Landau Theory of Social Clustering

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We discuss the Landau theory for a nonlinear equation that can describe social changes, such as an influence of the social environment on individual. The models can explain why the minority can survive inside the majority population. It is described in terms of the complex intermittent clusters behavior in the stationary limit.

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1 Introduction

1.1 Analytical models in the sociology

Social change is a fascinating topic in modern times. Social transitions depend to some degree on political and economical global factors. On the other hand, as recent election results in many countries of Central and Eastern Europe show, social changes are dependent to a high degree on public opinion, which is based on individual attititudes.

During social transitions there are some external factors acting on each individual, but social influence is also critical as each individual consults his or her opinion with others. Both processes are very important for understanding this kind of process.

The first mathematical approach to the analysis of opinion formation in groups was made by Abelson [1]. He has shown that a wide class of linear models of individual attitude change lead to complete uniformity of opinions which is not a case of real-world phenomena. The other model with self-organization of minority and majority members was published by Axelrod [2].

The other class of models was proposed by Nowak, Szamrej and Latané [3]. In this model one has a variety of stationary states with a well-localized and dynamically stable clusters (domains) of individuals who share minority opinions. The mathematical framework for these class of probabilistic cellular automata ([4],[5],[6], [7]) models was proposed by Lewenstein, Nowak and Latané [8] in terms of the mean-field theory with intermittent behavior. This approach was based on the theory of social impact formulated by Lataneé [9]. His main postulate is that the impact of a group of individuals on a given person is proportional to three factors: the "strength" of the members of the group, their social "distance" from the individual and their number N. This approach was successfully used in a variety of phenomena such as attitude change [10], conformity [10], social loafing [11], interest in news events [9], stage fight [12] tipping in restaurants [13], bystander intervention in emergencies [14].

1.2 Cellular automata model of social influence

The model of dynamics of the social impact is a generalization of the concepts known from physics of cellular automata. In the model of Nowak, Lewenstein and Latané one takes into account individual differences of all subsystems and the social influence decaying with distance (so it is unlocal cellular automaton). This assumptions are rather obvious from the empirical point of view.

This model is characterized by long- or moderate-range interactions and by an intristic disorder, which is essential for a dynamical, non-trivial theory of social influence in a variety of social geometries [8].

In terms of this model they easily obtain two main emergent phenomena:

polarization and clustering in terms of the staircase dynamics. Behavior of this model is quite general, and almost all critical features are included (for example: locality of influence)

The changes in social systems are quite similar to phase transitions. These changes are induced by some external factors. For example we can introduce the bias added to the local fields, so clusters of the minority start to grow. After some time we have new majority, but the strongest members of the old tend to survive. This is a kind of social memory.

In this article we try to describe social changes in terms of nonlinear Schrödinger equation. Our model is based on the cellular, discrete model model of the social space introduced by Nowak, Lantané and Levenstein ([3]).

The model describe a variety of social changes via theory of Landau functional and the nonlinear Schrödinger equation. In physics there are many papers concerning this kind of formalism, for example in the context of super-fluid ^4He [4], weakly interacting Bose gas in an external potential [15], quantum solitons in optical fibres [16]. The main difference is that in this approach one takes real function instead of complex, and we describe only stationary states of system.

The paper is organized as follows. In section 3 we describe postulates of the discrete and continuous models. In the section 4 we introduce the equation of dynamics and the Lyapunow functional of the system. In the section 5 we describe stationary solutions and the phases in the model.

2 Description of the Cellular Model and Its Continuous Generalization

The model of the social space introduced by Nowak, Lantané and Levenstein is based on several assumptions:

1. *Two-state elements.*

   We deal with the category of cellular automata consisting of N individuals, each holding one of two possible opinions ("Yes" or "No"), which can be described by the variable σ_i equal to 1 or -1 as in the standard theory of ferromagnet.

2. *Disorder and random "strength" parameters.*

   Each individual is characterized by two random strength parameters, which we call persuasiveness p_i and supportiveness s_i, which determine how a given individual may interact with other individuals. In our analysis we assume that $p_i = s_i$ for all i. To incorporate individual differences in persuasiveness and supportiveness one should choose p_i and s_i as a random variables with a probability density $p(p_i, s_i)$.

3. *"Social" space.*

   Each individual is localized in social space, so that each pair (i, j) of persons is characterized by a distance d_{ij}. Interaction between individuals tend to decrease with this distance, which is described by a function $1/g(d_{ij})$. In the case of the Euclidean cellular automata they take the simplified metric because of analytical difficulties in the form of the constant, finite-range interaction: $g(x) = g$ for $x < R$, and $g(x) = \infty$ otherwise. They additionally choose $g(0) = \frac{1}{\beta}$, with $\beta = kT$, where T is the temperature, in order to describe competition between social impact and self-supportiveness.

4. *Social impact theory.*

   Individuals are assumed to influence each others in terms of social impact which can be defined as:

$$I_I = \sum_j \frac{p_j}{(s+p)g(d_{ij})}(1 - \sigma_i\sigma_j) - \sum_j \frac{s_j}{(s+p)g(d_{ij})}(1 + \sigma_i\sigma_j), \qquad (1)$$

   where $g(.)$ is proper function of the distance d_{ij}, and $t(.)$ is the strength scaling function, and s and p are the means of random variables s_i and p_i.

   The equations of dynamics in the model are as follows:

$$\sigma_i' = -sgn(\sigma_i I_i), \qquad (2)$$

   where σ_i' denote opinion of the ith individual at the next time steps.

Now we are ready to introduce the continuous model (as the generalization of the cellular model) based on some postulates:

1. *Continuous field of social opinion.*

   We introduce a real field $v(\vec{x},t)$, which has an interpretation of opinion of a person localized in point of space $\vec{x}$ in time t. The values of this function are in the set of all real numbers.

2. *Social strength.*

   We describe social strength of a given person located in point $\vec{x}$ via a real function $f(\vec{x})$, with $f(\vec{x}) \geq 0$ for every $\vec{x} \in \Re$.

3. *Nonlinearity in a model.*

   Degree of nonlinearity in a model is given by a real parameter $\beta = 1/kT$. This term is introduced to ensure stability of two special values of opinion ($\sim \pm 1$), which are very important for a discrete models.

4. *Locality of social interactions.*

   We deal only with nearest-neighbors in a system. The strength of social interaction between this two persons is given by a real parameter α.

Using equation 2 we find:

$$\begin{aligned}\sigma_i(t+1) - \sigma_i(t) = -\sigma_i(t) + \\ sgn[\ \frac{s_{i+1}+p_{i+1}}{(s+p)g(d_{i,i+1})}\sigma_{i+1}(t) + \frac{s_{i-1}+p_{i-1}}{(s+p)g(d_{i,i-1})}\sigma_{i-1}(t) + \\ \frac{s_{i+1}-p_{i+1}+s_{i-1}-p_{i-1}}{(s+p)g(d_{i,i+1})}\sigma_i(t) + 2\beta\frac{s_i}{s+p}\sigma_i(t)],\end{aligned}$$

and go to the continuous limit, i.e.

$$\begin{aligned}\sigma_i(t+1) - \sigma_i(t) &\rightarrow \dot{v}(x,t), \\ -\sigma_i(t) &\rightarrow -v(x,t), \\ \frac{s_{i+1}+p_{i+1}}{(s+p)g(d_{i,i+1})}\sigma_{i+1}(t) &\rightarrow f(x+1)v(x+1,t), \\ \frac{s_{i-1}+p_{i-1}}{(s+p)g(d_{i,i-1})}\sigma_{i-1}(t) &\rightarrow f(x-1)v(x-1,t), \\ \frac{s_{i+1}-p_{i+1}+s_{i-1}-p_{i-1}}{(s+p)g(d_{i,i+1})}\sigma_i(t) &\rightarrow 0, \\ 2\beta\frac{s_i}{s+p}\sigma_i(t) &\rightarrow v(x,t)\beta f(x)\end{aligned}$$

3 Equation of Dynamics and Lyapunow function

After taking a continuous limit and rescaling we get in one dimension the continuous equation of the dynamics for this model:

$$\dot{v}(x,t) = -v(x,t) + f(x)v(x,t) - \gamma v^3(x,t) + \alpha \frac{\partial^2}{\partial x^2} f(x)v(x,t), \qquad (3)$$

This equation explains how the opinion in space point x and the time t changes its value. The first term describes decaying of opinion in lack of social interaction (there is only one person without self-supportivness). The second term describes the influence of social strength. This is only a one-person term. The third term is due to the nonlinearity effects in the model. We would like to get as a stable configuration (without social interaction) two stationary solutions with an interpretation "Yes" and "No" opinion. The strength of this term is governed by the γ parameter. The fourth term describes interaction with the nearest-neighbors in the model (governed by the α parameter).

Now we will try to describe various kinds of solutions in equation 3:

1. *When* $\alpha = \gamma = 0$

   Here we have equation in the form:

   $$\dot{v}(x,t) = [f(x) - 1]v(x,t), \qquad (4)$$

   which has a solution:

   $$v(x,t) = \exp[f(x) - 1]tv(x,0). \qquad (5)$$

   In subspaces where $f(x) > 1$ solution will blow up to infinity ($v(x,t) \to \infty$ with $t \to \infty$). In the parts of space where $f(x) = 1$ nothing changes, and if $f(x) < 1$ opinion comes to zero value.

2. *When* $\alpha = 0$ *;* $\gamma > 0$

   The equation in this case is as follows:

   $$\dot{v}(x,t) = [f(x) - 1]v(x,t) - \gamma v^3(x,t) \qquad (6)$$

We can divide space into subspaces with a different stationary solutions inside each of them when we take the equation:

$$f(x) - 1 = \gamma v^2(x,t) \tag{7}$$

(a) If $f(x) > 1$ for some x we have three stationary solutions:
$v(x,t) = 0$ (unstable solution),
$v(x,t) = +\sqrt{\frac{f(x)-1}{\gamma}}$ (stable), and
$v(x,t) = -\sqrt{\frac{f(x)-1}{\gamma}}$ (stable).
It has an obvious interpretation that persons which social strength is above average easily get and maintain opinion.

(b) If $f(x) \leq 1$ for some x we have only one stable, stationary solution: $v(x,t) = 0$. This means that average or below average person can not (in this model) maintain and even get stable opinion.

3. *When* $\alpha = 0$ *;* $\gamma < 0$

In this case there are two unstable solutions:

$$v(x,t) = \pm\sqrt{\frac{1-f(x)}{(-\gamma)}}, \tag{8}$$

and one stable trivial result $v(x,t) = 0$ in dependence of value of $f(x)$. This kind of result is uninteresting for us.

Now we are ready to give a functional description of the model.

When we introduce a new field $w(x,t) = \sqrt{f(x)}v(x,t)$ we get the equation:

$$\dot{w}(x,t) = -w(x,t) - \gamma\frac{w^3(x,t)}{f(x)} + f(x)w(x,t) + \alpha\sqrt{f(x)}\nabla^2\sqrt{f(x)}w(x,t). \tag{9}$$

Then we can describe the dynamic of the system via Lyapunow function:

$$\mathbf{H} = \int dx[\frac{w^2(x,t)}{2} + \frac{\gamma w^4(x,t)}{4f(x)} - f(x)\frac{w^2(x,t)}{2} + \frac{\alpha}{2}\nabla(\sqrt{f(x)}w(x,t))^2]. \tag{10}$$

So we get the dynamic equations in the following form:

$$\frac{\delta H}{\delta \dot{w}(x,t)} = \frac{\delta H}{\delta w(x,t)}, \tag{11}$$

with

$$H = \int dt \int dx [\frac{\dot{w}^2(x,t)}{2} - V(x,t)], \tag{12}$$

and

$$V(x,t) = \frac{w^2(x,t)}{2} + \frac{\gamma w^4(x,t)}{4f(x)} - f(x)\frac{w^2(x,t)}{2} + \frac{\alpha}{2}\nabla(\sqrt{f(x)}w(x,t))^2. \tag{13}$$

4 Stationary solutions and phases in the model

We look for stationary solutions ($\dot{v}(x,t) = 0$). We assume here Thomas-Fermi approximation neglecting the kinetic term $\alpha\nabla^2 f(x)v(x,t) \equiv 0$, as in the paper [15]. The equation then has a form:

$$\gamma v^2(x) = f(x) - 1, \tag{14}$$

with solutions:

$$v_0(x) = \begin{cases} \pm\frac{1}{\gamma^{\frac{1}{2}}}\sqrt{f(x)-1} & \text{if } f(x)-1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So we can divide space into parts, each with non-zero or zero solution. Clusters with non-zero solution has a mean diameter proportional to the correlation length in the system.

The kinetic term in this equation in some cases add a small term to this result. We can calculate this additional term in Thomas-Fermi approximation in quasi-one dimensional case. When we take an axis perpendicular to the border of the cluster (inside which we have $f(x) - 1 \geq 0$). we can write the approximate equation as follows:

$$\alpha\frac{\partial^2}{\partial x^2}[f(x)v(x)] + (f(x)-1)v(x) - \gamma v_0^2(x)v(x) = 0, \tag{15}$$

or

$$\alpha\frac{\partial^2}{\partial x^2}[\tilde{w}(x)] + (1 - \frac{1}{f(x)})\tilde{w}(x) - \gamma\frac{v_0^2(x)}{f(x)}\tilde{w}(x) = 0, \tag{16}$$

with $\tilde{w}(x,t) = f(x)v(x,t)$ as a opinion weighed by the strength of the person and the effective potential

$$\tilde{V}_{eff} = (f(x) - 1) - \gamma v_0^2(x) = \begin{cases} f(x) - 1 & \text{if } f(x) - 1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We assume here that the border of the cluster is placed in $x = x_0$ on the chosen perpendicular axis.

Linear approximation very near the edge of the cluster:

$$1 - \frac{1}{f(x)} \approx -\frac{f'(x_0)}{f^2(x_0)}(x - x_0), \tag{17}$$

produce the equation:

$$\alpha \frac{\partial^2}{\partial x^2}\tilde{w}(x) - \frac{f'(x_0)}{f^2(x_0)}(x - x_0)\tilde{w}(x) = 0. \tag{18}$$

When we substitute $x - x_0 = zh$ we get:

$$\alpha \frac{1}{h^2}\frac{\partial^2}{\partial z^2}\tilde{w}(z) - \frac{f'(x_0)}{f^2(x_0)}zh\tilde{w}(z) = 0. \tag{19}$$

If we take $h^3 = \alpha \frac{f^2(x_0)}{f'(x_0}$ the equation will be as follows:

$$\frac{\partial^2}{\partial z^2}\tilde{w}(z) - z\tilde{w}(z) = 0. \tag{20}$$

This is well known differential equation with functions Airy as the solutions, with asymptotic behavior:

$$A_i(z) \sim z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right), \tag{21}$$

which is our additional term (solution outside cluster). The thickness of the transitory layer is equal to $h = (\alpha|\frac{f^2(x_0}{f'(x_0)}|)^{1/3}$. So our solution is build from the clusters of nonzero opinion in form from equation 15 smoothed by the thin layer with thickness h of non-zero Airy like function as in equation 21 equation.

This approximation is valid only in the case of slow effective potential V_{eff} with small social interaction. This means that, the thickness of the

transitory layer should be much smaller than mean diameter of the cluster and the mean distance between clusters.

We can introduce some averaged with random function $f(x)$ description of the system:

$$P_{tot} =< \theta(f(x) - 1) > \tag{22}$$

which is the total area of the non-zero clusters,

$$h_{eff} =< h >=< (\alpha|\frac{f^2(x_0}{f'(x_0)}|)^{1/3} > \tag{23}$$

- thickness of the transitory layer,

$$P_{clust} =< \pi L^2 > \tag{24}$$

- mean area of one cluster,

$$I = \frac{P_{tot}}{P_{clust}} \tag{25}$$

- number of clusters, and finally

$$S = \sqrt{2\frac{P_{clust}}{P_{tot}}} - 2L \tag{26}$$

as the mean distance between clusters.

The approximation is valid if $h_{eff} \ll L$.

In the previous section we introduce a stationary solution for the model of the system with social interactions. We can divide the solution into three different phases depending on the value of the parameters and an order parameter for this system in form of $\eta = \frac{S}{2h_{eff}}$:

1. *Sparse phase* if $S \gg 2h_{eff}$. Here we have small clusters of non-zero opinion in the "sea" of zero opinion. There are some stable two-clusters configurations and of course unstable which fast decay into one-sign group of persons.

2. *Middle density phase* if $S \geq 2h_{eff}$. Some clusters are close to each other, some are quite far away from each other.

3. *Large density phase* if $0 < S \leq 2h_{eff}$. All clusters are very close to each other, system starts to become uniform in term of the sign of the opinion function $v(x,t)$ in the clusters.

We can easily calculate approximate time of the collapsing the minority cluster. When we take small negative cluster surrounded by the large positive one we can calculate diameter of this cluster by a formula:

$$R = \int x \frac{d}{dR} v(x,t) dx, \tag{27}$$

where $\frac{d}{dR} v(x,t) = \delta(x - R)$. We than have

$$\dot{R} = -\frac{2}{R} f(R_0), \tag{28}$$

and

$$R = \sqrt{R_0^2 - 2f(R_0)t}. \tag{29}$$

The time of collapsing the small bubble is equal to:

$$t_{max} = \frac{R_0^2}{2f(R_0)}, \tag{30}$$

which has a well defined, not infinite value.

5 Summary

In this article we introduce a nonlinear equation to describe a social changes in the stationary case. We distinguished three phases of the system - sparse (large social isolation), middle density (a lot of interesting transistent, metastable configurations) and the large density (large value of social interaction) near the uniformity edge. The society in the first case consists of a clusters of persons with well defined opinion (plus-"Yes" or minus-"No" sign) with no interaction between them because of the large number of very weak persons. In the second case we have variety of interesting geometrical shapes of clusters, some quite stable and robust. Our approximation of the interminent layer is not valid here, so we should look for exact solution. It is easy to guess that this stable result is the uniform state.

The various phases of the system are described by the order parameter η that characterize geometrical and dynamical features of this model. This parameter described also the approaching to the stationary state in terms of intermittent steps i.e. first the strongest person changes their opinion, then the weaker ones.

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References

[1] R. P. Abelson, in *Contributions to Mathematical Psychology*, edited by N. Frederksen and H. Gulliksen (Holt, Reinehart and Winston, New York, 1964).

[2] R. Axelrod, J. Conflict Resolution 24, 3 (1980); 24, 279 (1980);
R. Axelrod, *The Evolution of Cooperation* (Basic Books, New York, 1984);

[3] A. Nowak, J.Szamrej, B. Latané, Psychol. Rev. 97, 362 (1990);

[4] A.L. Toom, in *Multicomponent Random Systems*, edited by R. L. Dobrushin and Ya. G. Sinai (Dekker, New York, 1980);

[5] C.H. Bennett, G. Grinstein, Phys. Rev. Lett. 55, 657 (1985);

[6] T.M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985);

[7] J.L. Lebowitz, C. Maes, E.R. Speer, J. Stat. Phys. 59, 117 (1990);

[8] M. Lewenstein, A. Nowak, B. Latané, Phys. Rev. A 45, 763 (1992);

[9] B. Latané, Am. Psyvhol. 36, 343 (1981);

[10] B. Latané, S. Wolf, Psychol. Rev. 88, 438 (1981);
S. Wolf, B. Latané, J. Person. Soc. Psychol. 45. 282 (1983);

[11] B. Latané, K. Williams, S. Harkins, J. Person. Soc. Psychol. 37, 882 (1979);

[12] J. Jackson, B. Latané, J. Person. Soc. Psychol. 40, 73 (1981);

[13] S. Freeman, M. Walker, R. Borden, B. Latané, Person. Soc. Psychol. Bull. 1, 584 (1975);
M. Lynn, B. Latané, J. Appl. Soc. Psychol. 14, 551 (1984);

[14] B. Latané, J. Darley, *The Unresponsive Bystander: Why Doesn't He Help ?* (Prentice-Hall, New York, 1970);
B. Latané, S. Nida, Psychol. Bull. 89, 308 (1981);

[15] D. Huse, H. Siggia, J. Low. Temp. Phys. 46, 137 (1982)

[16] P.D. Drummond, R.M. Shelby, S.R. Friberg, Y. Yamamoto Nature 365, 307 (1993);