5th QEDMO (QED Mathematical Olympiad)

1. Let a, b and k be three positive integers.
We define two sequences (a_n) and (b_n) by the starting values $a_1 = a$ and $b_1 = b$ and the recurrent equations $a_{n+1} = ka_n + b_n$ and $b_{n+1} = kb_n + a_n$ for each positive integer n. Prove that if $a_1 \perp b_1$, $a_2 \perp b_2$ and $a_3 \perp b_3$ hold, then $a_n \perp b_n$ holds for every positive integer n.
Hereby, the abbreviation $x \perp y$ stands for "the numbers x and y are coprime".
(Darij Grinberg)

2. Let $ABCD$ be a (not self-intersecting) quadrilateral satisfying $\measuredangle DAB = \measuredangle BCD \neq 90°$. Let X and Y be the orthogonal projections of the point D on the lines AB and BC, and let Z and W be the orthogonal projections of the point B on the lines CD and DA.

   Establish the following facts:

   a) The quadrilateral $XYZW$ is an isosceles trapezoid such that $XY \parallel ZW$.

   b) Let M be the midpoint of the segment AC. Then, the lines XZ and YW pass through the point M.

   c) Let N be the midpoint of the segment BD, and let X', Y', Z', W' be the midpoints of the segments AB, BC, CD, DA. Then, the point M lies on the circumcircles of the triangles $W'X'N$ and $Y'Z'N$.
(Darij Grinberg)

3. Let a, b, c, d be four positive reals such that $d = a + b + c + 2\sqrt{ab + bc + ca}$. Prove that $a = b + c + d - 2\sqrt{bc + cd + db}$.
(Darij Grinberg, inspired by the Descartes-Soddy formula about the radii of four circles which touch each other in pairs)

4. Let n be a positive integer, and let $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n)$ and $(c_1, c_2, ..., c_n)$ be three sequences of integers such that for any two distinct numbers i and j from the set $\{1, 2, ..., n\}$, none of the seven integers

$$\begin{array}{ll} a_i - a_j \ ; & (b_i + c_i) - (b_j + c_j) \ ; \\ b_i - b_j \ ; & (c_i + a_i) - (c_j + a_j) \ ; \\ c_i - c_j \ ; & (a_i + b_i) - (a_j + b_j) \ ; \\ (a_i + b_i + c_i) - (a_j + b_j + c_j) & \end{array}$$

   is divisible by n.

   Prove that:

   a) The number n is odd.

b) The number n is not divisible by 3.
(generalization of one direction of Theorem 2.1 in: Dean Alvis, Michael Kinyon, ***Birkhoff's Theorem for Panstochastic Matrices****, American Mathematical Monthly, 1/2001 (Vol. 108), pp. 28-37; the original Theorem 2.1 is obtained if you require* $b_i = i$ *and* $c_i = -i$ *for all* i*, and add in a converse stating that such sequences* $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n)$ *and* $(c_1, c_2, ..., c_n)$ *indeed exist if* n *is odd and not divisible by* 3*)*

5. Let a, b, c be three integers. Prove that there exist six integers x, y, z, x', y', z' such that

$$a = yz' - zy'; \qquad b = zx' - xz'; \qquad c = xy' - yx'.$$

   (Darij Grinberg)

6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that the equality $f\left((f(x))^2 + f(y)\right) = x \cdot f(x) + y$ holds for any two rationals x and y.
   (http://www.mathlinks.ro/Forum/viewtopic.php?t=16667)

7. Given a positive integer n. One night, somewhere deep in the voids of the Internet, $2n$ SPAMmers SPAMmed each other. Hereby, each SPAMmer has sent SPAMs to at least n other SPAMmers. Prove that there were two SPAMmers each of whom has sent a SPAM to the other one.
   (Norway MO? Irish MO?)

8. Let A, B, C, A', B', C', X, Y, Z, X', Y', Z' and P be pairwisely distinct points in space such that

$$A' \in BC;\ B' \in CA;\ C' \in AB;\ X' \in YZ;\ Y' \in ZX;\ Z' \in XY;$$
$$P \in AX;\ P \in BY;\ P \in CZ;\ P \in A'X';\ P \in B'Y';\ P \in C'Z'.$$

   Prove that

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{YX'}{X'Z} \cdot \frac{ZY'}{Y'X} \cdot \frac{XZ'}{Z'Y}.$$

   (Darij Grinberg)