2nd QEDMO (QED Mathematical Olympiad) Bayreuth (2. - 5. January 2006)

1. Solve the equation $x^2 + y^2 = 10xy$ for integers x and y.
(ancient Greeks?)

2. Can a 1337×1337 chessboard be colored black and white (this means that each field of the chessboard is colored either black or white) such that for each field of the chessboard, the number of black neighbours of this field is odd?
Hereby, two fields are called neighbours if they have a common side or a common corner. This means that each of the inner fields has 8 neighbours.
(Iran TST 1996: http://www.mathlinks.ro/Forum/viewtopic.php?t=111249)

3. Prove the inequality
$$\frac{b^2 + c^2 - a^2}{a\left(b+c\right)} + \frac{c^2 + a^2 - b^2}{b\left(c+a\right)} + \frac{a^2 + b^2 - c^2}{c\left(a+b\right)} \geq \frac{3}{2}$$
for any three positive reals a, b, c.
(Darij Grinberg)

4. Let $ABCD$ be a cyclic quadrilateral. Let X be the foot of the perpendicular from the point A to the line BC, let Y be the foot of the perpendicular from the point B to the line AC, let Z be the foot of the perpendicular from the point A to the line CD, let W be the foot of the perpendicular from the point D to the line AC. Prove that $XY \parallel ZW$.
(Darij Grinberg, but actually just an application of a known fact)

5. For any natural number m, we denote by $\phi\left(m\right)$ the number of integers k relatively prime to m and satisfying $1 \leq k \leq m$. Determine all positive integers n such that for every integer $k > n^2$, we have $n \mid \phi\left(nk+1\right)$.
(Daniel Harrer)

6. On the 1 km long ridge of Mount SPAM, there are 2006 lemmings. In the beginning, each of them walks along the ridge in one of the two possible directions with speed $1\frac{\mathrm{m}}{\mathrm{s}}$. When two lemmings meet, they both reverse the directions they walk but keep their walking speed. When some lemming reaches the end of the ridge, he falls down and dies.
Find the least upper bound for the time it can take until all the lemmings are dead.
(probably some contest)

7. Let H be the orthocenter of a triangle ABC, and let D be the midpoint of the segment AH. The altitude BH of triangle ABC intersects the perpendicular to the line AB through the point A at the point M.

The altitude CH of triangle ABC intersects the perpendicular to the line CA through the point A at the point N.
The perpendicular bisector of the segment AB intersects the perpendicular to the line BC through the point B at the point U.
The perpendicular bisector of the segment CA intersects the perpendicular to the line BC through the point C at the point V.
Finally, let E be the midpoint of the side BC of triangle ABC.
Prove that the points D, M, N, U, V all lie on one and the same perpendicular to the line AE.
(most parts of this problem are due to Victor Thébault, 1950, but got rediscovered by many others - e. g. part of this problem was in Balkan MO 2003)

8. Show that for any positive integer $n \geq 4$, there exists a multiple of n^3 between $n!$ and $(n+1)!$.
(*Remark.* For any positive integer k, the abbreviation $k!$ is defined by $k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k$.)
(some low-level contest?)

9. In a one-player game, you have three cards. At the beginning, a nonnegative integer is written on each of the cards, and the sum of these three integers is 2006. At each step, you can select two of the three chards, subtract 1 from the integer written on each of these two cards - as long as the resulting integers are still nonnegative -, and add 1 to the integer written on the third card. You play this game until you can't perform a step anymore because two of the cards have 0's written on them. Assume that, at this moment, the third card has a 1 written on it. Prove that I can tell you which card contains the 1 without knowing how exactly you proceeded in your game, but only knowing the starting configuration (i. e., the numbers written on the cards at the beginning of the game) and the fact that at the end, you were left with two 0's and a 1.
(some Russian contest?)

10. Let X_1, Z_2, Y_1, X_2, Z_1, Y_2 be six points lying on the periphery of a circle, in this order.
Let the chords Y_1Y_2 and Z_1Z_2 meet at a point A; let the chords Z_1Z_2 and X_1X_2 meet at a point B; let the chords X_1X_2 and Y_1Y_2 meet at a point C. Prove that
$$(BX_2 - CX_1) \cdot BC + (CY_2 - AY_1) \cdot CA + (AZ_2 - BZ_1) \cdot AB = 0.$$
(Darij Grinberg, based on known facts)

11. $2n$ cards are aligned on a table, forming a row. Each card contains a positive integer. Albatross and Frankinfueter play a game: Albatross starts and takes either the leftmost or the rightmost card away from the row. Then, Frankinfueter does the same. Then, Albatross

continues the same way, and so on, until all cards are taken. Show that Albatross has a strategy to make sure that at the end of the game, the sum of the integers on the cards he has taken during the game is greater or equal to the sum of the integers on the cards Frankinfueter has taken. (Of course, the integers on the cards are visible to the players from the beginning of the game on.)
(Moldova MO 2006, but also earlier sources)

12. A number n is called *strange* if it can be written in the form $2^a 3^b$ with nonnegative integers a and b. Show that each positive integer can be written as a sum of strange numbers such that none of the strange addends is divisible by another one.
(Putnam 2005 problem A1, some Tournament of Towns(?), Erdös(?))

13. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any two reals x and y, we have
$f(f(x+y)) + xy = f(x+y) + f(x) f(y)$.
(MathLinks?)

14. On the sides BC, CA, AB of an acute-angled triangle ABC, we erect (outwardly) the squares BB_aC_aC, CC_bA_bA, AA_cB_cB, respectively. On the sides B_cB_a and C_aC_b of the triangles BB_cB_a and CC_aC_b, we erect (outwardly) the squares $B_cB_vB_uB_a$ and $C_aC_uC_vC_b$. Prove that $B_uC_u \parallel BC$.
(Moscow MO 2005, 10th grade)