4th QEDMO (QED Mathematical Olympiad)

Notation:

- All geometry problems happen in the plane, i. e. all points considered are assumed to lie on one plane.
- We denote the (non-directed) area of an arbitrary n-gon $P_1P_2...P_n$ by $|P_1P_2...P_n|$.

1. Find all primes p, q, r satisfying $p^2 + 2q^2 = r^2$.
(MathLinks?)

2. Let $ABCD$ be a trapezoid with $BC \parallel AD$, and let O be the point of intersection of its diagonals AC and BD. Prove that $|ABCD| = \left(\sqrt{|BOC|} + \sqrt{|DOA|}\right)^2$.
(exercise 8 in: V. Alekseev, V. Galkin, V. Panferov, V. Tarasov, Zadachi o trapezijah, Kvant 6/2000, pages 37-41)

3. Let n be a positive integer, and let $M = \{1, 2, ..., n\}$. Two players take turns at the following game: Each player, at his turn, has to select an element of M and remove all divisors of this element (including this element itself) from the set M.
a) Assume that the player who cannot move anymore (because the set M is empty when it's his move) wins. For which values of n does the first player have a winning strategy?
b) Assume that the player who cannot move anymore (because the set M is empty when it's his move) loses. For which values of n does the first player have a winning strategy?
(Daniel Harrer)

4. Prove that there is no positive integer $n > 1$ such that $n \mid 2^n - 1$.
(classical)

5. Let ABC be a triangle, and let X, Y, Z be three points on the segments BC, CA, AB, respectively. Denote by X', Y', Z' the reflections of these points X, Y, Z in the midpoints of the segments BC, CA, AB, respectively. Prove that $|XYZ| = |X'Y'Z'|$.
(classical; e. g.: [apparently Arthur] Engel, Praxis der Mathematik problem P144)

6. Any two islands of the Chaos Archipelago are connected by a bridge - a red bridge or a blue bridge. Show that at least one of the following two assertions holds:
$\mathcal{A}_1$: For any two islands a and b, we can reach b from a through at most 3 red bridges (and

no blue bridges).
$\mathcal{A}_2$: For any two islands a and b, we can reach b from a through at most 2 blue bridges (and no red bridges).
Alternative formulation: Let G be a graph. Prove that the diameter of G is ≤ 3 or the diameter of the complement of G is ≤ 2.
(Frank Harary, Robert W. Robinson, The Diameter of a Graph and its Complement, The American Mathematical Monthly, Vol. 92, No. 3. (Mar., 1985), pp. 211-212)

7. Let a, b, c be three nonnegative reals. Prove that

$$|ca - ab| + |ab - bc| + |bc - ca| \leq \left|b^2 - c^2\right| + \left|c^2 - a^2\right| + \left|a^2 - b^2\right|.$$

   (Darij Grinberg, but may be known)

8. Show that there are no integers x and y satisfying $x^2 + 5 = y^3$.
   (Daniel Harrer, but turned out to be classical)

9. A team contest between n participants is to be held. Each of these participants has exactly k enemies among the other participants. (If A is an enemy of B, then B is an enemy of A. Nobody is his own enemy.) Assume that there are no three participants such that every two of them are enemies of each other.
   A *subversive enmity* will mean an (un-ordered) pair of two participants which are enemies of each other and which belong to one and the same team. Show that one can divide the participants into two teams such that the number of subversive enmities is $\leq \dfrac{k(n-2k)}{2}$.
   (The teams need not be of equal size.)
   (Glenn Hopkins, William Staton, Maximal Bipartite Subgraphs, Ars Combinatoria 13 (1982), pp. 223-226)

10. Let ABC be a triangle.
    The A-excircle of triangle ABC has center O_a and touches the side BC at the point A_a.
    The B-excircle of triangle ABC touches its sidelines AB and BC at the points C_b and A_b.
    The C-excircle of triangle ABC touches its sidelines BC and CA at the points A_c and B_c.
    The lines C_bA_b and A_cB_c intersect each other at some point X.
    Prove that the quadrilateral AO_aA_aX is a parallelogram.
    Remark. The *A-excircle* of a triangle ABC is defined as the circle which touches the segment BC and the extensions of the segments CA and AB beyound the points C and B, respectively. The center of this circle is the point of intersection of the interior angle bisector of the angle CAB and the exterior angle bisectors of the angles ABC and BCA.
    Similarly, the B-excircle and the C-excircle of triangle ABC are defined.

(Theorem (88) in: John Sturgeon Mackay, The Triangle and its Six Scribed Circles, Proceedings of the Edinburgh Mathematical Society 1 (1883), pages 4-128 and drawings at the end of the volume)

11. Let S_1, S_2, ..., S_n be finitely many subsets of $\mathbb{N}$ such that $S_1 \cup S_2 \cup ... \cup S_n = \mathbb{N}$. Prove that there exists some $k \in \{1, 2, ..., n\}$ such that for each positive integer m, the set S_k contains infinitely many multiples of m.
(some contest?)

12. Let the incircle of a triangle ABC touch its sides BC, CA, AB at the points X, Y, Z, respectively. The line BY intersects this incircle at a point Y' (apart from Y). The parallel to the line CA through the point B intersects the line ZX at a point U. Show that $Y'U \perp BY$.
(MathLinks: http://www.mathlinks.ro/Forum/viewtopic.php?t=111468)

13. Let n and k be integers such that $0 \leq k \leq n$. Prove that

$$\sum_{u=0}^{k} \binom{n+u-1}{u} \binom{n}{k-2u} = \binom{n+k-1}{k}.$$

Note. We use the following conventions:

$\binom{r}{0} = 1$ for every integer r;

$\binom{u}{v} = 0$ if u is a nonnegative integer and v is an integer satisfying $v < 0$ or $v > u$.

(Darij Grinberg)

14. Let $(a_1, a_2, a_3, ...)$ be a sequence of reals such that

$$a_n \geq \frac{(n-1)\,a_{n-1} + (n-2)\,a_{n-2} + ... + 2a_2 + 1a_1}{(n-1) + (n-2) + ... + 2 + 1}$$

for every integer $n \geq 2$. Prove that

$$a_n \geq \frac{a_{n-1} + a_{n-2} + ... + a_2 + a_1}{n-1}$$

for every integer $n \geq 2$.
(Darij Grinberg)