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Sigma Notation (Summation Notation)

- **Review**

  1. Arithmetic sequence and its sum :
  2. Geometric sequence and its sum :
  3. **Note**:
     The partial sums of a sequence.
     For the sequence,
     $$a_1, a_2, a_3, \ldots, a_n, \ldots$$
     the partial sums(not infinite sums, or series) are
     $$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n \end{aligned}$$
     S_1 is called the first partial sum. S_n is called the nth partial sum.

- **Motivation of this section.**
  There is a convenient notation for the sum of the terms of a sequence. It is called summation notation or sigma notation. You should be very used to this notation.

- **Summation Notation**
  Definition : The sum of the first n terms of a sequence is represented by
  $$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$
  i is called the **index of summation** (or **summation(or index) variable**). n is the **upper limit of summation**, and 1 is the **lower limit of summation**.

  Example
  $$\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{8!} = 2.71828 \approx e$$
  $$\sum_{j=1}^{6} = 2 + 2 + \cdots + 2 = 12$$
  $$\sqrt{3} + \sqrt{4} + \cdots \sqrt{77} = \sum_{i=3}^{77} \sqrt{i}$$

  Properties of Sums(See page 140 for proofs.)
  $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k + \sum_{j=1}^{n} b_j$$
  $$\sum_{k=1}^{n} c \cdot a_k = c(\sum_{k=1}^{n} a_k)$$
  But, notice that
  $$\sum_{k=1}^{n} a_k \cdot b_k \neq \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} b_k$$

  Example (page 140)
  $$\sum_{i=1}^{20} (a_{2k-1} + a_{2k}) = 47$$
  Then what is $\sum_{i=1}^{40} a_i$?

Example

$$\sqrt{2}+\sqrt{2\sqrt{2}}+\sqrt{\sqrt{2\sqrt{2}}}+\sqrt{\sqrt{\sqrt{2\sqrt{2}}}}+\cdots$$

By using the sigma notation, the sum up to nth term is $2^{\frac{2n-1}{2}}$. Note that $a_{n+1}=\sqrt{2a_n}$

Example

$$\sum_{i=4}^{10} 10 = 70$$

Here, note that $\sum_{i=m}^{n} k = (n-m+1)k$

Now, let's try to find a general formula for $\sum_{i=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2$. (see page 142 for a derivation.)

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Example Triangular numbers.
The sequence $1, 3, 6, 10, 15, \ldots$ is called the triangular numbers. The formula for a_n is $a_n = \frac{n^2+n}{2}$. The sum is

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(\frac{k^2}{2}\right)+\sum_{k=1}^{n}\frac{k}{2} = \frac{n(n+1)(n+2)}{6}$$

How about the sum of the reciprocals of all the triangular numbers? Which is

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\cdots$$

Using the sigma notation and the formula for a_n,

$$\sum_{k=1}^{n}\frac{2}{k^2+k} = 2\sum_{k=1}^{n}\frac{1}{k(k+1)}$$

when n goes to infinity, this is intuitively,

$$\begin{aligned}&2\left(\tfrac{1}{1\cdot 2}+\tfrac{1}{2\cdot 3}+\tfrac{1}{3\cdot 4}+\cdots\right)\\ &=2\left(\tfrac{1}{2}+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots\right)\\ &=2\cdot 1=2\end{aligned}$$

Example Binomial Theorem

$$(x+y)^n = \frac{n!}{0!(n-0)!}x^n+\frac{n!}{1!(n-1)!}x^{n-1}y+\frac{n!}{2!(n-2)!}x^{n-2}y^2+\cdots+\frac{n!}{(n-1)!(n-(n-1))!}xy^{n-1}+\frac{n!}{(n)!(n-n)!}y^n$$

Express the sum using the sigma notation.

- **Homework**: Using the equation

$$(x+1)^4 = x^4+4x^3+6x^2+4x+1$$

derive the equation

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$