Signal Analysis by Time-Frequency Methods

• Ph.D. Thesis Abstract •

Author: Dan Ştefănoiu
Advisor: Professor Petre Stoica

April 1995

Thesis succinct contents

Foreword and Acknowledgements

Born almost 50 years ago, thanks mostly to Norbert Wiener and Claude Shannon from Massachusetts Institute of Technology and Bell Laboratories, the Signal Processing domain has known lately an explosive development. A rich collection of branches and applications extended the former concepts and generated new insights. Therefore, there is a mutual agreement among Signal Processing community regarding the difficulty to specify the boundaries of this research domain.

One of the most focused problem within the last decade regards non-stationary and finite energy signals processing. This problem appeared especially when signals like seismic data, radar, acoustics, speech or images have to be analysed in order to realise forecasting, data compression, automatic extraction of features and interpretation, etc. These are all non-stationary signals, i.e. with (rapid) variable spectrum in time, for which classical harmonic analysis methods usually fail. New methods, time-frequency type, were replaced the classical ones and even the old Fourier Transform was rejuvenated, adapted and re-adopted as a powerful tool in this framework.

The main goal of this thesis is to provide an overview of time-frequency analysis area and to describe at length a transformation that has known recently a very large impact not only within applied mathematics, but also in many applications inside or beyond Signal Processing domain: the *Wavelet Transform*.

Three main parts including 5 chapters in total are presented in this aim. The first part provides among 2 chapters a description of time-frequency area from the majority researchers point of view, but some personal interpretations and formulations are given as well. The second part constitutes the theoretical kernel of thesis. Here, within a chapter, the main contribution of author is presented in detail. It consists of a theory about discrete time multiresolution structure, based on orthogonal wavelets, that could endow the finite energy signals space, similarly to continuous time case. Within the third part, a very intense studied application is developed among a separated chapter: speech coding. The previous theory is used as main tool in designing a new coding method based on orthogonal wavelets and psycho-acoustic properties of human auditory system. The final chapter was devoted to concluding remarks and further research directions. A list of more than 250 bibliographical references completes the thesis.

This thesis cannot encompass the whole panorama of time-frequency analysis area, especially since it is rapidly growing by every week new results. But it reveals the basic concepts and the spirit of a new approach in Signal Processing, that significantly differs from the classical one.

⋆

I am very grateful to many people who gave me all necessary support to perform this research inside my country (Romania) or abroad.

Professor Petre Stoica from "Politehnica" University of Bucharest, Department of Automatic Control and Computer Science was my advisor and collaborator in Romania. He imposed a high level research standard, for the benefit of science, and I would like to thank him very much for his invaluable and competent guidance.

The discrete-time multiresolution theory would probably never be conceived without the passionate research environment offered to me by both *Institut National Polytechnique of Grenoble* (INPG) and *Institut de la Communication Parlée* (ICP) of Grenoble, in France, during my doctoral stages. Professor Gang Feng from ICP was one of my closest collaborators in France and he opened for me the door of his research group (*Speech Processing and Coding* (SPC)), where I could acquire an important research experience. I thank him for his unconditional support and enthusiasm.

Another close collaborator was Mr. Radwan Kastantin, Ph.D. student at ICP-INPG, from the same group SPC. We designed together the speech coding algorithm based on discrete time orthogonal wavelets and psycho-acoustic properties. His previous research experience in speech coding (subsuming almost 10 years) was crucial for completion of our method. I would like to thank him not only as a scientist but also as a friend.

Many interesting scientific working meetings (although in an informal manner), were realised in collaboration with Professor Nadine Martin from *Centre d'Etudes des Phenomènes Aléatoires of Grenoble* (CEPhAG), affiliated to INPG. She gave me numerous and extremely useful advises about my thesis subject, in which she was very interested. Moreover, she accepted to express in writing her opinion about my thesis, as one of abroad country referees. Many thanks.

The financial support for my doctoral stages abroad country was provided by European Community through TEMPUS and COST projects, as well as by French Region Rhône-Alpes through TEMPRA project. These founds would be never obtained without the contribution of Professor Petre Stoica (who convinced me to candidate for a TEMPUS bourse), Professor Gang Feng, Professor Nadine Martin (who both greatly supported me when I submitted my COST application) and especially of the exquisite and distinguished Professor Marie-Thérèse Janot-Giorgetti from INPG – the soul of TEMPUS Project in Romania – who sustained me in all my atempts without any hesitation.

I also would like to thank all the people that I could not mention here, but who helped me in a way or another to complete my tasks.

Finally, I am very grateful to my family, who stoically endured my absence from country and/or my fights against the time.

Dan Ştefănoiu
Grenoble, France
February, 1995

Signal Analysis by Time-Frequency Methods

• Ph.D. Thesis Abstract •

Dan Ştefănoiu*

1 Classical Harmonic Analysis versus Time-Frequency Analysis

This chapter constitutes an introduction within the terminology of Signal Processing domain. Besides, a comparison between the classical point of view, Fourier type, and the new current of time-frequency analysis is performed. The goal is to introduce the set of recent concepts of time-frequency framework, by relating them with well known ones.

1.1 Main classical transformations of Signal Processing

Although some problems of functions interpolation and approximation that interested the mathematicians 300 years ago were anticipating the development of Signal Processing (SP) domain, it is born only in 1948, when Claude Shannon from Bell Laboratories published his Theory of Communication [42]. In fact 1948 is considered *annus mirabilis* for Signal Processing, since many important results (especially in audio data compression and data encrypting) were presented. But the most credited researcher as giving life to this domain is Norbert Wiener from Massachusetts Institute of Technology, who was really concerned with signal processing problems. His book about numerical data processing published in 1949 [61] includes various approaches and methods that originated many modern tools of Signal Processing and System Identification [43] domains (such as the Levinson–Durbin Algorithm described in an annex of [61] and ignored for 20 years, but successfully used today).

One of the fundamental concepts in SP is *the signal* and the following definition is mutually accepted, even though, probably, it is not the most general one.

Definition 1
Any function:

$$f \;:\; \mathcal{T} \longrightarrow \mathcal{M}\;,$$

where:

- $\mathcal{T}$ *is a totally ordered non-void set referred to as* **instants set** *(but it could contain different entities than time instants, like, e.g. space distances);*
- $\mathcal{M}$ *is any non-void set referred to as* **values set** *(but it could contain symbols and not numbers);*

is a **signal**.

In wide sense, the *signal* is a map containing information about the behaviour of a system over the instants set. Usually, $\mathcal{T}$ consists of time instants and $\mathcal{M}$ is real or complex valued.

Mathematically, this concept lies naturally in the framework of Distributions Theory and Differential Equations Theory. But, for practical purposes, the signal is viewed as an element of Lebesgue spaces: $L^p(\mathcal{T})$ (in continuous time case, i.e. for *analogic signals class*) or $l^p(\mathcal{T})$ (in discrete time case, i.e. for *digital signals class*), where $p \geq 1$. From these, the most focused on are the *finite energy signals* spaces ($p = 2$) and the *stable (finite spectrum) signals* spaces ($p = 1$). The main properties of spaces above are: the *separability* (i.e. the possibility to construct countable bases) and the *Hilbert structure* (i.e. the capacity to be complete with respect to a scalar product and its canonical norm).

*Associate-Professor at "Politehnica" University of Bucharest, Department of Automatic Control and Computer Science, System Identification and Signal Processsing Group, 313 Splaiul Independenţei, Sector 6, 77206–Bucharest, ROMANIA.
@mail: danny@router.indinf.pub.ro

After defining the signal and the basic framework in SP, this section presents some classifications of signals. The class of *non-stationary* signals is pointed out as very important in context of this thesis.

Finally, a succinct panorama of classical transformations in SP is realised.

Thus, in continuous time case, the *Laplace Transform* is known as one of the most employed tool within Systems Theory, since it offers a very elegant manner to solve differential equations in weakly restrictive conditions. When the Laplace transform operates only with the imaginary axis of complex plane, it is referred to as *(Continuous) Fourier Transform*. Similarly, in discrete time case, the *$\mathcal{Z}$ Transform* is usually employed to operate with difference equations in Systems Theory. When the argument varies continuously on the unit circle of complex plane, it becomes the *(Continuous) Fourier Transform for Digital Signals*.

Both Systems Theory and Signal Processing domains are using these Fourier transformations, but they are considered specific tools mostly of the second one. Stationary or quasi-stationary signals are usually analysed by means of Fourier Transforms, which are based on sine and cosine functions. This analysis was therefore considered *harmonic type*.

In practice, the classical description of systems or signals behaviour is realised through the concept of *(frequency) spectrum*. This concept issued from Joseph Fourier's Theory regarding the non-uniform approximation of functions by harmonic elementary waveforms superposition. Accordingly, the signal spectrum reveals a distribution of its energy among frequency axis, where spectral power of each frequency is determined by the coefficient of elementary harmonic waveform within superposition. The accuracy of this description is intimately related to the accuracy of Fourier approximation for the original signal. A large set of spectral estimation techniques were designed in order to control and improve this accuracy, such as those proposed by Bartlett or Welch, described in [37]. They are trying to find a good trade off between time and frequency resolution of signals. The opposite variation of these two resolutions is due to Gabor-Heisenberg Uncertainty Principle [21] and to some results of Sampling Theory, obtained by Valée-Poussin, Kotel'nikov, Shannon et al. [33]. Spectral estimation techniques are usually implemented by using *Discrete Fourier Transform* (DFT), which is the discretized version of Continuous Fourier Transform for Digital Signals. The algorithms used to compute DFT are referred to as *Fast Fourier Transform* (FFT) techniques.

Another instrument employed within classical analysis (but almost for theoretical purposes) is *Hilbert Transform*. By using this transformation, a unique *analytic signal* can be associated to the original one. This signal has null spectral power for negative frequencies and preserves the spectrum shape of original signal for non-negative frequencies. The time-frequency analysis operates with analytic signal as well, as described in next section.

For all classical transformations, sound definitions are given and main properties are described inside this section. A detailed description (but in Romanian) is also presented within [50].

1.2 Drawbacks of harmonic analysis in case of non-stationary signals

The classical analysis has a good advantage for its intuitive interpretation of spectrum and for FFT class of algorithms implementing DFT (numerous implementation versions were proposed especially in the '70s). If the signal is stationary or quasi-stationary, DFT is an excellent analysis instrument, with a small amount of computations, if implemented by using a FFT Algorithm. The traditional part of SP community is strongly attached to DFT and is oriented to improve its capabilities rather than to replace it by new and more suitable analysis instruments, when signals lose the stationary behaviour.

In case of non-stationary signals, DFT does not work properly, since it offers an average image of their frequency contents, being unable to reveal the instantaneous characteristics of their spectrum. But, non-stationary behaviour means time variability of spectrum. The most part of real life signals are non-stationary and, moreover, many of them have also a fractal structure, such as: seismic ones, speech, images, etc. For all these signals, DFT is insensitive to modifications in time of their spectrum and even to simple time translations, if one focuses only on amplitude and not on phase. This is the main caveat of classical harmonic analysis and new instruments should be employed instead, such as joint time and frequency transforms.

1.3 Basic definitions and fundamental principle of time-frequency analysis

In this section, the new approach of time-frequency analysis is introduced. It generalises and includes the traditional one, based on elementary harmonic signals superposition. Actually, in this framework the concept of harmonic signal is replaced by *instantaneous frequency* in order to describe the signals intimate dynamics.

The time-frequency analysis is yet an older approach than one believes. In fact, it was foreseen by some researchers even before the SP domain was born, such as D. Gabor [21] (who elaborated one of the first approaches in Theory of Information) and E.P. Wigner [62] (who defined a notorious transformation for theoretical physics). Also, in some articles of J. de Ville, such as [60] in 1948, several time-frequency concepts are defined and they are used until today with the same meaning. But only after 1980 the research in time-frequency analysis became intense and systematic. The main goal of this research is to construct new transforms that are more suitable for non-stationary signals analysis.

The main characteristics of any stable signal (with finite spectrum) $f \in L^1(\mathbb{C})$, are described in a quite complete form by the following 7 concepts (from which the last 2 are specific to time-frequency analysis):

1. *The analytic signal*: $f_a(t) \stackrel{def}{=} f(t) + \frac{j}{\pi}\, vp \int \frac{f(\tau)}{t-\tau}\, d\tau\,, \quad \forall\, t \in \mathbb{R}.$
2. *The envelope*: $a_f(t) \stackrel{def}{=} |f_a(t)|\,, \quad \forall\, t \in \mathbb{R}.$
3. *The spectrum*: $\widehat{a_f}(\Omega) \stackrel{def}{=} |\widehat{f}_a(\Omega)|\,, \quad \forall\, \Omega \in \mathbb{R}.$
4. *The analyitc phase*: $\varphi_f(t) \stackrel{def}{=} arg(f_a(t))\,, \quad \forall\, t \in \mathbb{R}.$
5. *The classical phase*: $\widehat{\varphi_f}(\Omega) \stackrel{def}{=} arg(\widehat{f}_a(\Omega))\,, \quad \forall\, \Omega \in \mathbb{R}.$
6. *The instantaneous frequency*: $\nu_f(t) \stackrel{def}{=} \frac{1}{2\pi}\frac{d\varphi_f}{dt}(t)\,, \quad \forall\, t \in \mathbb{R}.$
7. *The group delay*: $t_f(\nu) \stackrel{def}{=} -\frac{1}{2\pi}\frac{d\widehat{\varphi_f}}{d\nu}(2\pi\nu)\,, \quad \forall\, \nu \in \mathbb{R}.$

Here, $\widehat{f}$ stands for the Fourier Transform of f and $vp \int$ indicates that the integral is evaluated in sense of Cauchy main value. Also, here and hereafter, $\int \equiv \int\limits_{-\infty}^{+\infty}$ (integral limits are eliminated for simplicity). For discrete signals, similar concepts are defined, but the derivative is replaced by a difference between successive values in time.

Beside these general concepts, a set of specific concepts for time-frequency analysis are defined, such as:

- the *time-frequency/scale plane* referred to as *phase space* as well;
- the *node phase* – any point of phase space, determined by the pair (t – time, ω – pulsation/frequency);
- the *state function* or *distribution/transformation time-frequency*, denoted by Φ;
- the *states system* and the *state surface* (replacing the classical spectrum).

1.4 Mathematical origins of time-frequency transformations

Since all time-frequency transformations defined until now are numerous, some classifications are useful in selection of the appropriate one for each specific application. The next chapter presents such a classification that the SP community seem to agree with. But, from our point of view (and it is not an isolated one), an interesting and useful primary classification could be realised starting from the mathematical origins of time-frequency transformations. The most part of them were defined inside theoretical physics, strongly connected with high level mathematics.

Thereby, two large classes of transformations are central within time-frequency analysis (although they cannot include all transformations):

1. *Integral transformations*, defined by using Lie's algebraic groups where lie their arguments.
2. *Bilinear transformations*, defined to represent the energy distribution over the phase space.

Each category is described at length in thesis in separated sub-sections.

The most representative transformations defined by using Lie's groups are enumerated below.

1. The *Short-Time (Windowed) Fourier Transform*.

   Its name becomes from the main operators of Lie's uncertainty group, which contains a frequency modulation (similarly to classical Fourier Transform) and a time translation. In fact, this transformation constitutes an adaptation of Fourier transforms to the framework of time-frequency analysis, as revealed by the following definition:

$$\left[\begin{array}{llll} \Phi(f) : & \mathcal{P} \subseteq \mathbb{R}^2 & \rightarrow & \Gamma \\ & (t,\omega) & \mapsto & \Phi(f)(t,\omega) \stackrel{def}{=} \langle\, f\,,\, g^{(t,\omega)}\,\rangle = \displaystyle\int f(\theta)\,\overline{g(\theta-t)}\,e^{-j\omega\theta}\,d\theta\,. \end{array}\right.$$

Here:

- $g \in L^2(\mathbb{R})$ is a *window-mother* weighting the original signal around a central point. This point is gliding among the signal and a Fourier Transform is practically evaluated for each one of its positions. The window-mother has to be essentially localised both in time and frequency (i.e. it has to have at least polynomial decay type in time and frequency or compact support in time).
- $g^{(t,\omega)}$ is a version of g obtained after time translation/shifting with period t and frequency modulation at pulsation ω. Its role is to focus only on a small zone around (t,ω) inside phase space. The focus quality is controlled only by the window-mother and not by the time shifting or modulation. One of the best windows was proposed by Gabor (the *gaborette-mother*). It is Gaussian type and has radial localisation inside the phase space.

2. The *Wavelet Transform*.

   This transformation appeared approximately 13 years ago, thanks to A. Grossmann and J. Morlet, who tried to solve the difficult problem of seismic data forecasting by introducing the concept of *variable scale representation* [22], [36]. In fact, this transform is defined by means of Lie's affine group, as below:

$$\left[\begin{array}{llll} \Phi(f) \; : & \mathcal{P} \subseteq \mathbb{R}_+^* \times \mathbb{R} & \rightarrow & \Gamma \\ & (a,b) & \mapsto & \Phi(f)(a,b) \stackrel{def}{=} \dfrac{1}{a}\,\langle\, f \,,\, h^{(a,b)}\,\rangle = \dfrac{1}{a\,\sqrt{a}} \displaystyle\int f(t)\,\overline{h\left(\dfrac{t-b}{a}\right)}\,dt\,. \end{array}\right.$$

   Here:

- $h \in L^2(\mathbb{R})$ is the *wavelet-mother* and acts as a gliding window among the signal, being able to focus on parts of signal at different scales. Usually, it has compact support and verifies the following *admissibility condition*:

$$C_h \stackrel{def}{=} \int \frac{|\widehat{h}(\omega)|^2}{|\omega|}\,d\omega < \infty$$

  The quantity C_h is referred to as *admissibility constant*. This condition involves a null spectrum of h in frequencies origin $\widehat{h}(0) = 0$, which means a high frequency localisation of wavelet-mother.
- $h^{(a,b)}$ is a *wavelet* at scale a, after a time shifting with period b. Its role is to focus only on a zone of phase space around the node $\left(ab,\, a^{-1}\right)$. The broadness is controlled not only by wavelet-mother, but also by scale factor a, which varies inverse proportionally with the frequency. Practically the argument of h is affected by the affine operator $\mathcal{W}_{t,a}$ which generates in this way the family of all wavelets born from wavelet-mother:

$$\{\mathcal{W}_{t,a}h\}_{t\in\mathbb{R},\; a\in\mathbb{R}_+^*} = \left\{\frac{1}{\sqrt{a}}\,h\left(\frac{\cdot - t}{a}\right)\right\}_{t\in\mathbb{R},\; a\in\mathbb{R}_+^*}.$$

   This transformation belongs also to *affine class*, due to the argument above.

Both aforementioned transformations use gliding windows in order to evaluate the values of state function. This involves an indirect intervention of Uncertainty Principle, which limits their representation resolutions in time and frequency.

From the second category of transformations, an old and very employed representative can be mentioned here: the *Vigner-Ville Transform*. It is defined as follows:

$$\left[\begin{array}{llll} \Phi(f) \; : & \mathcal{P} \subseteq \mathbb{R}^2 & \rightarrow & \Gamma \\ & (t,\omega) & \mapsto & \Phi(f,f)(t,\omega) \stackrel{def}{=} \Phi(f)(t,\omega) \stackrel{def}{=} \dfrac{1}{2\pi} \displaystyle\int f\left(t-\dfrac{\theta}{2}\right)\,\overline{f\left(t+\dfrac{\theta}{2}\right)}\,e^{-j\omega\theta}\,d\theta \end{array}\right.$$

and generates many other similar bilinear transformations. The term "bilinear" comes from the property of $\Phi(f,f)$ to be linear in both arguments f, like a scalar product. In fact, the general form of a bilinear transformation is not $\Phi(f,f)$, but $\Phi(f,g)$, where f is the signal and g is a glidding window. Beside this, another important class of bilinear transformations is constructed starting from *Ambiguity function* as shown in next chapter. The main required property of these transformations is the non-negativity, which allows us to consider the state function as an energy distribution over phase space. Notice that even Wigner-Ville Transform (which is similar to a canonical norm defined by a scalar product) is not positively defined for any signal. This requirement is difficult to verify for any signal, but the most part of transformations satisfy it for usual classes of signals.

This section is completed by a set of mathematical manipulations proving some interesting properties of these transformations and by 2 examples revealing how the discretization can be performed.

2 Time-Frequency and Time-Scale Transformations

This chapter is devoted to a presentation at length of time-frequency analysis kernel. However, the presentation of whole time-frequency area cannot be complete within a doctoral thesis[1]. Also, the description focuses here on continuous time and frequency case. In next chapter a multiresolution theory for discrete case is developed.

2.1 Overview of time-frequency/scale transformations. Main properties.

This section starts with a quasi-extended state of the art related to publications that contributed significantly to development of time-frequency methods. They evolved in a dynamical but non-linear manner.

Thus, although E.P. Wigner defined a bilinear transform in 1932 [62], he didn't realise its importance and meaningful outside quantum physics. The first germs of time-frequency approach appeared in late '30s, the radio era, when the concept of "instantaneous frequency" was introduced by J.R. Carson and T.C. Fry [8]. They tried to conceive a method for denoising the radio transmissions by using the phase first derivative of a complex signal in time and they referred this derivative to as *instantaneous frequency*. This concept was also considered by B. van der Pol in 1946 [59]. But mid '40s were tremendously influenced by the new born Information Theory. D. Gabor had an important contribution for initiating the time-frequency mentality, especially within his paper [21]. He suggested the use of a window for weighting the most important data and forgetting the useless ones, in order to extract the current information they encode. He also proposed an algorithm that, later, was emergent with the concept of analytic signal.

But the impact of weighting data was small until '70s, when NATO allocated important financial resources for civil and military research in SP. At that time, time-frequency methods were developed almost inside fundamental research projects, as prove publications like [1], [38] or [23]. The technological advances allowed implementation of these methods only starting with the next decade, in '80s. In the past 15 years, the old ideas of Wigner or Gabor were reconsidered, adapted, improved and new insights were revealed by papers like [25], [26], [39], [24], [20], [40]. Many generalisations and new results of 80's and '90s decisively contributed to the development of time-frequency analysis, as a solid branch inside SP domain.

One outstanding figure within this picture is the physicist Leon Cohen, who provided several systematic classifications of time-frequency/scale transformations and proved their properties [13], [10], [11].

After this historical presentation, the thesis enumerates the definitions of the most important time-frequency transformations, as they appear in publications. Several special remarks are given regarding the Wavelet Transform – one of the newest and most powerful analysis instruments for signals with severe non-stationary behaviour and/or intimate ruptures.

An extended list of applications where these transformations have been implemented so far is also provided in the end of section.

2.2 Mathematical foundation of time-frequency analysis

The mathematical background of time-frequency methods issued from 4 important branches: Mathematical Analysis, Distribution Theory, Probability Theory and Statistics.

This section starts with a presentation of the most employed mathematical terms and their interpretations related to SP domain.

Next, the following notions are defined and described in detail:

1. *Time-frequency transform, distribution.*
2. *Analytic signal, instantaneous frequency.*
3. *Group delay.*
4. *Local auto-correlation function.*
5. *Uncertainty Principle, resolutions trade-off, smoothness trade-off.*

Within this extended abstract only some of these concepts are defined and the most expressive interpretations are given.

D. Gabor was the first one who observed the importance of *instantaneous frequency* concept introduced by Carson and Fry in 1937, as mentioned. Moreover, he deduced that this concept requires the construction of a unique complex signal from any real valued one and proposed a method to construct it. His idea is founded on

[1] An excellent presentation is provided by Leon Cohen in [12], but this book appeared after my thesis completion.

the simple remark that the elementary harmonic signals $\sin \omega t$ and $\cos \omega t$ (with $\omega > 0$) can construct together the unique complex signal $e^{j\omega t}$, which has a unique spectral power line for $\Omega = \omega > 0$ and null spectrum for negative pulsation ($\Omega < 0$). He generalised this property for any real valued stable signal f, by proposing a procedure where the spectral power for negative frequencies is cancelled and the spectral power for positive frequencies is doubled in $\widehat{f}$ (the Fourier Transform of f). By applying the Inverse Fourier Transform to this new frequency signal, one obtains exactly the analytic signal f_a defined in previous chapter.

The original definition of instantaneous frequency/pulsation is the following:

Definition 2
The **instantaneous frequency** *and the* **instantaneous pulsation** *associated to a complex valued signal f are:*

$$\nu_f(t) \stackrel{def}{=} \eta^2 \frac{d}{dt} \arg\left(f(t)\right) = \eta^2 \varphi'(t) \, , \quad \forall\, t \in I\!R \, .$$

$$\omega_f(t) \stackrel{def}{=} 2\pi\, \nu_f(t) = \frac{d}{dt} \arg\left(f(t)\right) = \varphi'(t) \, , \quad \forall\, t \in I\!R \, .$$

If f is real valued, it is replaced by the corresponding analytic signal f_a. (Here, $\eta \stackrel{def}{=} \dfrac{1}{\sqrt{2\pi}}$.)

Initially, the instantaneous frequency ν_f was considered well-defined if, at any instant t, the signal reduces only to a unique elementary harmonic signal with frequency $\nu_f(t) > 0$. Thus, the necessity of analytic signal use (with non-null spectrum only for positive frequencies) becomes obvious. Negative instantaneous frequencies were inconceivable, since no real life correspondent can be imagined. Even today, many authors cannot renounce at this requirement, which limits the utility of definition above. Another segment of researchers have imposed the alternate and modern definition below:

Definition 3
The **instantaneous pulsation** *associated to a signal f and to its time-frequency distribution $\mathcal{P}$ is the first order conditional moment of pulsation, evaluated for any instant $t \in I\!R$:*

$$\langle\, \omega \,\rangle_t \stackrel{def}{=} \frac{1}{\mathcal{P}_1(t)} \int \omega\, \mathcal{P}(t,\omega)\, d\omega \, .$$

Here:

$$\mathcal{P}_1(t) \stackrel{def}{=} \int \mathcal{P}(t,\omega)\, d\omega \, .$$

Apparently, the signal f is not present in this definition. But this is false, since both $\mathcal{P}$ and $\mathcal{P}_1$ depend on it (in fact, $\mathcal{P}_1$ is aiming to normalise $\mathcal{P}$). If $\mathcal{P}$ is not positively defined, then it is possible that the instantaneous frequency of Definition 3 be negative, which does not correspond any more to our intuition.

In general it is possible that $\omega_f(t) \neq \langle\, \omega \,\rangle_t$, for some instant $t \in I\!R$. The equality is verified only if some requirements are fulfilled. The advantage of ω_f consists of its direct intuitive interpretation, whereas $\langle\, \omega \,\rangle_t$ can be evaluated without the associated analytic signal.

The correct definition of instantaneous frequency constitutes an open problem even today. It is strongly relied on spectral estimation techniques for non-stationary signals. Some important results on this subject were obtained lately by B. Bouachache [3], [4], [5], [27], [6], [7].

There is a duality between the concepts of *instantaneous frequency* and *group delay*, due to the dual relationship between a stable f signal and its Fourier Transform $\widehat{f}$.

Traditionally, the group delay is defined by means of $\widehat{f}$ (expressed in polar form: $\widehat{f}(\Omega) = \widehat{a_f}(\Omega) \exp\left[j\widehat{\varphi_f}(\Omega)\right]$) as follows:

Definition 4
The **group delay** *of stable signal f is:*

$$t_f(\nu) \stackrel{def}{=} -\,\eta^2 \frac{d}{d\nu} \left[\arg\left(\widehat{f}(\nu)\right)\right] = -\,\eta^2\, \widehat{\varphi_f}'(\nu) \;\; or \;\; t_f(\omega) \stackrel{def}{=} -\,\frac{d}{d\omega} \left[\arg\left(\widehat{f}(\omega)\right)\right] = -\,\widehat{\varphi_f}'(\omega) \, , \;\; \forall\, \nu, \omega \in I\!R \, .$$

The most common interpretation of group delay is the following: if f is the weighting function of a linear dynamic system, then $t_f(\omega)$ is the delay that the system applies to the elementary harmonic signal with pulsation

ω, provided that f includes a unique harmonic signal at each instant. A similar condition was imposed for interpretation of *instantaneous frequency* and the duality of Definitions 2 and 4 becomes obvious.

The second definition below is also usually accepted within time-frequency analysis framework:

Definition 5
The **group delay** *associated to a signal f and to its time-frequency distribution $\mathcal{P}$ is the first order conditional moment of time instant, evaluated for any pulsation $\omega \in I\!\!R$:*

$$\langle\, t \,\rangle_\omega \stackrel{def}{=} \frac{1}{\mathcal{P}_2(\omega)} \int t\,\mathcal{P}(t,\omega)\,dt\ ,$$

where:

$$\mathcal{P}_2(\omega) \stackrel{def}{=} \int \mathcal{P}(t,\omega)\,dt\ .$$

The Definitions 3 and 5 are dual as well. The quantity $\mathcal{P}_2$ is aiming to realise a normalisation of distribution $\mathcal{P}$. In general, it exists at least a pulsation $\omega \in I\!\!R$ such that $\langle\, t \,\rangle_\omega \neq t_f(\omega)$, but the identity of these two group delays could be also achieved.

The concept who allows us to give a rigorous definition of non-stationary signals is the *local auto-correlation*. Since the mathematical expression of this concept is technical and the interpretation of non-stationary behaviour is less intuitive than the previous one (variable spectrum in time), this part is skipped here.

A very interesting topic within time-frequency analysis framework is about Uncertainty Principle. Although defined by Heisenberg for quantum physics purposes, this principle transgressed the boundaries and became popular in many sciences and even in Philosophy. In SP domain, it is expressed by the *Gabor-Heisenberg Uncertainty Inequality/Principle*, which reveals a fundamental relationship between standard deviations of a stable signal and of its Fourier Transform. More specifically, if f is a stable signal and $\widehat{f}$ is its Fourier Transform (as usual), the standard deviations are defined as follows:

$$(\Delta t)^2 \stackrel{def}{=} \int \left(t-\widetilde{t}\right)^2 |f(t)|^2\,dt\ , \qquad \text{where} \quad \widetilde{t} \stackrel{def}{=} \int t\,|f(t)|^2\,dt\ ;$$

$$(\Delta \omega)^2 \stackrel{def}{=} \int \left(\omega-\widetilde{\omega}\right)^2 |F(\omega)|^2\,d\omega\ , \quad \text{unde} \quad \widetilde{\omega} \stackrel{def}{=} \int \omega\,|F(\omega)|^2\,d\omega\ .$$

Then the Gabor-Heisenberg uncertainty inequality is:

$$(\Delta t)\,(\Delta\omega) \geq \frac{1}{2}\ .$$

The interpretation of this relation is relied on the concept of *representation resolution*. This is defined as the inverse of standard deviation and has two facets: in time (for $(\Delta t)^2$) and in frequency (for $(\Delta\omega)^2$). The above inequality could be expressed equivalently as:

$$\frac{1}{\Delta t}\,\frac{1}{\Delta\omega} \leq 2\ .$$

This represents the *resolutions trade-off inequality*. It shows that the signal and its Fourier Transform cannot be simultaneously represented with an infinite accuracy (measured by their representation resolutions in time and, respectively, in frequency). Thus, an infinite accuracy for the signal or its Fourier Transform involves automatically a null resolution for the Fourier Transform or, respectively, the signal. But, unlike the original Heisenberg's Uncertainty Principle, here, if both resolutions are too small, they could be independently increased while the inequality is verified. This is the case of Wavelet Transform that allows (under certain requirements) to refine the representation in frequency without decreasing the accuracy of time representation. The property is due exclusively to the inequality sign replacing the equality one in traditional definition. Unfortunately, the inequality sign cannot be used for any time-frequency transformation. For example, within Short Time Fourier Transform, only the equality is available.

In 1980, M.I. Scol'nik provided in [41] an interesting interpretation easily adopted by SP community: any waveform with compact and sharp support has a wide-band spectrum and vice-versa. Practically, the supports of a signal and of its spectrum cannot be chosen simultaneously indefinitely sharp. Actually, compact support for an entity involves infinite support for its dual.

A similar and interesting trade-off is described in thesis between smoothness of a state surface (depicted by a time-frequency distribution) and its localisation accuracy in phase space: the more accurate is the localisation (i.e. the more focused on a certain point), the less smooth/accurate is the state surface describing the energy distribution of signal.

2.3 Classes of time-frequency/scale transformations. Examples.

As mentioned before, new time-frequency/scale transformations are defined today in a very dynamic manner, so that any classification could be only partial. But a set of stable classes including remarkable transformations can be emphasised. In this section, such classes are described, following Leon Cohen's point of view (one of the most advised in the area). (Two main categories of transformations are focused on.) Beside this, several analytical properties of mentioned transformations were proven and some insights concerning the construction of new transformations are specified.

An abstract of classes described inside the section is listed next. Two large categories of remarkable transformations can be specified:

- *Linear time-frequency/scale transformations*

  Denote by $\mathcal{T}_f$ any transformation of this category. Then, it has to verify the following *Linear Superposition Principle*:

  – *For any (non-stationary) signals f and g and any complex numbers α, β, the following implication holds:*

  $$h \equiv \alpha f + \beta g \Longrightarrow \mathcal{T}_h \equiv \alpha \mathcal{T}_f + \beta \mathcal{T}_g \,.$$

  The integral and affine classes belong to this category. The Short Time Fourier Transform and the Wavelet Transform are representative here.

- *Quadratic/Bilinear time-frequency/scale transformations*

  Within this category, transformations, denoted by $\mathcal{P}_f$, are meant to reveal the signal energy distribution over the phase space. They have to verify the following *Quadratic Superposition Principle*:

  – *For any (non-stationary) signals f and g and any complex numbers α, β, the following implication holds:*

  $$h \equiv \alpha f + \beta g \Longrightarrow \mathcal{P}_h \equiv |\alpha|^2 \mathcal{P}_f + |\beta|^2 \mathcal{P}_g + \alpha \overline{\beta} \mathcal{P}_{f,g} + \overline{\alpha} \beta \mathcal{P}_{g,f} \,,$$

  where:

  ∗ $\mathcal{P}_f \stackrel{def}{=} \mathcal{P}_{f,f}$ *is referred to as* **(auto-)transformation** *of f;*

  ∗ $\mathcal{P}_{f,g}$ *stands for an* **inter-transformation** *or* **cross-transformation** *of f and g.*

  Moreover, the cross-transformation must be bilinear, i.e., for any (non-stationary) signals f, g, h and complex numbers α, β, the following identities have to be verified:

  $$\varphi \equiv \alpha f + \beta g \Longrightarrow \begin{cases} \mathcal{P}_{\varphi,h} & \equiv & \alpha \mathcal{P}_{f,h} + \beta \mathcal{P}_{g,h} \\ \mathcal{P}_{h,\varphi} & \equiv & \overline{\alpha} \mathcal{P}_{h,f} + \overline{\beta} \mathcal{P}_{h,g} \end{cases} .$$

  The Quadratic Superposition Principle generates the parasite interference term:

  $$\alpha \overline{\beta} \mathcal{P}_{f,g} + \overline{\alpha} \beta \mathcal{P}_{g,f} \,,$$

  which could be responsible for undesirable negative values of state surface.

  Obviously, the Wigner-Ville transformation belongs to this category. But it is not the only one representative. Actually, at least 8 classes of transformations can be constructed within this category, as follows:

  A. *Energetic/Positive defined distributions*

  B. *Correlative distributions*

  C. *Distributions sensitive to time and or frequency shifting*

  D. *Distributions sensitive to correlative shifting*

  E. *Distributions sensitive to affine transformations*

  F. *Affine distributions sensitive to time and or frequency shifting*

  G. *Smoothed distributions*

  H. *Affine smoothed distributions*

  Beside Wigner-Ville Transformation, another remarkable representatives of this category are: the Ambiguity Function, the Spectrogram (quadratic absolute value of Short Time Fourier Transform) or the Scalogram (quadratic absolute value of Wavelet Transform).

2.4 General desirable properties

This section is concerned with the most important properties that a time-frequency transformation should verify. For example, it is important that a transformation: be sensitive to both translations in time and frequency and to affine operator applied on its arguments; verify a set of marginal equations; has a smooth behaviour.

All desirable properties are rigorously defined and explained in the section. Unfortunately, there is no transformation verifying all desirable properties, due mostly to Uncertainty Principle. But, some of time-frequency transformations are the best ones. Thus, Wigner-Ville distribution and Ambiguity function verify the most part of desirable properties and constitute prototype instruments for time-frequency analysis.

2.5 Continuous time Wavelet Transform and its discretization

The second chapter is completed by a short presentation of Wavelet Transform – one of the most recent and amazing tool within time-frequency analysis. (Many details on this subject could be found in [46] – a report written in French.) This presentation includes only the cases of continuous time and of discretized transform (i.e. still acting on continuous-time signals). The discrete time case (discrete transform, discrete signals) is described at length, by a personal theory, within next chapter.

After A. Grossmann and J. Morlet introduced the concepts of *variable scale representation* and *ondelette* (the French correspondent of *wavelet*) in [22], Y. Meyer realised the importance of this new discovery and started to construct a rigorous mathematical framework of wavelets [34], [35]. He was followed by another mathematicians in this attempt, such as P.G. Lemarié [31] and G. Battle [2]. The main step towards applications of the new born Wavelet Transform was made by S. Mallat, who used a wavelet based algorithm to process and compress images [32]. But, without any doubt, I. Daubechies had a large contribution (maybe the largest one) for the development of Wavelets Theory. She provided the first systematic study on wavelets in one of the most cited papers in publications, [16], concerning the construction and the properties of compactly supported wavelets. Many other papers such as [18], [17], [19], [9] (and another ones mentioned in thesis references list) enforced her credit as one of the Wavelets Theory founders. This very succinct historical review cannot be closed without mentioning the contribution of R. Coifman and M.V. Wickerhauser, who offered an important generalisation of Mallat's approach, by proving that wavelets are able to realise arbitrarily non-uniform splitting of signal spectrum [14], [15]. The contributors list above is by far not complete and a larger description is provided inside thesis.

One of the most important approach using wavelets was conceived by S. Mallat and I. Daubechies. This approach opened a large door to applications. The main concept founding their approach is the *multiresolution structure*. The Hilbert space $L^2(I\!\!R)$ can be endowed with a multiresolution structure, by using countable bases of wavelets. This structure consists of a family of orthogonal subspaces where the original signal is represented by partial signals at different constant scales and for different time shifting periods. Each subspace is spanned by an orthogonal sub-basis consisting of temporal translations of a unique wavelet-mother represented at a certain scale. Inside a subspace, the time resolution of representation is constant, but this resolution varies among different subspaces, according to Uncertainty Principle. Thus, when wavelets are contracted, they are focusing on sharp parts of the original signal in time, but they cover a broad high frequency zone. Microscopic ruptures of signal can be detected, but the effect they produce within local spectrum of signal is poorly characterised. Conversely, dilated wavelets are able to focus on sharp zones of low frequency spectrum, but time resolution is poor. A partial signal is a linear combination of orthogonal wavelets within a subspace and looks like a blurred image of the original one. The coefficients multiplying wavelets within the linear combination are referred to as *wavelet coefficients*. They are computed by projecting the original signal on each wavelet of the orthogonal basis. The clarity degree of each partial signal is controlled by the scale of representation and the selected wavelet-mother (that generates the wavelets basis). By aggregating together all partial signals, the original signal can be perfectly recovered (in theory). Moreover, due to wavelets properties, many wavelet coefficients have insignificant values and could be ignored. The "compressed" signal constructed by using only significant wavelet coefficients looks often very close to the original one and even no distinction can be made between the shapes of these two signals if the wavelet-mother was appropriately chosen.

This theory generated one important algorithm devoted to time-frequency/scale analysis and synthesis of signals, by using wavelets: Mallat's Algorithm. This algorithm involves an implementation by using QMF (*Quadratic Mirror Filter*) banks. Such a filter bank has two main branches: analysis and synthesis, as illustrated by Figures 1 and 2.

The analysis branch provides the wavelet coefficients, denoted by D^0, C^1, C^2, $\ldots$, C^L, D^L, by repeatedly applying 2 operations: filtering ($\star\widetilde{h}$, $\star\widetilde{g}$) and decimation with factor 2 ($\downarrow 2$). A maximum depth $L \geq 1$ is imposed and it points to the coarser subspace where the signal can be analysed. The original signal ($x_d = f$ discretized) is identified with the best time resolution and poorer frequency resolution partial signal D^0, represented at scale 0. Next, the wavelet mother is dilated with scale factor 2^1 and the signal space is split in two subspaces. A partial signal lies inside of each subspace, their wavelet coefficients being D^1 (for the remaining low frequency signal) and C^1 (for the high frequency signal - a blurred version of the original one, with 2 times better/poorer frequency/time resolution than the original). Next, the remaining signal, represented by means of wavelet coefficients D^1, suffers the same treatment by dilating the wavelet mother with scale factor 2^2. One generates another pair of wavelet coefficient sets, D^2 and C^2. The partial signal corresponding to C^2 has now 4 times better/poorer frequency/time resolution than the original one. The analysis continues until the coarser pair of subspaces is reached.

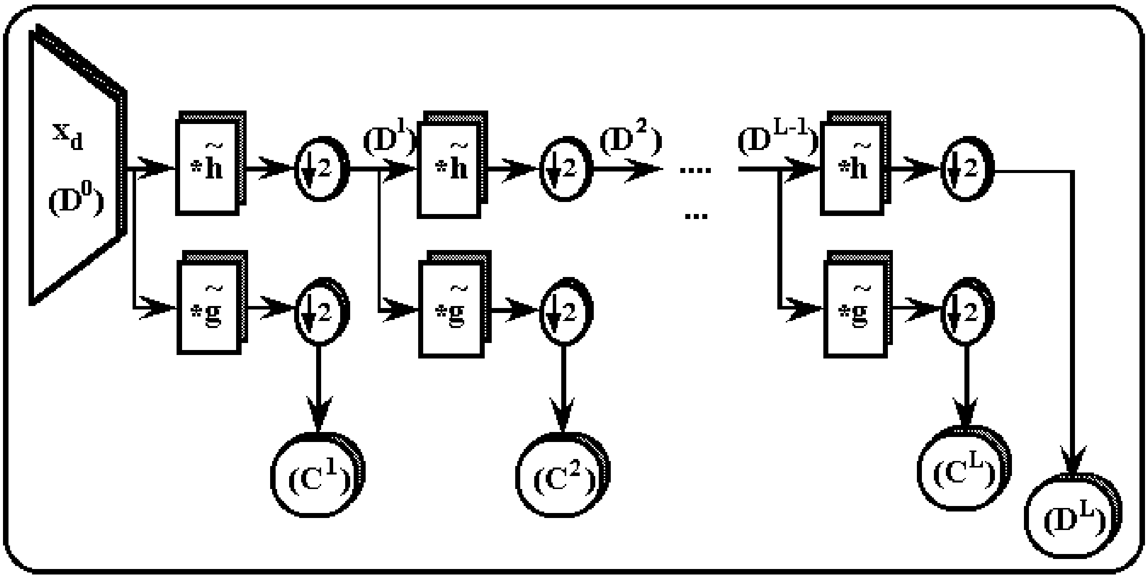

Figure 1: *Analysis branch of Mallat Algorithm.*

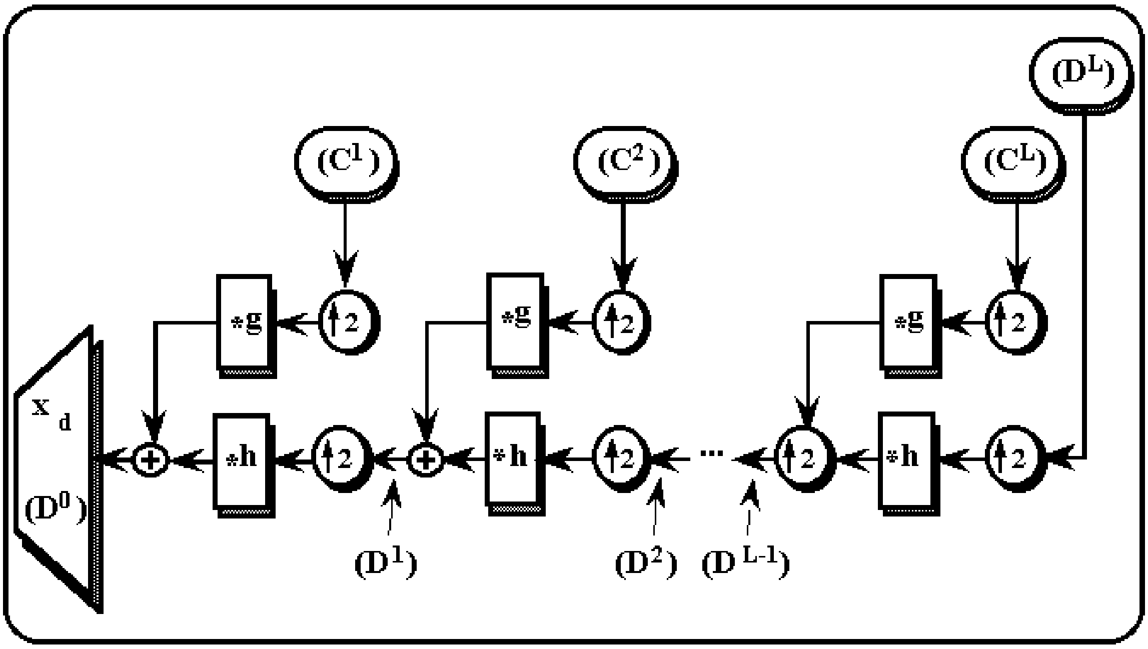

Figure 2: *Synthesis branch of Mallat Algorithm.*

The synthesis branch of QMF bank just aggregates together the partial signals above, in order to recover the original signal, by repeatedly applying one interpolation with factor 2 ($\uparrow 2$) and one filtering with mirror filters. If some of wavelet coefficients are removed, then the reconstructed signal could differ from the original one. But if only the insignificant wavelet coefficients are removed, this signal could approximate very well the original one. The synthesis is necessary especially in signal compression applications.

This algorithm reveals an interesting frequency effect produced by wavelets. For example, the QMF bank of Figure 3 (where $L = 2$) provides the spectrum splitting scheme of Figure 4. Thus, each partial signal s^0, s^1 and s^2 is responsible for a segment of original signal spectrum. The puzzle has 3 pieces (see Figure 4) and each one is half length of its right neighbour. At high frequencies, an entire half-band is covered by s^0 spectrum. Its time resolution is 2 times smaller than the original, but its frequency is 2 times larger, since it is focusing only on high frequency half-band of original spectrum. The low frequency half-band is covered by the 2 equally spaced spectra of remaining partial signals. Now, s^1 is responsible for the middle frequency sub-band and s^2 for the low frequency one. Both signals have a time resolution 4 times smaller than the original. However, their frequency resolution is 4 times improved, since their spectra focus on quarter bands of the original spectrum. One says that frequency are grouped in *octaves* (like for musical instruments) and this idea was originated, in fact, the wavelets approach in [22].

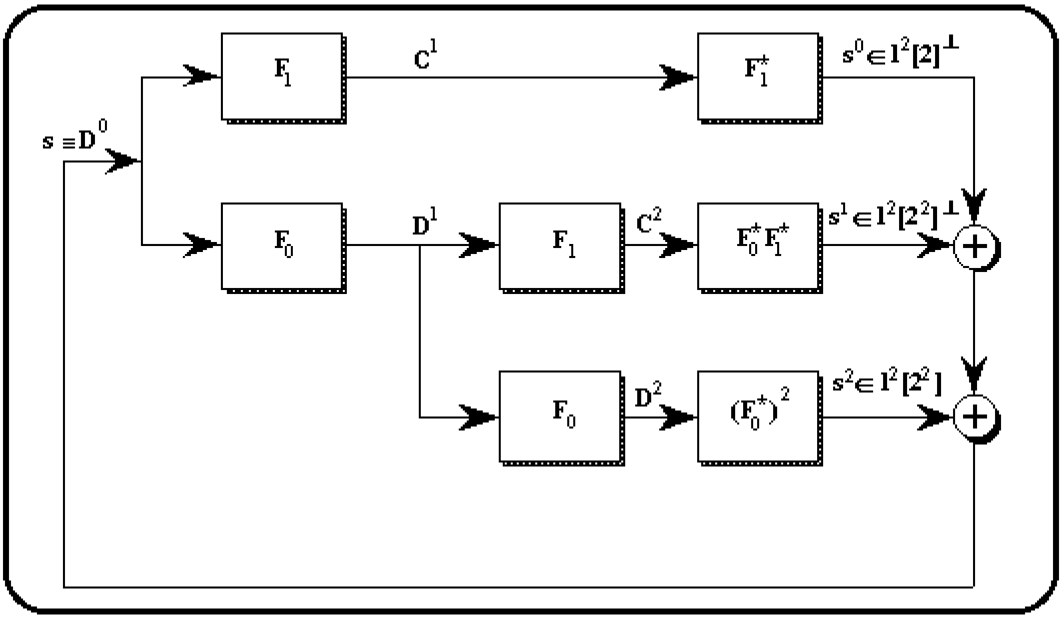

Figure 3: *A two step analysis-synthesis loop of Mallat Algorithm.*

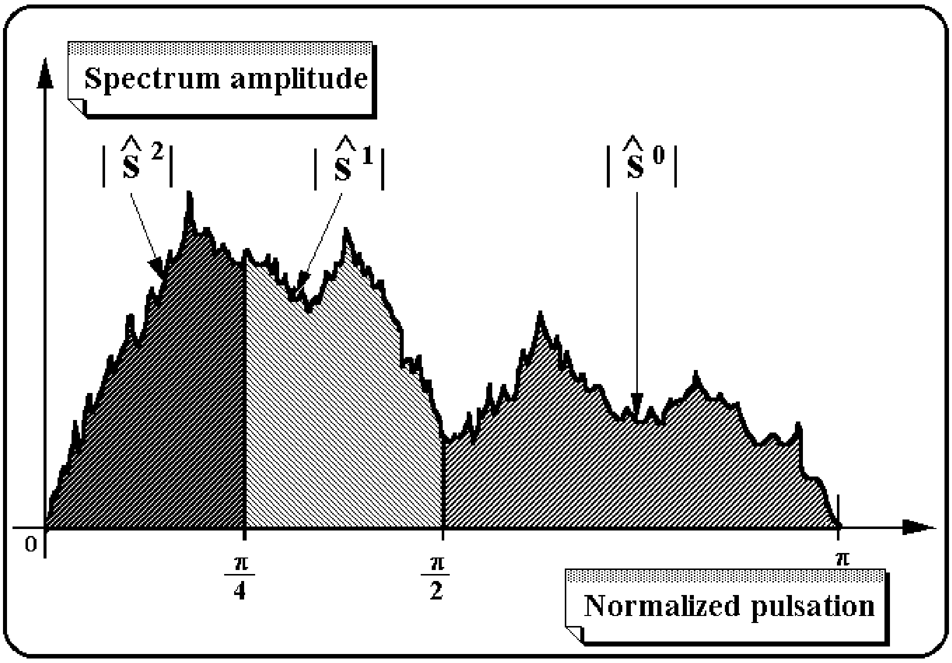

Figure 4: *Frequency effect of two step analysis-synthesis loop above.*

This section also includes a description of wavelets properties (as described in most of related articles), gives various details concerning Wavelet Transform discretization and presents our personal approach concerning time-frequency localisation of wavelet coefficients over the phase space. Thus, unlike the Fourier coefficients obtained by using Short Time Fourier Transform (which are localised within rectangular identical cells), the wavelet coefficients lie over a hyperbolic network that covers the state space, due to scaling operation. For practical purposes, this network can be replaced by a rectangular one, where rectangles have different dimensions, depending on representation scale, according to Uncertainty Principle.

3 A Theory of Discrete Time Orthogonal Wavelet Transform

This chapter constitutes the theoretical kernel of thesis. A personal theory of discrete time signal processing with orthogonal discrete wavelets is developed here. A detailed version of this approach was published in [54] and [55], in French, but the author could provide, if necessary, an English version as well. An English short version (3 pages) was also published in [53]. Many details of this theory were described in [57] and [56] (both in French).

Therefore, the description of this chapter is skipped here.

4 High Quality Speech Coding by Using the Discrete Wavelet Transform

The application developed during my stages at Institut de la Communication Parlée of Grenoble, in France was concerned with the problem of high quality and low rate speech coding. The solution was based on discrete time wavelets theory above and included a human psycho-acoustic model.

Beside this chapter thesis, all details of application were published in [47], [57] and [56] (all in French). The obtained results and short presentations of speech coding algorithm appeared in [29], [28] (both in French) and [30], [58] (both in English).

Therefore, the description of this chapter is skipped here.

5 Concluding remarks and further research directions

A thesis overview and a brief research prognosis for the future (influenced by some open problems) is realised in the final chapter.

Thus, within this thesis, a quasi-exhaustive presentation of time-frequency signal analysis field and a focus on a sharp zone (the wavelets) were provided at the same time. The area is rapidly evolving and new results are expected very soon. Without doubts, today, time-frequency analysis represents the most revolutionary part of SP domain. However, it seems that, in spite of all possible new approaches, the fundamental concepts and results (especially theoretical) have been already presented. Applications could provide new approaches and, actually, we assist at an explosion of publications in this field. They are due not only to the theoretical results obtained so far, but also to important technological advances, especially in computer science and fabrication, that open the way for efficient implementation of many complex algorithms specific to time-frequency analysis.

Wavelets are complex and very useful instruments in processing of many difficult and rich signals such as voice or image, but, as expected, they are not universal tools. For some signals (including seismic ones), wavelets analysis is not yet satisfactory. They have various limits (as proven, for example, by I. Daubechies) that requires many improvements and a careful selection, depending on signal intimate structure (fractal if the signal has microscopic ruptures, smooth if the signal is so). However, they constitute an important discovery and, moreover, they are hiding an interesting general idea: when an entity has to be studied, the appropriate tools aiming to alleviate this task should have a similar nature to the entity.

⋆

Beside the publications mentioned in previous chapters, many parts of this thesis were published before thesis completion in: [45], [44], [52] (all three in Romanian), [46], [51] (both in French), [48], [49] (both in English).

References

[1] **Ackroyd M.H.** *Time–Dependent Spectra: The Unified Approach.* In Proceedings of the NATO Advanced Study Institute Signal Processing, pages 1–9. Academic Press London & New York, 1973.

[2] **Battle G.** *A Block Spin Construction of Ondelettes: I. Lemarié Functions.* Communications in Mathematical Physics, 110:601–615, 1987.

[3] **Bouachache (Boashash) B.** *Instantaneous Frequency of Signals: Concepts, Estimation Techniques and Applications.* In Proceedings of SPIE -- The International Society for Optical Engineering, Vol. 1152, pages 382–400, August 1989.

[4] **Bouachache (Boashash) B.** *Algorithms for Instantaneous Frequency Estimation: a Comparative Study.* In Proceedings of SPIE -- The International Society for Optical Engineering, Vol. 1348, pages 126–148, July 1990.

[5] **Bouachache (Boashash) B.** *Time–Varying Higer Order Spectra* . In Proceedings of SPIE -- The International Society for Optical Engineering, Volume No. 1566 , pages 98–108, July 1991.

[6] **Bouachache (Boashash) B.** *Estimating and Interpreting the Instantaneous Frequency of a Signal: I. Fundamentals.* Proceedings of the I.E.E.E., 80(4):520–538, April 1992.

[7] **Bouachache (Boashash) B.** *Estimating and Interpreting the Instantaneous Frequency of a Signal: II. Algorithms and Applications.* Proceedings of the I.E.E.E., 80(4):540–568, April 1992.

[8] **Carson J.R, Fry T.C.** *Variable Frequency Electric Circuit Theory with Application to the Theory of Frequency Modulation.* Bell Systems and Technology Journal, 16:513–540, 1937.

[9] **Cohen A., Daubechies I., Feauveau J.C.** *Biorthogonal Bases of Compactly Supported Wavelets.* Preprint, 1991.

[10] **Cohen L.** *Time–Frequency Distributions – A Review.* Proceedings of the I.E.E.E., 77(7):941–981, July 1989.

[11] **Cohen L.** *The Scale Representation.* I.E.E.E. Transactions on Signal Processing (Special Issue on Wavelets and Signal Processing), 41(12):3275–3292, December 1993.

[12] **Cohen L.** Time-Frequency Analysis. Prentice Hall, New Jersey, USA, 1995.

[13] **Cohen L., Posch Th.E.** *Positive Time–Frequency Distribution Functions.* I.E.E.E. Transactions on A.S.S.P., ASSP–33(1):31–37, February 1985.

[14] **Coifman R., Meyer Y., Quake S., Wickerhauser M.V.** *Signal Processing and Compression with Wavelet Packets.* In Proceedings of the $3^{\underline{rd}}$ International Conference on Wavelets and Applications, pages 77–93, Toulouse, FRANCE, June 8–13, 1992.

[15] **Coifman R., Wickerhauser M.V.** *Entropy–Based Algorithm for Best Basis Selection.* I.E.E.E. Transactions on Information Theory, 38(2):713–718, March 1992.

[16] **Daubechies I.** *Orthonormal Bases of Compactly Supported Wavelets – I. First Approach.* Communications on Pure and Applied Mathematics, XLI:909–996, 1988.

[17] **Daubechies I.** *The Wavelet Transform, Time–Frequency Localisation and Signal Analysis.* I.E.E.E. Transactions on Information Theory, 36(9):961–1005, September 1990.

[18] **Daubechies I., Lagarias J.** *Two Scale Difference Equations: I. Existence and Global Regularity of Solutions.* Preprint, 1989.

[19] **Daubechies I., Lagarias J.** *Two Scale Difference Equations: II. Local Regularity, Infinite Products of Matrices and Fractals.* Preprint, 1990.

[20] **Farkash S., Raz S.** *Linear Systems in Gabor Time–Frequency Space.* I.E.E.E. Transactions on Signal Processing, 42(3):611–617, March 1994.

[21] **Gabor D.** *Theory of Communication.* Journal of Instute of Electrical Engineering (London), 93(3 (III)):429–457, 1946.

[22] **Groupillaud P., Grossmann A., Morlet J.** *Cycle–Octave and Related Transforms in Seismic Signal Analysis.* Geoexploration, 23:85–102, 1984–1985.

[23] **Harmuth H.F.** *Applications of Walsh Functions in Communications : State of the Art.* In Proceedings of the NATO Advanced Study Institute Signal Processing, pages 43–61. Academic Press London & New York, 1973.

[24] **Heil C., Walnut D.** *Gabor and Wavelet Expansions.* In Proceedings of the NATO Advanced Study Institute Fourier Analysis and its Applications, Il Ciocco, ITALY, July 1989.

[25] **Janssen A.J.E.M.** *Gabor Representation of Generalised Functions.* Journal of Mathematical Analysis and Applications, 83(2):377–394, October 1981.

[26] **Janssen A.J.E.M.** *Gabor Representation and Wigner Distribution of Signals.* In Proceedings of the I.E.E.E. International Conference on A.S.S.P., April 1983.

[27] **Jones G., Bouachache (Boashash) B.** *Instantaneous Quantities and Uncertainty Concepts for Signal Dependent Time–Frequency Distributions.* In Proceedings of SPIE -- The International Society for Optical Engineering, Vol. 1566, pages 167–178, July 1991.

[28] **Kastantin R., Ştefănoiu D., Feng G.** *Codage de la parole basé sur la Transformation en Ondelettes Discrètes. Apport des modèles psycho-acoustiques.* In Quinzième Colloque sur le Traitement du Signal et des Images, GRETSI'95, pages 337–340, Juan-les-Pins, FRANCE, 18–21 September 1995.

[29] **Kastantin R., Ştefănoiu D., Feng G., Martin N., Mrayati M.** *Codage à bas débit de la parole de haute qualité basé sur une transformation en ondelettes discrètes.* In Colloque "Temps-fréquence, ondelettes & multirésolution: théorie, modèles & applications", pages (21–)1–4, Lyon, FRANCE, March 1994.

[30] **Kastantin R., Ştefănoiu D., Feng G., Martin N., Mrayati M.** *Optimal wavelets for high quality speech coding.* In Proceedings of the Symposium EUSIPCO '94, pages 399–403, Appleton Tower, University of Edinburgh, SCOTLAND, UNITED KINGDOM, September 13–16, 1994.

[31] **Lemarié P.G.** *Ondelettes à localisation exponentielle.* Journal des Mathématiques Pures et Appliqués, 67:227–236, 1988.

[32] **Mallat S.** *A Theory for Multiresolution Signal Decomposition: the Wavelet Representation.* I.E.E.E. Transactions on Pattern Analysis and Machine Intelligence, 11(7):674–693, July 1989.

[33] **Marcus J.** Échantillonnage et Quantification, pages 11–46. Gauthier--Villars, Paris, FRANCE, 1965.

[34] **Meyer Y.** *Les Ondelettes.* Preprint, 1990.

[35] **Meyer Y.** *Wavelets – Their Past and Their Future.* In Proceedings of the $3^{\underline{rd}}$ International Conference on Wavelets and Applications, pages 9–18, Toulouse, FRANCE, June 8–13, 1992.

[36] **Morlet J., Grossmann A.** *Wavelets – Ten Years Ago.* In Proceedings of the $3^{\underline{rd}}$ International Conference on Wavelets and Applications , pages 3–7, Toulouse, FRANCE, June 8–13, 1992.

[37] **Oppenheim A.V., Schafer R.** Digital Signal Processing. Prentice Hall, 1985.

[38] **Pichler F.** *Walsh Functions: Introduction to the Theory.* In Proceedings of the NATO Advanced Study Institute Signal Processing, pages 23–41. Academic Press London & New York, 1973.

[39] **Porat B., Friedlander B.** *Detection of Transient Signals by the Gabor Representation.* I.E.E.E. Transactions on A.S.S.P., ASSP–37(2):169–179, February 1989.

[40] **Qian S., Chen D.** *Optimal Biorthogonal Analysis Window Function for Discrete Gabor Transform.* I.E.E.E. Transactions on Signal Processing, 42(3):694–697, March 1994.

[41] **Scol'nik M.I.** Introduction to Radar Systems. McGraww--Hill, New York, 1980.

[42] **Shannon C.E.** *The Mathematical Theory of Communication.* The Bell Systems Technical Journal, (27):379–423; 623–656, 1948.

[43] **Söderström T., Stoica P.** System Identification. Prentice Hall, London, 1991.

[44] **Ştefănoiu D.** *The Dilation Equations in Signal Processing. II. Wavelets. Mallat's Algorithm.* In SNITES IV, pages 37–44, Craiova, ROMANIA, May 10–12, 1991.

[45] **Ştefănoiu D.** *The Dilation Equations in Signal Processing. I. Scale functions and Fourier Analysis.* In SNITES IV, pages 30–36, Craiova, ROMANIA, May 10–12, 1991.

[46] **Ştefănoiu D.** *Sur la Construction et les Propriétés des Ondelettes à Support Compact.* Technical report, Institut de la Communication Parlée, I.N.P. Grenoble, FRANCE, July 1992.

[47] **Ştefănoiu D.** *Étude comparative des méthodes temps–fréquence et temps–échelle sur des signaux de parole réels.* Technical Report Research Contract CIPA Number 351OCT921072, Institut de la Communication Parlée & Université Stendhal, Institut National Polytechnique de Grenoble, FRANCE, December 1993.

[48] **Ştefănoiu D.** *Introduction to Signal Processing with Wavelets.* In International Conference CSCS-9, pages 143–155, Bucharest, ROMANIA, May 25–28, 1993.

[49] **Ştefănoiu D.** *The Classical Solution of Signal Processing Problem and the Wavelets Solution.* Journal of Studies in Informatics and Control (publisher: Institute of Research in Informatics, Bucharest, ROMANIA), 3(1):97–110, 1993.

[50] **Ştefănoiu D.** *The Transformations Theory in Signal Processing.* Technical Report UPB-FAC-150793, "Politehnica" University of Bucharest, ROMANIA, July 1993.

[51] **Ştefănoiu D.** Journée d'évaluation du Projet TEMPUS PEC 2736-91, chapter *Méthodes temps–échelle appliquées au codage de la parole.* Universitatea "Politehnica" Bucureşti, ROMÂNIA, June 2, 1994.

[52] **Ştefănoiu D.** *The Algorithm of Wavelets Packets Applied to Speech Coding.* In SNITES VII, pages 111–114, Craiova, ROMANIA, May 20–21, 1994.

[53] **Ştefănoiu D.** *Endowing the l^2 space with flexible multiresolution structures.* In IASTED International Conference SIPA'96, Annecy, FRANCE, June 12–14, 1996.

[54] **Ştefănoiu D.** *Une théorie de la structure multirésolutive discrète: I. Bases canoniques d'ondelettes discrètes orthogonales.* Révue Roumaine des Sciences Techniques, Série Électrotechnique et Énergétique, 41(4):521–542, 1996.

[55] **Ştefănoiu D.** *Une théorie de la structure multirésolutive discrète: II. Paquets d'ondelettes discrètes orthogonales.* Révue Roumaine des Sciences Techniques, Série Électrotechnique et Énergétique, 42(1):105–126, 1997.

[56] **Ştefănoiu D.** Avancés dans le Traitement Numérique des Signaux – Approches temps–fréquence. Printing House of "Politehnica" University, Bucharest, ROMANIA, June 1997.

[57] **Ştefănoiu D.** Traitement du signal avec des ondelettes. Application au codage de la parole. Cursus universitaire, INPG-ENSERG-ICP, Grenoble, March 1995.

[58] **Ştefănoiu D., Kastantin R., Feng G.** *High Quality Speech Coding Based on the Generalized Discrete-Time Wavelet Transform and Human Auditory System Properties.* In Fourth European Conference on Speech Communication and Technology, EUROSPEECH'95, pages 661–664, Madrid, SPAIN, 18–21 September 1995.

[59] **Van der Pol B.** *The Fundamental Principles of Frequency Modulation.* Journal of Instute of Electrical Engineering (London), 93:153–158, 1946.

[60] **Ville J. de**. *Théorie et Applications de la Notion de Signal Analytique.* Cables et Transmission, 2A:61–74, 1948.

[61] **Wiener N.** Forecasting, smoothing and filtering the data strings. Prentice Hall, London, 1949.

[62] **Wigner E.P.** *On the Quantum Correction for Termodynamic Equilibrum.* Physics Review, 40:749–759, 1932.