n}$$

Solution: Let $S = \sum_{n=0}^{\infty} \frac{5n+7}{6^n}$. $6S = \sum_{n=0}^{\infty} \frac{5n+7}{6^{n-1}} = \sum_{n=-1}^{\infty} \frac{5(n+1)+7}{6^n}$. Subtracting, $5S = \sum_{n=0}^{\infty} \frac{5}{6^n} + 42 = \frac{5}{1-\frac{1}{6}} + 42 = 48$. So $\boxed{S = \frac{48}{5}}$.

9. *A stick of length 10 is marked with 9 evenly spaced marks (so each is one unit apart). An ant is placed at every mark and at the endpoints, randomly facing either right or left. Suddenly, all the ants start walking simultaneously at a rate of 1 unit per second. If two ants collide head-on, they immediately reverse direction (assume that turning takes no time). Ants fall off the stick as soon as they walk past the endpoints (so the two on the end don't fall off immediately unless they are facing outwards). On average, how long (in seconds) will it take until all of the ants fall off of the stick?*

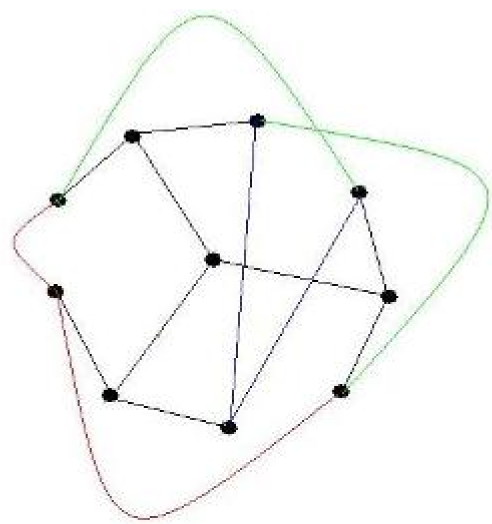
   Solution: The key is to recognize that two ants colliding and reversing directions is equivalent to two ants passing each other and not affecting each other's paths.

   $$\mathbb{E}(\max T_{\text{wait}}) = n \times (\mathbb{P}(\text{either endpoint faces inwards}) + (n-1) \times \mathbb{P}(\text{both endpoints face outwards + either of 2nd closest points face inwards}) + \dots$$

   Since $n = 10$ is even, we have that

   $$\begin{aligned}\mathbb{E}(\max T_{\text{wait}}) &= \left(\sum_{i=0}^{\frac{n}{2}-1} \frac{3}{4} \cdot \left(\frac{1}{4}\right)^i \cdot (n-i) \right) + \left(\frac{1}{2}\right)^n \cdot \frac{n}{2} \\ &= \frac{2^{-n}}{3} + \frac{3n-1}{3} \\ &= \boxed{\frac{9899}{1024}}.\end{aligned}$$

10. *What is the largest possible number of vertices one can have in a graph that satisfies the following conditions: each vertex is connected to exactly 3 other vertices, and there always exists a path of length less than or equal to 2 between any two vertices?*



   Solution: Pick one vertex. This is connected to three distinct vertices. Each of those can be connected to at most 2 other distinct vertices, for a total of 1 + 3 + 6 = 10. It is therefore impossible to satisfy the path-length requirement with more than 10 vertices. We verify that $\boxed{10}$ works by construction.

1.2.5 Calculus

1. *Evaluate the limit:*

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - \sqrt{x^2-x-1}$$

Solution:

$$\sqrt{x^2+x+1} - \sqrt{x^2-x-1} = \frac{x^2+x+1-x^2+x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}}$$
$$\lim_{x\to\infty} \frac{2x+2}{\sqrt{x^2+x+1}+\sqrt{x^2-x-1}} \approx \frac{2x}{x+x} = \boxed{1}.$$

2. *The Three Stooges are standing at the vertices of an equilateral triangle with side length* 1*. At the same time Mo starts running towards Curly, Curly starts running towards Larry, and Larry starts running towards Mo. They all travel at the same speed* $s = 1$ *unit per second. How long, in seconds, does it take before the three of them collide?*

Solution: From the symmetry in the problem we can see that the three guys will always be the vertices of an equilateral triangle. Suppose at a certain time this triangle has side length L. When they next travel a very short distance dx, we can approximate their motion by a straight line. This forms a new, slightly smaller triangle with side length L'. We know they travel this distance in time $\frac{dx}{s}$. We want to determine how quickly the distance between them decreases, i.e., $s\frac{L-L'}{dx}$. Using Law of Cosines, we find that $L'^2 = (L-dx)^2 + dx^2 - 2Ldx\cos(60) = L^2 - 3Ldx + 3dx^2$. We discard the dx^2 term and find $L' = \sqrt{L^2 - 3Ldx}$. Now we want

$$\lim_{dx\to 0} \frac{L-L'}{dx} = \lim_{dx\to 0} \frac{L-\sqrt{L^2-3Ldx}}{dx}.$$

Multiply by the conjugate of the numerator to get

$$\frac{L^2-L^2+3Ldx}{dx(L+\sqrt{L^2-3Ldx})} = \frac{3L}{L+\sqrt{L^2-3Ldx}}.$$

It should now be clear that as $dx \to 0$, we have the closing speed $s' = s\frac{3}{2}$. Then the time it takes for them to collide is $\frac{1}{s'} = \frac{2}{3s} = \boxed{\frac{2}{3}}$.

3. *Find all real m such that there exists a sequence* $a_1, a_2, a_3, \ldots$ *where* $0 < a_k < 1$ *for all k and*

$$\lim_{n\to\infty} (a_1 a_2 a_3 \cdots a_n) = m$$

Solution: We can find a_i such that the product m is any number in $[0,1)$. Consider the sequence $b_n = 1 - \frac{1}{(n+2)2} = \frac{(n+1)(n+3)}{(n+2)2}$. Then

$$\prod_{n=0}^{\infty} b_n = \frac{1}{2}$$
$$\prod_{n=k}^{\infty} b_n = \frac{k+1}{k+2}$$
$$\lim_{k\to\infty} \frac{k+1}{k+2} = 1$$

Suppose $m \in [0,1)$. Let k be large enough that $1 > \frac{k+1}{k+2} > m$. Let $a_2 = b_k, a_3 = b_{k+1}, \ldots$, and let $a_1 = \frac{m(k+2)}{k+1}$ so that $0 < a_1 < 1$ and we have the desired result. So $\boxed{m \in [0,1)}$

4. *Let $a_0 = 2006$, $a_1 = 2007$, and $a_n^2 = a_{n-1} + a_{n-2}$ for $n \geq 2$. Assume that $a_n > 0$ for all n. Evaluate the limit $\lim_{n\to\infty} a_n$.*

   Solution: If there is a limit L, it must satisfy $L^2 = L + L$, so $L = 0, 2$. Now, note that if $a_{n-1} > 2$ and $a_{n-2} > 2$ then $a_n 2 > 4$ so $a_n > 2$. As $a_0 > 2, a_1 > 2$, every term in our sequence is greater than 2, so the limit must be $\boxed{2}$.

5. *Suppose $x + y = 3e^2$. Find the maximum value of x^y.*

   Solution: $y = 3e^2 - x$ so let $f(x) = x^y = x^{3e^2-x}$. Then $\ln(f(x)) = (3e^2 - x)\ln(x)$ and so $f'(x) = x^{3e^2-x}\left(\frac{3e^2-x}{x} - \ln(x)\right)$. By inspection we see that $f'(e^2) = 0$. Sign analysis shows that this is the only relative extremum, and it is a maximum. So the maximum value is $f(e^2) = \boxed{e^{2e^2}}$.

6. *Evaluate the limit:*

$$\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{(-1)^k}{k}$$

   Solution 1: $\sum_{k=0}^{\infty} \frac{(-1)^k}{k}$ is a convergent series. Therefore, for large enough n, we know that $\sum_{k=n}^{\infty} \frac{(-1)^k}{k}$ is arbitrarily small. Therefore the limit is $\boxed{0}$.

   Solution 2:

$$\left|\sum_{k=n}^{2n} \frac{(-1)^k}{k}\right| \leq \left|\sum_{k=n}^{2n} \frac{1}{k(k+1)}\right| \leq \left|\sum_{k=n}^{2n} \frac{1}{k^2}\right| \leq \left|\sum_{k=n}^{2n} \frac{1}{n^2}\right| \leq \frac{1}{n}$$

   Clearly the RHS goes to 0 as n gets large. So the desired limit is $\boxed{0}$.

7. *Evaluate the sum:*

$$\sum_{n=1}^{\infty} \frac{n^2}{(n-1)!}$$

   Solution: Let $S_k = \sum_{n=0}^{\infty} \frac{(n+1)^k}{n!}$. We already know $S_0 = e$, and we want to find S_2. Note that $\frac{(n+1)^2}{n!} = \frac{n^2+2n+1}{n!} = \frac{n}{(n-1)!} + \frac{2}{(n-1)!} + \frac{1}{n!}$. But the $n = 0$ term is 0, so those first two terms are just those in the summations for S_1 and $2S_0$. So we have $S_2 = S_1 + 3S_0$. Now note that $\frac{n+1}{n!} = \frac{1}{(n-1)!} + \frac{1}{n!}$ and we get $S_1 = 2S_0$ for the same reason. So $S_2 = 5S_0 = \boxed{5e}$.

8. *Evaluate the limit:*

$$\lim_{x\to 0} \frac{1}{x^2}\left(\frac{\cos(x)-1}{x} + \frac{\ln(1+x)}{2}\right)$$

   Solution: Using Taylor series for cosine and logarithm, we have

$$\begin{aligned}\frac{1}{x^2}\left(\frac{\cos(x)-1}{x} + \frac{\ln(1+x)}{2}\right) &= \frac{1}{x^2}\left(\frac{1-x^2/2+O(x^4)-1}{x} + \frac{x-x^2/2+O(x^3)}{2}\right) \\ &= \frac{1}{x^2}(-x/2+O(x^3)+x/2-x^2/4+O(x^3)) \\ &= -1/4+O(x)\end{aligned}$$

So the limit as $x \to 0$ is $\boxed{-\frac{1}{4}}$.

9. *Suppose points A, B, C are chosen at random on the circumference of a circle of radius 1. What is the expected value of the perimeter of triangle ABC?*

   Solution: On a circle, the average distance between two random points on the circle is

$$\frac{1}{2\pi}\int_0^{2\pi} 2r\sin\frac{\theta}{2} = \frac{4r}{\pi}.$$

   A triangle is three random points on a circle, so on average, the perimeter of the triangle will be

$$\boxed{3 \cdot \frac{4(1)}{\pi} = \frac{12}{\pi}}$$

10. *Evaluate the sum:*

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}$$

   Solution: We write $2S(1/2) = \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}$, so $S(1/2) = \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+1}}$. Write the summand as $\frac{x^{n+1}}{n+1}$, so we have a general $S(x)$ and observe that it is the same as

$$S(x) = \sum_{n=0}^{\infty} \int_0^x \frac{d}{dy}\frac{y^{n+1}}{n+1} dy.$$

   Now we exchange the order of summation and integration and take the derivative:

$$S(x) \int_0^x \sum_{n=0}^{\infty} y^n dy = \int_0^x \frac{1}{1-y} dy = -\ln(1-x).$$

   We want $2S(1/2) = 2(-\ln(1/2)) = \boxed{2\ln(2)}$.

Chapter 2

Team Test

2.1 Rules

1. The 2006 Team Test will consist of 10 short-answer questions, and will last 30 minutes.
2. The first five problems are worth 8 points each and the last five problems are worth 12 points each. Under no circumstances will partial credit be given for an answer.
3. Proctors will give 5-minute and 1-minute warnings, and no other information.
4. At the end of the 30 minutes, teams must submit the completed official answer sheet to their Proctor. Copying of answers from the blackboard or other sheets of paper to the official answer sheet after the time limit will not be allowed.
5. During the Team Test, any team member found communicating with anyone other than his or her teammates or Proctor will result in disqualification of the entire team from the Team Test.

2.2 Problems

1. *Find the smallest positive integer n such that* $2n+1$ *and* $3n+1$ *are both squares.*

   Solution: We can do this by mods. If $2n+1$ and $3n+1$ are both perfect squares, they are restricted in what they can be mod various numbers. 4 and 5 are the first interesting ones.

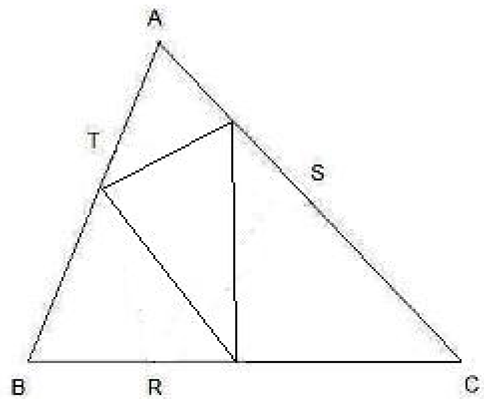
n	a	b
0	1	1
1	3	0
2	1	3
3	3	2

So $n \equiv 0 \pmod{4}$.

n	a	b
0	1	1
1	3	4
2	0	2
3	2	0
4	4	3

Perfect squares are 0, 1, or 4 $\pmod 5$, so we must have $n \equiv 0 \pmod 5$. Thus $20|n$. Obviously $n = 20$ doesn't work, but $\boxed{n = 40}$ does, so that's the answer. (One can actually check $\pmod 8$ and find that necessarily $40|n$.)

2. *In triangle ABC, R is the midpoint of BC and $CS = 3SA$. If x is the area of CRS, y is the area of RBT, z is the area of ATS, and $y^2 = xz$, then what is the value of $\frac{AT}{TB}$? Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive and as small as possible, and c is squarefree.*



Solution: Let $CR = BR = a$, $CS = 3b$, $AS = b$, $AT = kc$, and $BT = c$. We are required to solve for k. Now $x = [CRS] = \frac{1}{2}a \cdot (3b) \cdot \sin\angle C = \frac{3}{8} \cdot (\frac{1}{2}(2a) \cdot (4b) \cdot \sin\angle C) = \frac{3}{8}[ABC]$, $y = [BRT] = \frac{1}{2}ac \cdot \sin\angle B = \frac{1}{2(k+1)} \cdot (\frac{1}{2}(2a) \cdot ((k+1)c) \cdot \sin\angle B) = \frac{1}{2(k+1)}[ABC]$, and $z = [AST] = \frac{1}{2}b \cdot (kc) \cdot \sin\angle A = \frac{k}{4(k+1)} \cdot (\frac{1}{2}(4b) \cdot ((k+1)c) \cdot \sin\angle A) = \frac{k}{4(k+1)}[ABC]$. Also, $y^2 = xz$. Substituting, $(\frac{1}{2(k+1)}[ABC])^2 = \frac{3}{8}[ABC] \cdot \frac{k}{4(k+1)}[ABC]$, which simplifies to the quadratic equation $3k^2 + 3k - 8 = 0$, which has positive root $\boxed{k = \frac{\sqrt{105} - 3}{6}}$.

3. *Find all real solutions (x, y) to the equation $y^4 + 2y^2 + 8x^2 + 16x^4 = 24xy - 8$.*

Solution: Write the expression as $y^4 + 2y^2 + 8x^2 + 16x^4 + 4 + 4 = 24xy$. The purpose of this is to have the same number of terms on the LHS as the sum of the powers of each variable on the LHS. We then see if the AM-GM inequality can provide any information. Since all variables on the LHS are raised to even powers, the terms are all positive, so the geometric mean of the LHS is $\sqrt[6]{2^{12}y^6x^6} = 4xy$. Then by the AM-GM inequality, LHS $\leq$ |RHS|. But we are given that LHS = RHS, and equality occurs in the AM-GM inequality only when all terms are equal. So $y^4 = 16x^4 = 4$, whence $y = \pm\sqrt{2}$ and $x = \pm\frac{1}{\sqrt{2}}$. But we also need the RHS to be positive, so x and y have the same sign. Thus the solutions are $\boxed{(x, y) \in \left\{ \left(\frac{\sqrt{2}}{2}, \sqrt{2}\right), \left(-\frac{\sqrt{2}}{2}, -\sqrt{2}\right) \right\}}$.

Alternatively, substitute $2x = z$ to obtain $y^4 + 2y2 + z^4 + 2z^2 = 12yz - 8$. We'd like to find a nice factorization of this, such as $(y^2 + a)^2 + (z^2 + a)^2 + b(y + cz)^2$. An obvious choice for a is 2, to get rid of the 8. Then we have $(y^2 + 2)^2 + (z^2 + 2)^2 - 2y^2 - 2z^2 - 12yz = 0$. This doesn't seem too helpful, but if $a = -2$ then we have $(y^2-2)^2 + (z^2-2)^2 + 6y^2 - 12yz + 6z^2 = 0$, which is $(y^2-2)^2 + (z^2-2)^2 + 6(y-z)^2 = 0$. Since all of these are squares, every term must equal 0 individually, so $y^2 = 2, z^2 = 2, y = z$, which gives us the same answers as before.

4. *Suppose that $n > 1$ and $P_n(x)$ is a polynomial of degree n. For $k = 1, 2, \ldots, n$ we have $P_n(k) = k(k+1)$. Also $P_n(0) = 1$. For all n there exists an integer $m > n$ such that $P_n(m) = P_{n+2}(m)$. Find the value of m for $n = 10$.*

   Solution: First note that we can actually find the polynomial $P_n(x)$. This will make finding the desired result much easier. We know that $P_n(x) - x(x+1)$ has n zeros at $1, 2, \ldots, n$. Therefore $P_n(x) = x(x+1) + c(x-1)(x-2)\cdots(x-n)$. Then $P_n(0) = 1 = 0 + c(-1)(-2)\cdots(-n) = c(-1)^n n!$, so

$$\begin{aligned} P_n(x) &= x(x+1) + \frac{(-1)^n}{n!}(x-1)(x-2)\cdots(x-n) \\ P_n(m) &= P_{n+2}(m) \\ m(m+1) + \frac{(-1)^n}{n!}\frac{(m-1)!}{(m-n-1)!} &= m(m+1) + \frac{(-1)^{n+2}}{(n+2)!}\frac{(m-1)!}{(m-n-3)!} \\ \frac{1}{(m-n-1)(m-n-2)} &= \frac{1}{(n+1)(n+2)} \end{aligned}$$

   Clearly $m = 2n+3$ will make the two sides equal, regardless of n. So when $n = 10$, $\boxed{m = 23}$.

5. *How many pairs of positive integers (a, b) are there such that $a < b$ and a, b can be the legs of a right triangle with hypotenuse 340?*

   Solution: Any Pythagorean triple with even hypotenuse is a multiple of a primitive triple with odd hypotenuse. So it's sufficient to find all triples with hypotenuses 5, 17, or 85 (the odd multiples of 340). For 5 and 17 we get $(3, 4, 5)$ and $(8, 15, 17)$. For 85, we need to perform some calculations. All primitive Pythagorean triples can be written in the form $(m^2 - n^2, 2mn, m^2 + n^2)$. If $m^2 + n^2 = 85$, then the only possible pairs are (9, 2) and (7, 6). These give the triples $(36, 77, 85)$ and $(13, 84, 85)$. In all, we have found $\boxed{4}$ solutions.

6. *Consider the sequence*

$$1, 1, 2, 1, 2, 4, 1, 2, 4, 8, 1, 2, 4, 8, 16, 1, \ldots$$

   formed by writing the first power of two, followed by the first two powers of two, followed by the first three powers of two, and so on. Find the smallest positive integer N such that $N > 100$ and the sum of the first N terms of this sequence is a power of two.

   Solution: Let S_n be the sum of all the terms up to the first appearance of 2^{n-1}. Then

$$S_n = n \cdot 1 + (n-1) \cdot 2 + \ldots + 2 \cdot 2^{n-2} + 1 \cdot 2^{n-1} \tag{2.1}$$

$$2S_n = n \cdot 2 + (n-1) \cdot 22 + \ldots + 2 \cdot 2^{n-1} + 1 \cdot 2^n \tag{2.2}$$

   Subtracting (1) from (2), $S_n = -n + 2 + 22 + \ldots + 2^{n-1} + 2^n = 2^{n+1} - 2 - n$. So we need the first k powers of 2 (which are the next k terms of the series) to add up to $n+2$ for the sum of the series to be 2^{n+1}, as required. If N_n is the number of terms in some S_n, then $N_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. The smallest $N_n > 100$ occurs when $n = 14$ and $N_{14} = 105$. The sum of the first k powers of 2 is $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$. If we equate this to $n - 2$ for $n \geq 14$, then $n = 29$ is minimal, with $N_{29} = 435$, and $k = 5$. So the minimal $N > 100$ such that the sum of the first N terms of the series is a power of two is $N = N_{29} + k = \boxed{440}$.

7. *S is a subset of* $\{1,2,\ldots,100\}$. *What is the maximum number of elements in S such that the product of any two of them is not a perfect square?*

   Solution: If $ab = c^2$ for some $c > 1$, then either a or b is divisible by d^2 for some $d > 1$. To show this, first note that there must be at least one prime factor of c greater than 1. Let p be one such factor. Either $p^2|a, p^2|b$, or $p|a,b$, and c but p^2 does not divide any of them. If the last case is true, since a is not equal to b, there must be another prime factor of c, and this process can be repeated. Since c is finite, the process must end somewhere, and either a or b is divisible by a square. We can have at most one of the numbers d^{2n}, n squarefree, in our set. Thus, if we exclude all numbers from our set that are divisible by squares, we are guaranteed to have the maximum possible number of elements. Between 1 and 100, there are 25 numbers divisible by 4. There are 11 numbers divisible by 9, but two of these are divisible by 36. There are 4 numbers divisible by 25, but one of these is divisible by 100. Finally, there are 2 numbers divisible by 49. This gives $(100-25-11+2-4+1-2) = \boxed{61}$.

8. *Evaluate the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

   Solution: Write

$$\frac{1}{n^2(n+1)} = \frac{1}{n}\left(\frac{1}{n(n+1)}\right)$$

   The inner term is a condensation of a familiar telescoping series, so let's expand it:

$$\frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

   Distributing the $\frac{1}{n}$ back in gives us $\frac{1}{n^2} - \frac{1}{n(n+1)}$. So we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

   The first sum is $\frac{\pi^2}{6}$, a result of Euler which we will not show here. The second sum telescopes to 1 (look at the expansion above). So our answer is thus $\boxed{\frac{\pi^2}{6} - 1}$.

9. *Suppose* a,b,c *are real numbers so that* $a+b+c = 15$ *and* $ab+ac+bc = 27$. *Find the range of values that may be obtained by the expression abc.*

   Solution: Vieta's formulas may suggest to us to consider the roots of a cubic polynomial. That is, if $p(x) = (x+a)(x+b)(x+c)$ then $p(x) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc = x^3 + 15x^2 + 27x + abc$. We want to find the range of possible values of abc for which a,b,c will be real. In other words, the local maximum of the polynomial must be nonnegative and the local minimum must be nonpositive. Observe that there can only be one point where the local minimum is 0, and there can only be one point where the local max is 0 (these are not the same point). This and the two given conditions are enough to determine a,b,c. Therefore, the interval between these is the one we want. If two of the variables are equal, then we have $2a+b = 15$ and $a^2 + 2ab = 27$. This gives $a^2 + 2a(15-2a) = 27$ so $3a^2 - 30a + 27 = 0$. This factors to $3(a-1)(a-9) = 0$, so $a = 1,9$. Similarly $a = 1 \implies b = 13 \implies abc = 13$. $a = 9 \implies b = -3 \implies abc = -243$. Therefore, $\boxed{abc \in [-243, 13]}$.

10. *The names of 8 people are written on slips of paper and placed in a hat. Each of the 8 people then randomly draw a piece of paper (without replacement). Then, the people are formed into groups satisfying the following requirements:*
(i) Each person is in the same group as the person who drew his piece of paper.
(ii)There are as many groups as possible while still satisfying condition (i).
On average, how many groups will there be? (There might be "groups" of only one person.)

Solution: Pick one person to draw first. If he gets his own name, that will close off one group. If he draws person x's name, let person x draw next. If person x draws person one's name, it will close off a group. If person x draws person y's name, let person y draw next... etc. If a group is closed off, start a new group in the same way. Note that each time a person draws, there is exactly one draw he can make that will close off the group. Since each group must be closed off exactly once, on average, there are $\frac{1}{8}+\frac{1}{7}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+\frac{1}{1}=\boxed{\frac{761}{280}}$ groups.

Chapter 3

Power Test

3.1 Rules

1. The Power Test will present an interesting definition of an advanced concept. The Power Test will last 60 minutes, and students will be required to submit full solutions.

2. The Power Test will be graded out of 110 points.

3. The Power Test will have several parts. Results from previous parts may be assumed true in later parts, even if the team fails to prove theses previous results. Students should cite these results when they use them. For example, From Part II, Question 1,

4. Proctors will give 5-minute and 1-minute warnings, and no other information.

5. At the end of 60 minutes, teams must submit their solutions written on official solution paper, in order, with numbered pages within each solution (if that solution has more than one page). Collecting, copying, and/or re-ordering the solutions after the time limit will not be allowed. Solutions written on pages other than the provided official solution paper will not be accepted.

6. Solutions should be written on one side of the paper only.

3.2 Problems

A function $f(x)$ is called *finitely 2-refinable* (we will just say "refinable" from now on) if there are constants $\{..., c_{-1}, c_0, c_1, c_2, ...\}$ so that

$$f(x) = \sum_{j \in \mathbb{Z}} c_j f(2x - j) = \ldots + c_{-1} f(2x+1) + c_0 f(2x) + c_1 f(2x-1) + \ldots$$

and only finitely many of the constants c_j are nonzero. For example, the function $f(x) = x$ is easily refinable with $c_0 = \frac{1}{2}$ the only nonzero constant. Also, $f(x) = k$ a constant is refinable in infinitely many different ways, so long as the sum of the c_j is 1.

3.2.1 Characteristic functions of intervals

The function

$$f(x) = \begin{cases} 1 & a \le x < b \\ 0 & \text{otherwise} \end{cases}$$

is called the characteristic function or indicator function for the interval $[a,b)$. We often write $\chi_{[a,b)}$ to denote this function.

1. *Show that $\chi_{[0,1)}$ is refinable.*

   Solution: Denote $f(x) = \chi_{[0,1)}$. Then $f(2x)$ is 1 if $x \in [0,1/2)$ and 0 otherwise. Also $f(2x-1)$ is 1 if $x \in [1/2,1)$ and 0 otherwise; it's a right translate by 1/2. Then $f(2x) + f(2x-1) = f(x)$, so $f(x)$ is refinable with $c_0 = c_1 = 1$.

2. *Show that $\chi_{[0,\frac{3}{2})}$ is not refinable.*

   Solution: Denote $f(x) = \chi_{[0,\frac{3}{2})}$. We note that $f(2x) = 1$ when $x = 1/5$ (say), as is $f(2x+1)$. Suppose that c_{-1} is not zero. $f(2x+1)$ is nonzero for some $x < 0$ too, so we will need $c_{-2} = -c_{-1}$. But then $f(2x+2)$ is nonzero for $-1 \le x < -1/2$, where $f(2x+1)$ is zero. So we need to make $c_{-3} = -c_{-2}$. Clearly this process will not terminate, so we won't have a finite refinement. So c_{-1} must be 0. But then $c_0 = 1$, because $f(1/5) = 1$. $f(2x) = 1$ up until $x = 3/4$, and $f(2x-1)$ is 1 when $1/2 \le x < 3/4$ also, so c_1 must be 0. But then no other translate of the function is nonzero for $3/4 \le x < 1$, so we won't be able to achieve the proper value of 1 on that interval. So this alternative doesn't work either, and we conclude that there's no finite refinement of $f(x)$.

3. *For which a and b is $\chi_{[a,b)}$ refinable?*

   Solution: The function $f(x) = \chi_{[a,b)}$ is refinable if and only if a and b are integers. If the endpoints are integers, then $f(x) = f(2x-a) + f(2x-b)$. If the endpoints are not integers, there are two possible situations. If $b - a < 1$, then $f(2x-j)$ is nonzero on an interval of length less than 1/2, but we can only translate by half-integers, so we can't cover the entire line by translations of interval $[a/2, b/2)$. If $b - a = 1$, then the translations of the intervals cover the real line, but only one translate is nonzero at any given point, so each one that overlaps the original interval must have coefficient 1. But as a is not an integer, $a/2 + j/2$ never equals a for integers j, so the leftmost interval that overlaps $[a,b)$ will extend too far to the left, and so we cannot make $f(x)$ out of translations of $f(2x)$. If $b - a > 1$, then we may have overlap. But for the same reason as previously, the leftmost interval that overlaps $[a,b)$ extends too far left. So the next interval will have the opposite coefficient to cancel out, but that one extends too far left again, and so on, so our refinement will not be finite. If the leftmost interval has coefficient 0, then the next farthest left interval either doesn't cover a or still extends too far left, and we repeat this finitely many times to conclude that the function is not refinable (since $b - a$ is finite).

3.2.2 Polynomials

1. *Show that $f(x) = x^2 + 1$ is refinable.*

   Solution: $f(2x-j) = (2x-j)^2 + 1 = 4x^2 - 4xj + j^2 + 1$. Try $j = -1, 0, 1$. We want $c_{-1}(4x^2 + 4x +$

$2) + c_0(4x^2+1) + c_1(4x^2-4x+2) = x^2+1$, so

$$\begin{aligned} c_{-1}+c_0+c_1 &= \frac{1}{4} \\ c_{-1} &= c_1 \\ 2c_{-1}+c_0+2c_1 &= 1 \end{aligned}$$

Using the second equation, the first and last become

$$\begin{aligned} 2c_1+c_0 &= \frac{1}{4} \\ 4c_1+c_0 &= 1 \\ 2c_1 &= \frac{3}{4} \implies c_1 = c_{-1} = \frac{3}{8} \\ c_0 &= -\frac{1}{2} \end{aligned}$$

So the function is refinable, as desired.

2. *Show that all polynomials of degree 2 are refinable in infinitely many distinct ways.*

Solution: This is exactly the same as the previous problem, except more general. Let $f(x) = ax^2+bx+c$ and let j,k,m be distinct integers. We want to find refinement coefficients c_j, c_k, c_m so that

$$\begin{aligned} f(x) &= c_j f(2x-j) + c_k f(2x-k) + c_m f(2x-m) \\ a &= 4c_j + 4c_k + 4c_m \\ b &= -4jc_j - 4kc_k - 4mc_m \\ c &= j^2c_j + k^2c_k + m^2c_m \end{aligned}$$

At this point there are two approaches. The elementary approach is to solve these equations directly. We will not work this solution here because it requires lengthy algebraic manipulations, but the answer is

$$\begin{aligned} c_j &= \frac{amk+b(m+k)+4c}{4(j-m)(j-k)} \\ c_k &= \frac{amj+b(m+j)+4c}{4(k-m)(k-j)} \\ c_m &= \frac{ajk+b(j+k)+4c}{4(m-k)(m-j)} \end{aligned}$$

The alternative solution requires some linear algebra. We use the fact that a system of linear equations has a unique solution if and only if the determinant of the matrix of coefficients is nonzero. So we compute

$$\begin{aligned} det\begin{vmatrix} 4 & 4 & 4 \\ -4j & -4k & -4m \\ j^2 & k^2 & m^2 \end{vmatrix} &= 16(-m^2k+mk^2+m^2j-mj^2-k^2j+kj^2) \\ &= 16(k-m)(m-j)(j-k) \neq 0 \end{aligned}$$

since all three integers were assumed to be distinct. So the refinement coefficients are uniquely determined by the coefficients of the polynomial. We have a different refinement for any three integers we choose, so there are infinitely many distinct refinements.

3. *In fact, all polynomials are refinable. Prove that for some class of polynomials, the entire class of polynomials is refinable. You may receive full credit for your solution even if you don't prove the most general possible result.*

   Solution: Letting n be the degree of the polynomial, we may write $f(x) = \sum_{i=0}^{n} a_i x^i$ with $a_n \neq 0$ since the zero polynomial is trivially refinable. Substituting, we need to satisfy

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{i=0}^{n} c_j a_i (kx-j)^i.$$

   By the binomial formula, we get

$$f(x) = \sum_{j\in\mathbb{Z}, 0\le l\le i\le n} c_j a_i \binom{i}{l} k^l x^l (-j)^{i-l} = \sum_{l=0}^{n} x^l k^l \sum_{i=l}^{n} \sum_{j\in\mathbb{Z}} a_i \binom{i}{l} c_j (-j)^{i-l}.$$

   Comparing with the definition of $f(x)$ and setting coefficients equal gives us:

$$k^l \sum_{i=l}^{n} \sum_{j\in\mathbb{Z}} a_i \binom{i}{l} c_j (-j)^{i-l} = a_l$$

   for $0 \le l \le n$. Making the change of summation index $m = i - l$, the sum is rewritten:

$$k^l \sum_{m=0}^{n-l} a_{m+l} \binom{m+l}{l} \sum_{j\in\mathbb{Z}} c_j (-j)^m = a_l.$$

   This is a triangular $n+1 \times n+1$ system of equations in the unknowns $\sum_{j\in\mathbb{Z}} c_j (-j)^m$ with diagonal coefficients $a_n k^l$. Since $a_n, k \neq 0$, a solution exists. Now, pick any set of $n+1$ distinct integers, e.g. 1 to $n+1$, to be the set of j's with non-zero c_j's. Then our problem of finding c_j's reduces to the matrix equation $Ac = b$ where A is the $n+1 \times n+1$ matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -j_1 & -j_2 & \dots & -j_{n+1} \\ (-j_1)^2 & (-j_2)^2 & \dots & (-j_{n+1})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-j_1)^n & (-j_2)^n & \dots & (-j_{n+1})^n \end{pmatrix}$$

   c is the column vector of the c_j's, and b is the column vector of the $\sum_{j\in\mathbb{Z}} c_j (-j)^m$ found above. But A is the transpose of a Vandermonde matrix with distinct row vectors, so its determinant is non-zero. Thus, we can compute $A^{-1}b$ to find the values of the c_j's.

3.2.3 Rational functions

1. *Show that the function $\frac{1}{x^2-1}$ is refinable.*

Solution: The methods from before obviously won't work for this kind of function. With characteristic functions of intervals, we considered the endpoints of the intervals. With polynomials, we considered equating coefficients on each power of x. For rational functions, the obvious distinguishing characteristics to consider are the locations of the poles (the points where the denominator is 0). In this case, $x^2 - 1 = (x-1)(x+1)$, so we might as well decompose the function using partial fractions. We get

$$f(x) = \frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$
$$f(2x) = \frac{1}{4x-2} - \frac{1}{4x+2}$$

As with the characteristic functions, we see that the poles have been "pulled twice as close together." We can try to shift the function over left and right so that the poles of the refinement match the poles of the original function:

$$f(2x-1) + f(2x+1) = \left(\frac{1}{4x-4} - \frac{1}{4x}\right) + \left(\frac{1}{4x} - \frac{1}{4x+4}\right)$$
$$f(x) = 4f(2x-1) + 4f(2x+1)$$

So the function is in fact refinable, as desired.

2. *Show that the function* $\frac{1}{x^2+1}$ *is not refinable.*

   Solution: The difference between this and the previous problem is that this function does not have poles (or at least, they're not on the real line). One approach is to flagrantly factor using complex numbers and do partial fractions again, which gives

$$f(x) = \frac{1}{x^2+1} = \frac{1}{2i(x-i)} - \frac{1}{2i(x+i)}$$
$$f(2x-j) = \frac{1}{2i(2x-j-i)} - \frac{1}{2i(2x-j+i)}$$

   But clearly no integer j will recreate the old denominator. To make this proof precise, observe that the locations of the poles are in fact at $x = i$ and $x = -i$. But in the refinement, they are at $x = \frac{j}{2} + \frac{1}{2}i$ and $x = \frac{j}{2} - \frac{1}{2}i$. These have imaginary part $\pm\frac{1}{2}$, regardless of j. So we can never obtain the original function by translations of $f(2x)$.

3. *Show that if* $r(x) = \frac{p(x)}{q(x)}$, *where* $p(x)$ *and* $q(x)$ *are polynomials with no common factors, then* $r(x)$ *is refinable only when all of the roots of* $q(x)$ *are real integers.*

   Solution: First consider the case that r has a complex pole, then it is not refinable. Suppose that r has at least one complex pole. Assume for contradiction that r is refinable. If this is the case, then both $r(x)$ and $\sum_j r(2x-j)$ have exactly the same poles. Let h be the maximum of $|\Im(z)|$ over all poles z of r. Then all poles of $r(2x-j)$ will be contained in the horizontal strip $|\Im(z)| \leq h/2$. But r has a pole z_0 such that $|\Im(z_0)| = h$, which is not contained in this strip. This is a contradiction and so r is not refinable.

   We now show that if r has a pole that does not occur at an integer, then r is not refinable. There are only three possible situations where the translates of the poles of $\sum_j c_j r(2x-j)$ can have a non-zero c_j.

(a) The left-most pole of $r(2x-j)$ occurs in the same location as the left-most pole of $r(x)$.

(b) All of the poles of $r(2x-j)$ occur at locations in between the left and right-most poles of $r(x)$.

(c) The right-most pole of $r(2x-j)$ occurs in the same location as the right-most pole of $r(x)$.

In other words, to include translates of $r(2x)$ that have poles outside the region bounded by the left and right-most poles of $r(x)$ would require inifitely many c_j to ensure that $r(x)$ and $\sum_j c_j r(2x-j)$ had exactly the same poles. From this we can assume without loss of generality that the left-most pole of $\sum_j c_j r(2x-j)$ lines up with the left-most pole of $r(x)$, and any translate of $r(2x)$ that shifts the poles to the left must have $c_j = 0$.

Suppose r has at least one non-integer pole. Then there is some pole P such that P is the left-most non-integer pole. Since P is a non-integer, the mapping $r(2x)$ will have a pole at $P/2$, which is not an integer or half-integer (this pole at $P/2$ will be the left-most pole in the function $r(2x)$ that is not an integer or half-integer). Thus, no translation of poles occuring at integers can establish a pole at $x = P$ in the function $\sum_j c_j r(2x-j)$. Since P itself is non-integer, the distance between the points P and $P/2$ is not an integer or half-integer. Thus, translating a pole at $x = P/2$ by integers will never result in a pole at $x = P$. The only other poles that can occur left of $x = P$ are those that may fall in between $x = P/2$ and $x = P$. If one of these poles can be integer translated to line up with the pole at $x = P$, it must leave a new pole at the integer translate of the pole at $x = P/2$. No translates of $r(2x)$ can cover this new pole because it is the integer translate of the left-most non-integer pole. Note also that there are no other non-integer poles to the left of P. We cannot left-translate the poles on the right side of P due to the reasons given above. Therefore, there is no way to establish a pole at $x = P$ in the function $\sum_j c_j r(2x-j)$, without introducing a new, uncoverable pole at the translate of $x = P/2$. Hence, $r(x) \neq \sum_j c_j r(2x-j)$ and r is not refinable.

Therefore, r is refinable only when all roots of $q(x)$ are real integers.

4. *Show that if*

$$r(x) = \sum_k \frac{p_k(x)}{(x-k)^{m_k}}$$

where the sum is finite, the m_k are integers, and the p_k are polynomials, then $r(x)$ is refinable only if $p_k(x)$ is a constant for all k and $m_j = m_k$ for all j,k.

Solution: We will prove this by contradiction. If $\exists j,k \in \mathbb{Z}$ such that $m_j \neq m_k$, then r is not refinable. In other words, every pole of r must be of the same order and r must be a proper rational function.

Assume that $p_i(x) = \alpha_i$. Since r has finitely many poles and at least one pole of positive order, it must have a left-most pole of positive order. Let the place of occurence of this pole be $x = i$ and the order of the pole be $m_i > 0$. Since r has a pole of order m_i at $x = i$, the mapping $\sum_j c_j r(2x-j)$ must also have an order m_i pole at $x = i$, if r is to be refinable. The translation of poles that are not of order m_i can never establish an order m_i pole. And from the same argument as in the previous problem, we cannot use any pole occuring to the right of $x = i$ to establish a pole at $x = i$. This leaves only the option of using translates of the contracted version of the pole at $x = i$ itself to create a pole at $x = i$ in $\sum_j c_j r(2x-j)$, requiring the equation

$$c_i \frac{\alpha_i}{((2x-i)-i)^{m_i}} = \frac{\alpha_i}{(x-i)^{m_i}}.$$

Thus, $c_i \alpha_i = 2^{m_i} \alpha_i$ and so $c_i = 2^{m_i}$. In order to establish a pole of order n, only translates of other poles of order n can be used. This means that if r has poles of varying order, r can be separated into a sum of functions r_t, where each r_t is a function that has poles of all the same order. Consequently, in order for r to be refinable, all of the functions r_t must be refinable with the same refinement sequence. However, as we noted a moment ago, the left-most pole of each order, m_n, will necessarily have a refinement coefficient of $c_n = 2^{m_n}$. Thus, none of the functions r_t can have the same first refinement coefficient unless they have poles of the same order. Therefore, if r has poles of varying order, r is not refinable. Moreover, since r is assumed to have at least one pole of positive order, any refinable r cannot have a pole of negative order. This means that r cannot be reduced to the sum of a polynomial and a rational function, which implies that all rational refinable functions must be proper rational functions.

3.2.4 Almost-refinability

A function $f(x)$ is called almost-refinable if we can write

$$f(x) = k + \sum_j c_j f(2x - j)$$

where k is a nonzero constant.

1. *Show that if $f(x) = g(x) + d$, where $g(x)$ is refinable and d is a constant, then $f(x)$ is almost refinable.*

   Solution: We are given that $g(x)$ is refinable, so it has a refinement with refinement coefficients c_j. That is,

$$\begin{aligned} g(x) &= \sum_j c_j g(2x - j) \\ g(x) + d\sum_j c_j &= \sum_j c_j (g(2x - j) + d) \\ f(x) - d + d\sum_j c_j &= \sum_j c_j f(2x - j) \\ f(x) &= d\left(1 - \sum_j c_j\right) + \sum_j c_j f(2x - j) \end{aligned}$$

   So $f(x)$ is almost refinable, as desired.

2. *Show that there exists a function $f(x)$ that is almost refinable but does not differ from a refinable function by a constant (that is, there is no constant d so that $f(x) - d = g(x)$ is refinable). Your function need not be defined on the entire real line.*

   By looking at the previous problem, we see that if $f(x)$ is almost refinable and the sum of its refinement coefficients is C, then we can reverse the process above and find a refinable $g(x)$ unless $\sum_j c_j = 0$. So we want to find a function that satisfies

$$\begin{aligned} f(x) &= k + \sum_j c_j f(2x - j) \\ \sum_j c_j &= 1 \end{aligned}$$

   It's simplest to assume that only one refinement coefficient is nonzero. Then we only need to find a solution to the functional equation

$$f(x) = k + f(2x)$$

Taking the derivative of this (assuming the function is differentiable; why not?), we get $f'(x) = 2f'(2x)$, so it's homogeneous of degree -1, like $\frac{1}{x}$. The function whose derivative is $\frac{1}{x}$ is $\ln(x)$. So we try $f(x) = \ln(x)$. Since $ln(2x) = ln(2) + ln(x)$, we find $k = -\ln(2)$, and $f(x) = \ln(x)$ is an almost refinable function which does not differ from a refinable function by a constant.

Chapter 4

Relay Test

4.1 Rules

1. Each team will be broken down into two sub-teams of five members each.

2. Each of the five people will receive a different problem.

3. When a person in relay positions 1, 2, 3, or 4 solves his or her problem, the answer should be written on an Official Relay Passing Slip, and passed to the right.

4. Every contestant is to work on his or her own problem. The only means of communication will be through the Official Relay Passing Slip, and the only thing that can be written on the Official Relay Passing Slip is the answer that the contestant has arrived at. This answer may be double-underlined to avoid ambiguity (for example, to differentiate between a 6 and a 9), but no other markings may be made on the Official Relay Passing Slip.

5. There is no limit to the number of official slips that may be passed. If, for example, a contestant realizes that a previous answer was incorrect, he or she is allowed to pass another slip with a different answer.

6. Relay positions 2, 3, 4, and 5 will have problems that use the result of the previous persons question. For example, relay question 3 involves the answer to relay question 2.

7. When the person in relay position 5 gets his or her answer, it should be filled out on an Official Relay Answer Slip. This is the only answer that will be considered for credit.

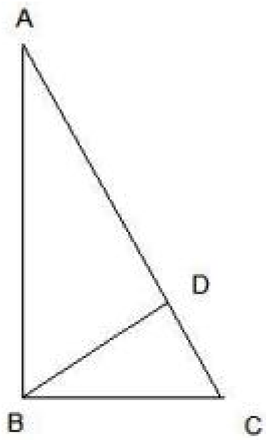
8. A Relay will last 12 minutes. Proctors will accept answers from the person in relay position 5 in the 15-second windows before the 6-minute mark and before the 12-minute mark.

9. If the answer submitted at 6 minutes is correct, and no answer is handed in later, 10 points will be awarded. If an answer is handed in at 12 minutes, any answer previously handed in will be discarded by the Proctor. Correct answers submitted at 12 minutes will be awarded 5 points. Obviously, do not submit the same answer twice! Under no circumstances will partial credit be given for an answer.

4.2 Problems

4.2.1 Round 1

1. *Given the triangle as shown,* $ABC = ADB = 90^\circ, AD = 4x, BD = 3x, BC = y$, *and* $CD = 9$. *Find* y.



Solution: Using the Pythagorean Theorem on $\triangle ABD$, $AB = 5x$. Now $\angle CDB = 180^\circ - \angle ADB = 90^\circ = \angle BDA$ and $\angle CBD = \angle ABC - \angle ABD = 90^\circ - \angle ABD = 180^\circ - \angle ADB - \angle ABD = \angle BAD$, so $\triangle BCD$ $\triangle ABD$ (AA). Then $\frac{BC}{AB} = \frac{CD}{BD}$, so $\frac{y}{5x} = \frac{9}{3x}$. Therefore, $\boxed{y = 15}$.

2. *Let* x *be TNYWR, and let* $y = \frac{x}{3}$. *A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at* y *miles per hour. When his pursuers had been 8 hours on the way, they met a train going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken?*

Solution: Let the criminal's rate be r. After 18 hours, he's gone 18r and the cops have gone $8(y+r)$. We are told $2(y+r)(2.4)$ was the distance between them 2.4 hours before. So $(18-2.4)r - (8-2.4)(y+r) = 4.8(y+r)$ and $10.4(y+r) = 15.6r.104y = 52r \rightarrow 2y = r$. So we want $(10+t)2y = 3yt$, which means $20 + 2t = 3t$ and $t = 20$. So the answer is $\boxed{20}$.

3. *Let* z *be TNYWR, and let* $y = \frac{z}{4}$. *Evaluate* $\binom{y}{1} + \binom{y}{3} + \ldots + \binom{y}{y}$, *where* $\binom{m}{n}$ *is the number of distinct teams of* n *people chosen from a pool of* $m \geq n$ *people.*

Solution: $2^{y-1} = 2^4 = \boxed{16}$.

4. *Let* a *be TNYWR. Compute the sum of the first 99 terms of* $\log_4 a - \log_4 a^2 + \log_4 a^3 - \ldots$, *where* $\log_x y$ *stands for the exponent that you put on* x *to get* y. *So if* $z = \log_x y$, *then* $x^z = y$.

Solution: $\log_4 16 = 2$. $\log_4 a^n = n\log_4(a)$. So we are looking for $2(1-2+3-\cdots+99) = 2(50) = \boxed{100}$.

5. *Let* d *be TNYWR. Find the focal length of the ellipse* $\frac{x^2}{d} + \frac{y^2}{36} = 1$.

Solution: $\sqrt{100-36} = \boxed{8}$.

4.2.2 Round 2

1. *How many ordered pairs of integers* (x,y) *satisfy the equations* $x^y = 1$ *and* $x+y=3$*?*

   Solution: If $x+y=3$ then $x^y = x^{3-x}$. Then $x^{3-x}=1$ means $x=\pm1$ or $3-x=0$, so $x = \in \{-1,1,3\}$. So there are $\boxed{3}$ such pairs of integers.

2. *Let k be TNYWR. Suppose ABC is a right triangle with integer sides and one leg of length k. Find the area of the inscribed circle for triangle ABC.*

   Solution: This is a $3-4-5$ right triangle. The area of a triangle is the semiperimeter times the inradius, $sr = A$. But we know s and A: $6r = 6 \implies r = 1$. So the area of the circle is $\boxed{\pi}$.

3. *Let S be TNYWR, where S is the surface area of a right circular cone. Given that the slant height of the cone is* $\frac{5}{6}$*, what is the radius of the base of the cone?*

   Solution: The surface area of a cone is $\pi rs + \pi r^2$. We are given that the surface area is π and the slant height is $\frac{5}{6}$. So $\pi = \pi r \frac{5}{6} + \pi r^2$; $6r^2 + 5r - 6 = 0$ factors to $(3r-2)(3r+3) = 0$. Only the positive root makes sense; $r = \boxed{\dfrac{2}{3}}$.

4. *Let x be TNYWR. If a solid has* $9x$ *congruent faces and* $18x$ *edges of length* $12x$*, what is the greatest distance between two vertices on the solid?*

   Solution: The solid has 6 congruent faces and 12 edges of length 8; this is a cube of side length 8. The greatest distance between two vertices of a cube is the distance between opposite vertices, which is $\sqrt{3}s = \boxed{8\sqrt{3}}$.

5. *Let n be TNYWR. A (very) unfair coin has a probability* $\frac{p}{q}$ *of coming up heads, where* $\frac{p}{q}$ *is in lowest terms. If* $2pq = n^2$ *what is the greatest possible probability that of two throws, at least one will come up heads?*

   Solution: We are given that $2pq = (8\sqrt{3})^2 \implies pq = 96$. As we are given that p and q have no common factors, the only possible pairs are $(1,96),(3,32),(32,3),(96,1)$. But since $\frac{p}{q}$ is a probability, $p \le q$, so only the first two make sense. Then, we want to find the greatest possible probability that out of two throws at least one is heads. That's the same as the probability that in two throws, not both are tails, which is $1 - \left(\frac{q-p}{q}\right)^2$. This is larger if $q-p$ is smaller, so $(3,32)$ is the right pair. So we want $1 - \frac{29^2}{32^2} = \frac{1024-841}{1024} = \boxed{\dfrac{183}{1024}}$.