Risk Drivers Revealed: Quantile Regression and Insolvency

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Abstract

Most stochastic corporate models examine only resultant output (specifically the surplus), and thereby divorce the model results from key input model assumptions, such as the term structure of interest rates. A simple relationship of the economic rate scenarios to the surplus could be a multivariate regression. However, one drawback of using ordinary least squares regression (OLSR) is that the regression is on the mean of the surplus results. With insolvency we are concerned with the tail of the distribution of surplus results (i.e., relatively infrequent extreme values) or where the surplus is negative, which may be far from the mean. The second drawback to multivariate regression is that it is very sensitive to outliers. The values of the associated coefficients are severely distorted when an extreme outlier is used within the regression. Koenker and Basset [13] resolve both of these difficulties when they introduce Quantile Regression (QR). As the name implies, QR allows one to conduct a regression on specific conditional quantiles. These specific conditional quantiles can be chosen so that they are located near the region of interest. Also, QR is a robust regression method that reduces or removes the influence of outliers on the regression coefficients. We will use QR to determine the location in time and the impact of specific risk drivers.

Key Words:

Insolvency, Quantile Regression, Risk Drivers, Surplus.

1 Introduction

In basic ruin analysis, Bowers et al. [5] set up a stochastic process with the following assumptions:

1. Claims count distribution,
2. Claims amount distribution,
3. No interest or asset performance,
4. Constant premiums,
5. Constant expense loads.

Even with these simple assumptions, there is no closed formula for the probability of ruin (that is, when surplus drops below zero), except for one special case. The one exception occurs when the stochastic process is compound Poisson with an exponential claims amount distribution. See De Vylder [9] for a further discussion of traditional ruin/risk theory approaches. In the life insurance industry, regulation and/or professional standards require us to conduct computer simulations on different lines of business to determine when the business performs poorly. We model our business as accurately as possible, allowing for interest and asset performance, changing premium and expense loads. We may or may not make assumptions on the claims count or amount distributions. In addition, we often make many other assumptions such as the term structure of interest rates, future interest rates, projected stock market returns, asset default probabilities, policyholder psychology, and the relationships of our decrements to the level of interest rates or the stock market. Computer simulations reveal the behavior of the business relative to these assumptions. We do not know the actual statistical distribution of our business model results. We assume that the computer simulation results are representative (within some degree of confidence) in certain areas of interest, such as the extreme tail. First, we need to determine if our models are valid (again within some degree of confidence). If valid, then we calculate the probability of "ruin," however defined (e.g. reserves are inadequate or all surplus is consumed) within the accuracy of these computer models, or observe the potential risks associated with that product or line of business.

Computer simulations of complex corporate models become very expensive in processing time as the number of scenarios increase. The need to obtain a timely answer often outweighs the need for information from additional scenarios.

Most computer business models are limited by the knowledge that we have about the basic assumptions used. We must be careful in how we think about and use these models. At a fundamental level, the models are neither correct nor assumed to be accurate. However, the benefit of using the computer to model actual business products and lines is that we can obtain an understanding of the different risks to which that product or line is exposed. Once we have this understanding, we can consider several methods to reduce the effect of any given risk. Such methods include product redesign, reserve strengthening, deferred expense write downs, asset hedging strategies, stopping rules (rules that recommend when to get out of a market), derivative positions and reinsurance.

However, once we have gained the basic understanding of the risks and have designed, say, a hedge strategy, we must remember that these models are not accurate, due to oversimplification of the model, lack of knowledge and insight, lack of confidence in the assumptions, or incorrect computer code. We cannot trust the model output as the "truth," but we can trust the knowledge and insight that we have gained from the process of modeling. If done correctly we know both the strengths and weaknesses of the model. For instance, when constructing a hedge to protect against the risks demonstrated by the model, we must not implement a hedge that optimizes against areas of model weakness. Ultimately, the model does not tell us what to do, but the model does make us more comfortable in finally making business decisions.

It is important to keep a clear perspective when using multiple economic scenarios in computer simulations. We can gain significant insight about the risk exposure from the economy using stochastic simulation. By examining multiple possibilities, we can protect ourselves as best as possible. However, we realize that only one path actually emerges. Therefore, the actuary must continually evaluate the economy and make reasoned business decisions to maintain existing business and to acquire new business.

The risk aversion of company management must also govern these business decisions. Insolvency must be considered and avoided. However, the actuary cannot remove all risk of insolvency, because the cost of the associated hedges would become so prohibitive that the company could not afford to conduct business. Accordingly, the actuary should understand where the product or business line places the company at risk and be able to communicate to upper management the specific risk exposure. For a further discussion of the balancing act between company profit and insolvency risk see Craighead [7].

Valuation actuaries, asset/liability management actuaries, and CFOs of insurance companies confront issues that are vast and complex, including:

1. Calculating the probability and/or impact of bankruptcy either by scenario testing or by determining the company's value at risk.

2. Helping to determine the initial capital allocation for a new line of business.

3. Making sure that reserves are adequate for new and existing lines of business.

4. Understanding how different lines of business are sensitive to the level of interest rates, corporate spreads, volatility of other economic indicators (such as stock indices), and the changes in the levels of these variables.

5. Estimating other risks to which the company is exposed in a timely fashion.

6. Pricing complex policy features to obtain profitability, while maintaining a competitive market position.

7. Aiding in the design and pricing of dynamic hedges to reduce the risk of extreme events.

8. Designing and pricing the securitization of various cashflows to reduce risk based capital requirements.

All of the above issues require timely and accurate valuation of different complex corporate models. When conducting the analysis on models the actuary goes through the following steps:

1. Collect relevant data.

2. Make relevant assumptions.

3. Construct the model.

4. Validate the model for reasonableness.

5. Revise the model.

After a corporate model is constructed an actuary uses the results in several ways. Some of these are:

1. Gain insight on the business modeled.

2. Determine risks to the company.

3. Observe the scenarios that give adverse model results.

4. Increase reserves, or create hedges or make product enhancements to reduce the risk exposure or adverse results.

The internal company standards and the external regulatory requirements require the actuary to determine risk levels from corporate models. It is of paramount importance to understand the impact that different economic drivers, product designs or investment/disinvestment strategies have on the behavior of a corporate model. This includes the determination of when (and how often) model results from scenarios fall in 'bad' locations. This knowledge is very important to understand the potential magnitude of the company's risk exposure. While adverse results occur relatively infrequently in scenario testing, the actuary would like to gain more knowledge of these adverse results without paying the cost of projecting enough scenarios to get the number of "hits" in the region of adverse results needed for statistical validity.

These adverse locations are discovered by first placing a valuation on the company's position, scenario by scenario. These valuations are then sorted and put in an increasing or decreasing order. From these ordered results, the location of the adverse results is found at either the highest or lowest valuations. The area of statistics, which analyzes sorted samples, is order statistics. The study of extreme order statistics is called extreme value statistics or the theory of records. (Note: The theory of records originally grew out of an actuarial problem. See Gumbel [11] for a statement of this problem and its subsequent discussion.)

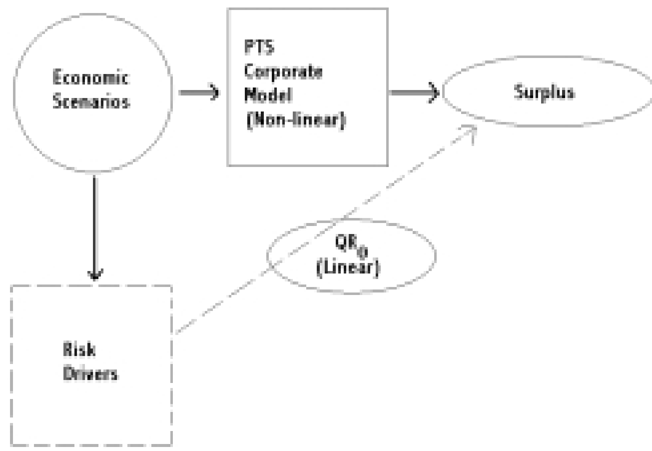

Figure 1: Concept of Risk Drivers

We have a need to approximate the relationship between the input economic scenarios and the surplus output results without additional computer processing. Also, if one is able to target the location of adverse results when developing this relationship, all the better.

In Figure 1 we have a non-linear computer corporate model which takes economic scenarios as input and produces certain model output, which represents the surplus of the corporate model. Next, we define a risk driver to be a function of the economic scenarios through time, which distills the most informative characteristics of the economic scenarios that have an impact on the model output. The QR function is an linear approximation at a specific quantile of an relationship between the risk drivers and the surplus output of the original non-linear computer model. For example, extracting the time series of 90-day Treasury Bill from each scenario is an example of a risk driver. Another example, is the time series of the spread of the 5-year Treasury Note less the 90-day Treasury Bill rate. In Section 2 we will list seven separate risk drivers that we will examine in Section 3.

Koenker and Bassett [13] developed Quantile Regression (QR) which has two major advantages over ordinary linear regression. Specifically, QR is conducted on specific conditional quantiles instead of the conditional mean as in OLSR. Also OLSR is very sensitive to data outliers, whereas QR is much more robust and extreme data values do not influence the regression results as greatly as in OLSR. Subsequently, we will use QR to approximate the relationship between the input economic scenarios and the surplus output.

See Appendix A for a brief discussion of the theory underpinning Quantile Regression. Also, see Bassett and Koenker [2], Koenker [14], Koenker and Bassett [13], Koenker and Portnoy [16], Portnoy [21], Portnoy and Koenker [22] for further discussions on QR. Also, see Koenker and D'Orey [15, 17] and Portnoy and Koenker [22] for computer algorithms to implement QR analysis. Buchinsky [6] also has an excellent overview of the theory and applications of QR.

The remainder of this paper will take the following path:

In Section 2 we discuss the variable annuity business model and both the input scenario and output surplus results that will be the basis of our following analysis.

In Section 3 we create a report to display the various risk drivers and the sensitivity of model results to these drivers.

Finally, in Section 4 we end the paper with a list of strengths and weaknesses of the QR method and a discussion of potential future research.

The QR Methodology Appendix then follows with a discussion of the theoretical underpinning of quantile regression.

2 Business Model-Input Economic Scenarios and Surplus

The economic scenarios used with the corporate model are generated from a process that replicates the Double Mean Reverting Process^{TM} (DMRP^{TM}) of Tenney [27, 28], coupled with a stock process. These economic scenarios are generated in such as fashion as to simulate the real probability measure underlying the U.S. Treasury Bill and Bond Market and that of the S&P 500 stock index, corporate bonds, and money market funds. The projection horizon is 20 years with yield curves varying every quarter. However, the index reflecting the behavior of the stock, corporate bond, and money market varies on a monthly basis. (Note: Below we will roll up this index series to be also on a quarterly basis.) We will examine only the results of 249 separate scenarios.

The business model is of a variable annuity modeled in the actuarial modeling environment PTS 2000^{TM} of SS&C [25]. The model consists of a single block of issues with 10% of the premium allocated to the fixed account and the remaining 90% allocated to the variable account. The equity return for the variable account is based on a blend of equity, bond, and money market returns, and the fixed account is credited the portfolio rate on the underlying assets less a 150 basis point spread. The credited rate is reset annually. The business model does not reflect dynamic policyholder behavior, and transfers between accounts are not modeled. The product has a seven-year surrender charge period. The surplus held in the business model is 250% of Risk Based Capital (RBC) requirements.

The specific output of the model will be the Option-Adjusted Value of Distributable Earnings (OAVDE). The formulation of OAVDE is further discussed in Becker [3, 4] and Babcock and Craighead [1]. A graph of the S-curve of these OAVDE results is Figure 2.

Basic statistics on OAVDE values are: Minimum $-2,531,000, First Quartile $260,000, Median $685,100, Mean $636,600, Third Quartile $1,115,000, Maximum $3,990,000 and the Standard Deviation is $823,268. Also, 40 of the 249 scenarios result with OAVDE values below zero, which states that the probability of ruin is approximately 16%.

The risk drivers from the input scenarios are:

1. The change in the 90-day Treasury bill rates in the input scenarios. These will be denoted Y_i^{90}.

2. The change in the 5-year Treasury bond rates in the input scenarios. These will be denoted Y_i^5.

3. The spread between the 5-year Treasury bond rates and the 90-day Treasury bill rates. These will be denoted S_i.

4. The equity returns from the stock index in the input scenarios. These will be denoted V_i.

5. The change in the 5-year Treasury bond rates from the rolling average 5-year Treasury Bond rates. These will be denoted RA_i^5.

6. The change between the equity returns and an assumed compound return of 8%. These will be denoted D_i.

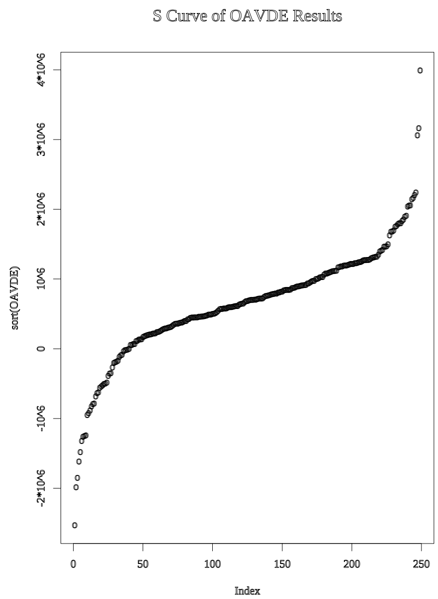

Figure 2: S-Curve of OAVDE

7. The change in the current 12-month rolling equity returns from the 12-month equity returns from the prior year. Denoted RA_i^{EQ}

3 Modeling of QR

In the analysis of corporate models, the need to observe the effect of an economic scenario on the model output (specifically surplus) gives the actuary a critical understanding of the underlying economic risks contained in the model.

Observe the formula

$$R_q = B_{0,q} + B_{1,q}X_1 + B_{2,q}X_2 + \cdots + B_{79,q}X_{79} + B_{80,q}X_{80} + U_q. \quad (1)$$

R_q is the OAVDE response (specifically at the qth quantile), and the X_i are the various risk drivers mentioned in Section 2 at the end of each quarter i. The $B_{i,q}$ are the related coefficients for the specific quantile q and U_q is the error. The assumption that $Quant(U_q) = 0$ leads to the formula of the conditional quantile regression:

$$Quant(R_q) = B_{0,q} + B_{1,q}X_1 + B_{2,q}X_2 + \cdots + B_{79,q}X_{79} + B_{80,q}X_{80}. \quad (2)$$

Koenker and Machado [19] have developed a goodness of fit statistic for QR, which they refer to as the R^1 statistic that corresponds to the OLSR R^2 statistic.[1] The R^1 statistic has similar attributes to that of the R^2 statistic for OLSR. They also discuss another statistic called a Wald estimator, which can be used in a fashion similar to the Student t statistic to indicate whether a specific variable in the design matrix is significant. (Also see Press et al. [24] for an additional discussion of the Wald estimator.) Below, we will use theR^1 and the Wald estimator as a goodness of fit measure and as a test for variable significance. Please refer to [19] for a further discussion of the use and interpretation of these and other statistics.

Our interest for a specific quantile is in its sensitivity to the coefficients through time. By treating the coefficients as a time series, we can observe this effect.

In this section we develop a table, which reveals the relevant information that is needed for the actuary. The actual value of the QR coefficients is not as critical to our understanding as is their relative magnitude when compared to all of the coefficients. We use the absolute magnitude of the coefficients to locate the year of a specific risk exposure as defined in the design matrix of the regression. This approach takes on a qualitative nature in that we do not try to predict the actual percentiles as in the other methods, but we use it to report risk or profit. The pricing actuary can use this qualitative approach to determine design flaws when examining low quantiles and positive upside design features in high quantiles. The valuation actuary can use this type of report to locate various risks and locations of those risks in existing lines of business. This also allows the actuary and the financial engineer to de-

[1] The design matrix of a regression demonstrates its completeness in how well the inner product of the coefficients against the design matrix replicates the responses R_q. In ordinary least squares regression (OLSR) the effectiveness (or goodness of fit) is measured by the R^2 statistic. As one adds relevant variables to the design matrix then R^2 will move closer to 1, thus indicating that the design matrix contains sufficient variables. However, if R^2 is close to zero, this implies that variability within the residuals is not well explained with the OLSR model. This implies that additional variables should added to the design matrix or one should try other types of regression. Note: By the use of the Student t test, one can determine if a variable is significant to the model even when R^2 is small. However, low values of R^2 still point to model ineffectiveness. However, an OLSR model can still be ineffective with high R^2 due to other problems with the residuals. For instance, if the residuals are serially correlated or if the variance of the residuals is not constant then other problems ensue with the model effectiveness. See Venables and Ripley [29] for a further discussion and for other references relative to the use of OLSR.

Time	Coeff.	Scaled Coeff.	Wald Estimator	Significant Scaled Coeff.
0	381170.0000000	0.000000	7892.2998	0.000000
1	-307.1900024	-0.031647	6.3950	-0.031647
2	157.4900055	0.016225	0.9947	0.000000
3	501.1799927	0.051631	27.7060	0.051631
4	276.2399902	0.028458	18.3690	0.028458
5	-493.2399902	-0.050813	25.2400	-0.050813
6	111.3700027	0.011473	1.2566	0.000000
7	588.8699951	0.060665	30.8740	0.060665
8	-84.7119980	-0.008727	0.6461	0.000000
9	33.7570000	0.003478	0.1064	0.000000
10	310.9299927	0.032032	8.2458	0.032032
11	-305.2000122	-0.031441	20.6110	-0.031441
12	519.7299805	0.053542	8.5731	0.053542
13	278.6000061	0.028701	4.7847	0.028701
14	-171.5000000	-0.017668	2.9636	0.000000
15	460.1300049	0.047402	21.5170	0.047402
16	-217.0500031	-0.022360	2.4092	0.000000
17	-64.3909988	-0.006634	0.1630	0.000000
18	21.9799995	0.002264	0.0214	0.000000
19	-71.5380020	-0.007370	0.6513	0.000000
20	256.5299988	0.026428	5.2164	0.026428
21	129.3999939	0.013331	4.1881	0.013331
22	49.6040001	0.005110	0.3311	0.000000
23	70.5400009	0.007267	1.1034	0.000000
24	80.5000000	0.008293	2.8748	0.000000
25	-100.0999985	-0.010312	2.5254	0.000000
26	264.0199890	0.027199	4.8925	0.027199
27	-144.1000061	-0.014845	3.6632	0.000000
28	39.0579987	0.004024	0.1574	0.000000
29	173.6300049	0.017888	5.7105	0.017888
30	-153.4400024	-0.015808	13.4550	-0.015808
31	209.4199982	0.021574	12.3870	0.021574
32	-104.7500000	-0.010792	2.2149	0.000000
33	83.0029984	0.008551	2.4024	0.000000
34	229.5000000	0.023643	29.3900	0.023643
35	-84.3809967	-0.008693	1.2283	0.000000
36	-13.0109997	-0.001340	0.0223	0.000000
37	33.9710007	0.003500	0.5827	0.000000
38	-29.6889992	-0.003058	0.1816	0.000000
39	-137.6499939	-0.014181	2.9087	0.000000
40	187.4199982	0.019308	7.1901	0.019308
41	-49.6599998	-0.005116	0.7498	0.000000
42	0.6496200	0.000067	0.0002	0.000000
43	5.6441002	0.000581	0.0209	0.000000
44	35.0629997	0.003612	0.3905	0.000000
45	36.4949989	0.003760	1.4998	0.000000
46	-44.1959991	-0.004553	1.2061	0.000000
47	31.0139999	0.003195	0.2465	0.000000
48	-17.4290009	-0.001796	0.0542	0.000000
49	-131.9400024	-0.013593	5.8767	-0.013593
50	21.1119995	0.002175	0.3294	0.000000
51	66.9000015	0.006892	3.2106	0.000000
52	149.7299957	0.015425	12.7200	0.015425
53	-83.3519974	-0.008587	5.9966	-0.008587
54	-0.0157490	-0.000002	0.0000	0.000000
55	94.5049973	0.009736	6.8752	0.009736
56	-95.4810028	-0.009836	6.6825	-0.009836
57	-55.9650002	-0.005766	2.0826	0.000000
58	17.8820000	0.001842	0.3357	0.000000
59	71.1320038	0.007328	4.9548	0.007328
60	-151.1600037	-0.015573	11.2670	-0.015573
61	61.6080017	0.006347	1.5131	0.000000
62	-6.5314002	-0.000673	0.0515	0.000000
63	133.1000061	0.013712	40.5830	0.013712
64	-34.0789986	-0.003511	1.2163	0.000000
65	60.3559990	0.006218	3.6596	0.000000
66	-81.2529984	-0.008371	6.4249	-0.008371
67	83.1750031	0.008569	6.0705	0.008569
68	-88.2490005	-0.009091	10.7080	-0.009091
69	-25.3040009	-0.002607	1.5522	0.000000
70	10.4099998	0.001073	0.1535	0.000000
71	12.3500004	0.001272	0.1193	0.000000
72	19.2859993	0.001987	0.5427	0.000000
73	-63.8139992	-0.006574	4.9589	-0.006574
74	107.8600006	0.011111	16.0900	0.011111
75	-52.7280006	-0.005432	4.7484	-0.005432
76	-29.9920006	-0.003090	0.5376	0.000000
77	-28.5690002	-0.002943	0.8533	0.000000
78	-22.3290005	-0.002300	0.9391	0.000000
79	69.3109970	0.007140	8.7981	0.007140
80	8.4715996	0.000873	0.3457	0.000000

Table 1: Drift D_i sensitivity for OAVDE

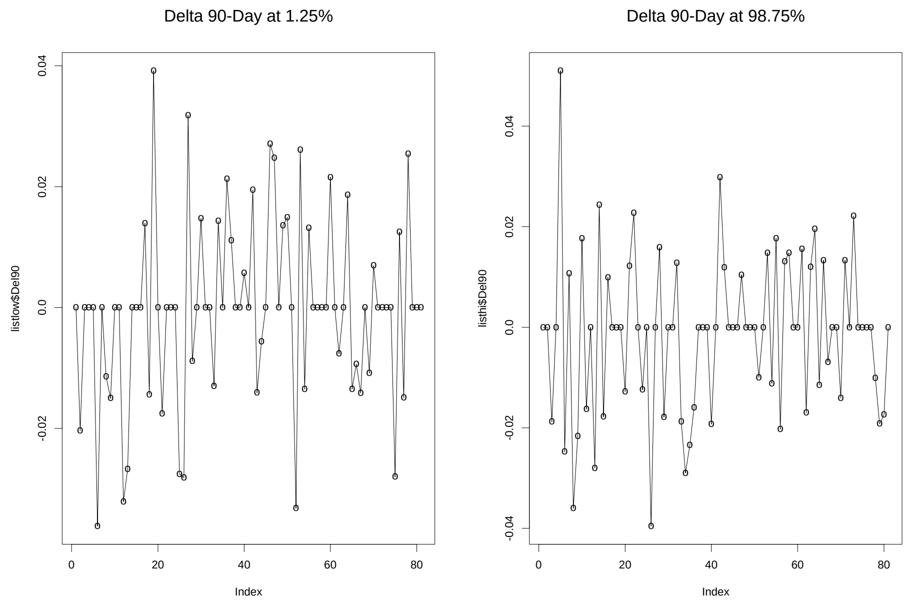

Figure 3: Significant Scaled Coefficients for Y_i^{90}

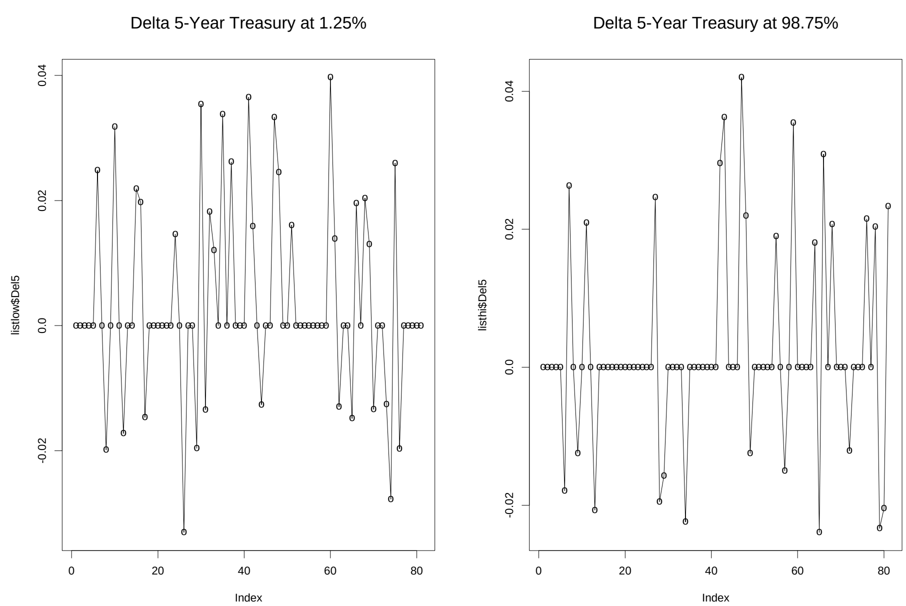

Figure 4: Significant Scaled Coefficients for Y_i^5

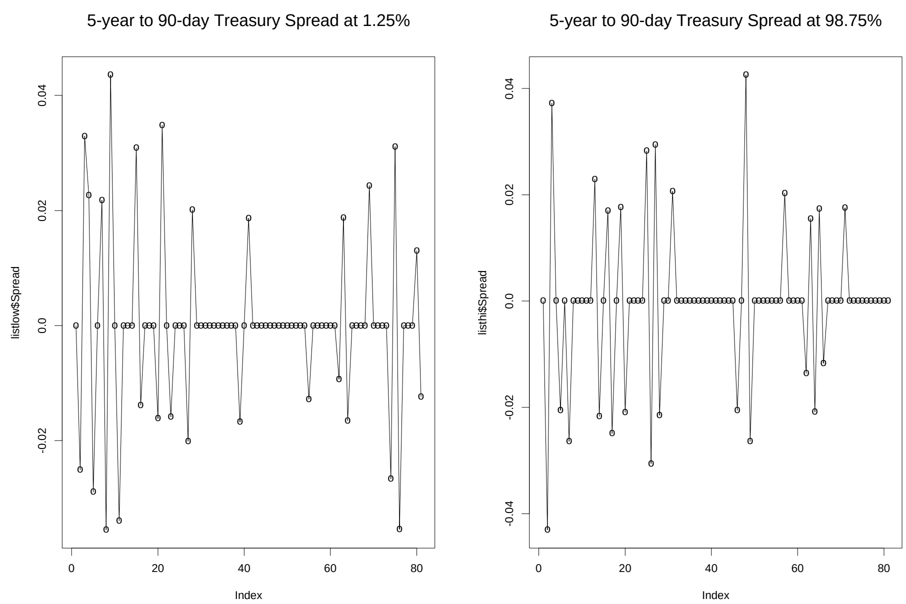

Figure 5: Significant Scaled Coefficients for S_i

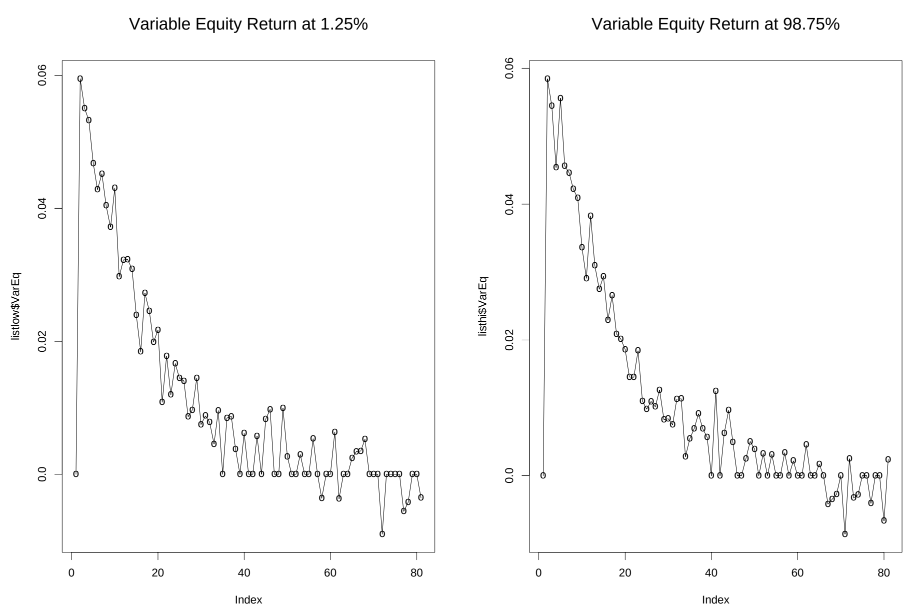

Figure 6: Significant Scaled Coefficients for V_i

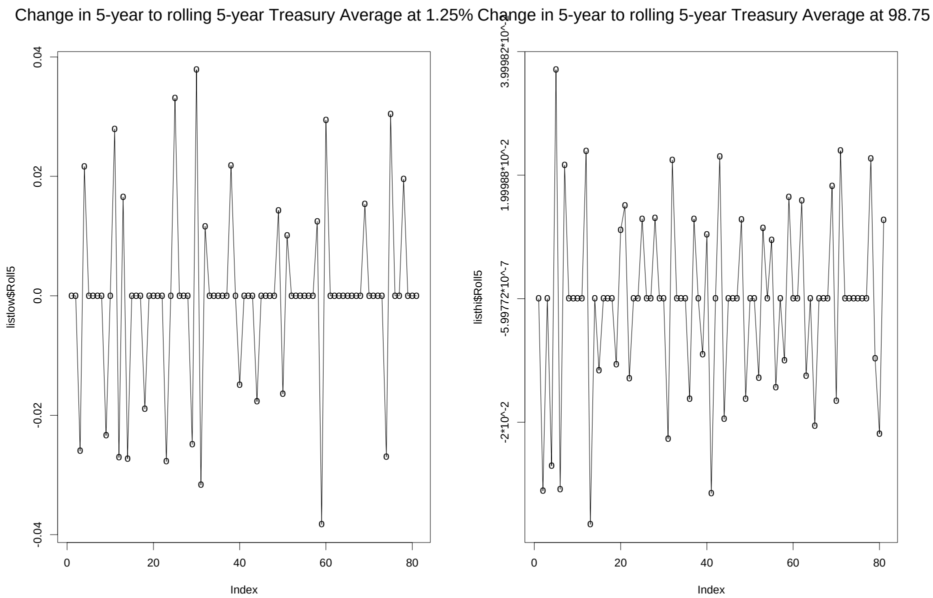

Figure 7: Significant Scaled Coefficients for RA_i^5

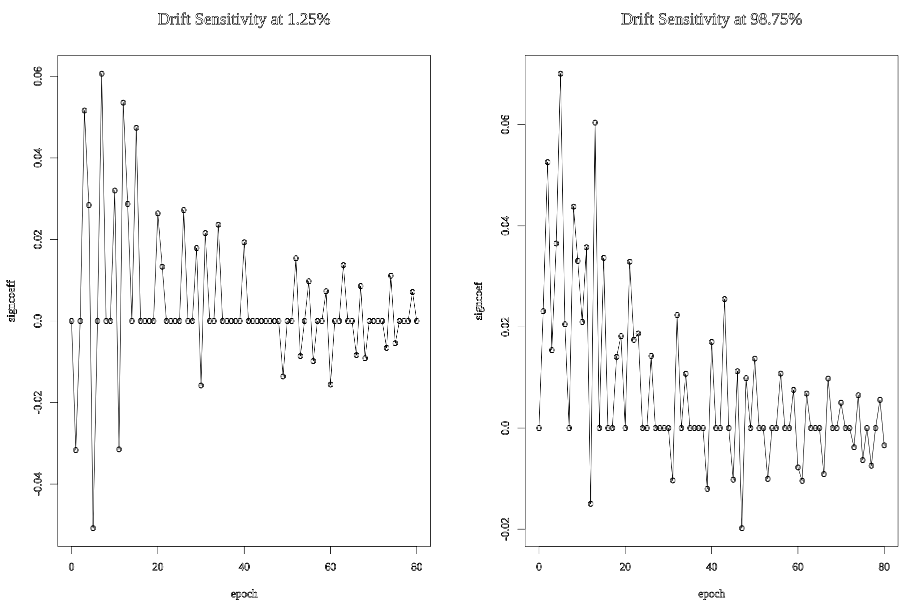

Figure 8: Significant Scaled Coefficients for D_i

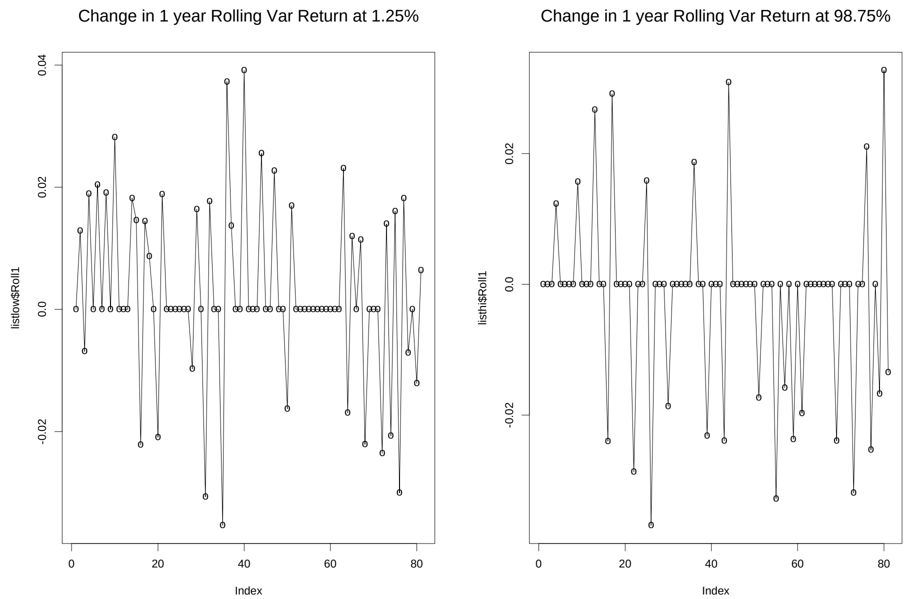

Figure 9: Significant Scaled Coefficients for RA_i^{EQ}

termine risk exposure from embedded options in the business. The financial engineer can also use these methods to improve his or her derivative hedge. This method also examines the effect from all scenarios on the specific R_q. In the past the actuary may have used different deterministic scenarios to determine the direction of the markets that created risk exposure to the business. However, the deterministic scenarios do not indicate the significance or aid in the determination of the exact location in the projection period that the business is at the highest risk.

Note the following relationship for R_q to the various risk drivers X_i. If value of the X_i can have both positive and negative values, we need only to examine the large $|B_{i,q}|$. Here if one is studying where profit is enhanced at the specific percentage being studied, if X_i is positive and $|B_{i,q}|$ is large and $B_{i,q}$ is positive, R_q increases. If X_i is negative and $B_{i,q}$ is negative, R_q also increases. Just reverse this reasoning if one is interested in determining when the business model is at risk.

In Table 1 we display the QR results that correspond to 1.25% of the Drift risk driver D_i mentioned in Section 2. This quantile regression has a R^1 value of .95, which indicates a very significant regression model. Note in column one that the epoch corresponds to the coefficient subscript. Also note that Epoch 0 is the intercept. The other coefficients correspond to the value of the specific risk driver at different quarters. In the second column, we have the actual QR coefficients. In the third column, each coefficient $B_{i,q}$ contributes to the performance of $Quant(R_q)$ as seen in Equation 2. Since we denote the influence of the sign of the $B_{i,q}$ by the risk direction, we are only concerned with the absolute magnitude of the $B_{i,q}$. In Principal Components Analysis (PCA) one understands the influence of specific eigenvalues by ranking the ratios of the eigenvalues to the sum of the eigenvalues. See Dillon [10], Johnson and Wichern [12], and Mardia et al. [20] for a further discussion of PCA. Using this concept we measure the significance S_i of a specific coefficient B_i by

$$S_i = |B_{i,q}| / \sum_{i=1}^{n} |B_{i,q}| \tag{3}$$

expressed as a percent where n is the total number of coefficients. Also note that the significance does not include the intercept $B_{0,q}$ in the formula. This is because the intercept does not contribute to the variability of the R_q, as well possibly obscuring the influence of the other coefficients of the predictors. In the fourth column, the Wald estimator is derived and indicates the significance of the specific coefficient in the second column. Since the Wald estimator is χ_1^2 the significant value for a χ_1^2 distribution at 95% is 4.00. Hence, the coefficients are significant if Wald estimator is greater than 4.00. Finally, in column five, the S_i is copied if the Wald estimator is significant. Note that the first graph in Figure 8 corresponds to this column.

The use of QR is tailored to aid in the analysis of the tails of the distribution of OAVDE. Due to the fact that QR minimizes a absolute sum instead of a absolute sum of squares, we obtain the same QR for multiple quantiles. For instance, in Table 2, we see that for D_i that the regressions from have the same $B_{i,q}$ for $q = 1.25\%$ to 5.00% and for Y_i the $B_{i,q}$ are the same for $q = 92.50\%$ to 98.75%. Also we observe in Table 3 that the highest R^1 values are at the extreme tails and reduces as one examines central quantiles. We believe that this is due to the fact that a large number of scenarios place the surplus results in the central quantiles. The wide blend of characteristics in these 'central' scenarios makes it more difficult to relate risk drivers to the surplus results in the central quantiles. Hence the lower R^1 values. Since (hopefully!) relatively few scenarios place our results in the distribution's

Type	Y_i^{90}	Y_i^5	S_i	V_i	RA_i^5	D_i	RA_i^{EQ}
Low	3.75	8.75	5.00	8.75	5.00	5.00	8.75
High	92.50	91.25	93.75	92.50	92.50	91.25	93.75

Table 2: Repeated QR results

Type	Y_i^{90}	Y_i^5	S_i	V_i	RA_i^5	D_i	RA_i^{EQ}
Left	0.68487	0.6773	0.64493	0.93947	0.66118	0.95002	0.68867
Minimum	0.25628	0.18372	0.25729	0.85947	0.19629	0.90744	0.22015
Right	0.72221	0.71323	0.68177	0.94891	0.70524	0.97681	0.70707

Table 3: Range of R^1 for various QR

tails, it is easier to discern the effects of the risk drivers on profits or risk in the tails.

The second graph in Figure 8 gives the significant scaled coefficients of the Drift model D_i at 98.75%. The R^1 statistic for this regression is 97.7%, which indicates a very good regression model.

Of the risk drivers we tested, D_i (See Table 1 and the two graphs of Figure 8) has the greatest impact on the surplus model, with an R^1 ranging from 0.90 to 0.98. This matches our intuition about the model. Since the premium is initially allocated 90% to the variable account, we expect profits to be driven by the variable account asset fees earned on the product. When we experience above average equity returns we collect higher asset fees and thus earn higher profits, and when equity returns are below average our asset fees decline and we are more at risk. Also, we can see from the graphs for the lowest and highest quantiles that the D_i are most significant in the first 4 to 5 years of the projection. This is partly due to the fact that the product has a 7-year surrender charge period with relatively low surrender charges in years 6 and 7. Since the surrender charge decreases, lapses increase and units in force decrease, which means the magnitude of profits or losses are lower in later years in the projection period, lessening their impact on OAVDE. Discounting the results at interest also reduces the impact of later projection years on OAVDE.

One must be careful when interpreting the results from the graphs. For example, there are some large negative coefficients in the second, sixth and twelfth quarters in the graph for the lowest quantile. At first glance, we might conclude that above average equity returns in those quarters lowers our profits. However, these coefficients explain the results for the worst scenarios, and the worst scenarios have below average equity returns in the long run. Given that we are in a scenario that has below average equity returns, if we have above average or even average returns early on in the scenario, we must have very low returns later in the scenario. Thus, a high return in a low quantile scenario is a leading indicator that we are about to have very low returns and ultimately low profits. These negative coefficients may also be due to noise, since we used only 249 scenarios in our model. We have observed in other studies that as the number of scenarios increase the volatility between the coefficients decrease. See Craighead [8] for further examples and a discussion of how sample size impacts the coefficient volatility.

We see from Figure 6 that another significant risk driver is V_i, the quarterly equity returns, which in Table 3 has an R^1 ranging from 0.85 to 0.95. These graphs appear to exhibit exponential decay, with the most significant coefficients appearing in the first few quarters of the projection. This is consistent with our expectation that the results from the first five to seven years of the projection have the greatest impact on OAVDE. These graphs indicate that we are very likely to experience losses in our business model if we have negative equity returns in the first 20 quarters of the projection. Consistent with our earlier results, we find our surplus is at risk when we experience declining equity returns.

Despite the fact that our results are heavily dependent on the equity returns, RA_i^{EQ}, as one can observe in Figure 9, has more negative coefficients than D_i. Observe in Table 3, that RA_i^{EQ} has an R^1 ranging from 0.23 to 0.71. This is because RA_i^{EQ} does not contain as much information as D_i. If RA_i^{EQ} is positive, it means that our return over the last 12 months was higher than the prior year 12-month equity return. If the prior year 12-month return was low, the return over the last 12 months could still be below average. Even if the current 12-month return is high, we still may have had a low return to date for the scenario, so we may not earn high profits for that scenario. However, if D_i is positive, then we know that we have had an above average return since the beginning of the projection, and it is much more likely that we will turn a profit. As a result, D_i shows a better correlation of profits to equity return than RA_i^{EQ}. This shows the importance of testing various drivers in QR to determine the risk drivers of your model, as we cannot be sure in advance which drivers will prove to have a significant impact on OAVDE.

We expect the Y_i^5 in Figure 4 risk driver to have less impact on profitability since our model had only 10% of the premium deposited in the fixed account. But Y_i^5 is a leading indicator for our portfolio return, as we would expect our portfolio return to increase in the future if the 5-year Treasury bond rate is currently increasing. Since we set the credited rate annually, an increasing portfolio return indicates that we will earn extra spread during the next year, whereas a decreasing portfolio rate indicates we will earn less spread than expected during the next year. There is some indication of this in the graph for Y_i^5, as it shows more positive coefficients than negative coefficients, which means OAVDE is positively affected if we have a positive change in Y_i^5. However, the magnitude of the contribution for the coefficients stays fairly level throughout the study period. Due to lapses and interest discounting, we would expect the magnitude of the contribution to OAVDE for later years to be smaller than the magnitude of the contribution for early years in the study. The fact that the magnitude stays relatively constant throughout the study period may indicate that the Y_i^5 risk driver does not have a significant impact on profits, or on our risk of loss. This might change if our business model included dynamic policyholder behavior, so that our credited rate would have an impact on lapses and transfers between accounts, or could change if we allocated a higher percentage of the initial premium to the fixed account.

We are unable to make further conclusions from the other three risk drivers. We believe that the coefficient volatility is so high due to insufficient scenarios, we are unable to draw further insight from the results.

QR analysis allows the actuary to conduct risk analysis on several different risk measures. In fact in the past the actuary did not consider some of the above analyses without extensive additional computer runs. This increased ability may initially raise more questions for the actuary to analyze, but this type of risk analysis appears to be an excellent tool to conduct these analyses.

4 General Comments, Conclusions and Future Research

Dr. Stephen Portnoy made us aware of a new QR algorithm that he and Koenker developed in [21]. This can obtained from the Internet site referenced in [18], and the authors.

Koenker and Machado [19], Portnoy [23], and Craighead [8] displays several ways to display Quantile Regression results.

Other issues surrounding subsampling and data dependency issues involving QR are discussed in Craighead [8]

We will next develop a list of strengths and weaknesses of the QR methodology and finish with a list of further research topics and concluding remarks.

4.1 Strengths and Weaknesses

The strengths of the QR methodology are:

1. Ties the input scenarios to the output.

2. Sign and magnitude of coefficients give insight into risk exposures.

3. Targets specific regions of the output's behavior.

4. Holistic. The QR results are determined across all scenarios not just a restricted subgroup.

5. We can determine the influence of a specific period in time to the surplus results from a specific risk driver.

6. Extreme outliers do not affect the results as much as in OLSR.

7. Very fast. We only had to use a single PTS computer projection to model the OAVDE. Afterward we conducted 79 separate quantile regressions for each risk driver.

8. Very good goodness of fit statistics available (finally).

9. Repeated regressions in the extreme tails.

The weaknesses of the use of QR are:

1. Complex.

2. Specialized software.

3. Linear vs. Nonlinear. The nonlinear corporate model is oversimplified with QR.

4.2 Future Research

In the process of examining the QR methodology we have found the following questions and areas that require further development and research:

1. Are there methods to reduce sample sizes required with QR?
2. Can the use of Low Discrepancy Sequences be used with the QR methodology to improve its subsample results?
3. Can QR analysis be used to help model complex surplus models, (e.g. target surplus)?
4. Can QR analysis be used in modeling complex exotic derivatives? Can it help in the design of dynamic asset strategies? It appears that Taylor is the first to consider the use of QR in derivative analysis in [26].

4.3 Concluding Remarks

We have taken a data set representing the surplus performance of one variable annuity line of business. This data set contains 249 distinct values that were obtained from a corporate study. We proceed to apply the QR methodology to this data set. Next, we develop a QR report and graphs that reveal risks at specific times for seven separate risk drivers. We find that the use of the QR methodology gives us a deeper understanding of the financial risks that our variable annuity product is exposed.

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Finally, I dedicate this paper to my late father-in-law, John Robert Bird, Jr. We sorely miss you and fondly recall your influence on all of our lives.

A The QR Methodology[2]

In multivariate linear regression a column vector of T responses $\{Y_t\}$ are related to a design matrix X of predictors in the following way:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + u_t. \tag{4}$$
$$E[Y_t] = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} \tag{5}$$

Let $t = 1, \cdots, T$ and denote the 'errors' as $\{u_t\}$. These 'errors' are where the predicted value from the formula in X_{ti} does not exactly correspond to the observation Y_t. The $\{u_t\}$ are considered to be independent and identically distributed with a distribution F_u and $E[u_t] = 0$.

Another way to look at the problem is a comparison between a model

$$\hat{Y}_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}, \tag{6}$$

which tries to predict the responses Y_t from that of some linear combination of the elements of the design matrix. The residuals $u_t = Y_t - \hat{Y}_t$ then are how well or how poorly the model fits the actual responses. In ordinary least squares regression (OLSR) the expectation of the residuals are considered to be zero. Also since the expectation operator is linear then $E[u_t] = E[Y_t] - E[\hat{Y}]$.

In multivariate linear regression, the β_i are determined by minimizing the following sum

$$\sum_1^T (Y_t - (\beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}))^2. \tag{7}$$

The determination of the $\{\beta_i\}$ is referred to as an OLSR or as a l_2 regression estimator. See Portnoy and Koenker [21] for a further discussion of l_2-estimation. After the $\{\beta_i\}$ are determined, the equation relates the sample mean of the Y_t to the predictors.

However, one major difficulty of using OLSR is that the values of the $\{\beta_i\}$ can be very sensitive to outliers in the responses $\{Y_t\}$. The area of robust statistics has arisen to deal with this outlier sensitivity. See Venables and Ripley [29] for a series of references on robust statistics.

[2] A further presentation of the following material is contained in Buchinsky [6].

Koenker and Basset [13] develop quantile regression (QR), where the regression is related to specific quantiles instead of the mean. We will now describe the process.

Let $\mathbf{x_t} = \{1, X_{t1}, X_{t2}, \cdots, X_{tk}\}$, and $\beta_\theta = \{\beta_0, \beta_1, \cdots, \beta_k\}$ and consider the following:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + u_{\theta_t} \tag{8}$$

or in matrix format:

$$Y_t = \beta_\theta \mathbf{x_t'} + u_{\theta_t}. \tag{9}$$

$$Quant_\theta(Y_t|\{X_{t1}, X_{t2}, \cdots, X_{tk}\}) = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} \tag{10}$$

or in matrix format:

$$Quant_\theta(Y_t|\mathbf{x_t}) = \beta_\theta \mathbf{x_t'}. \tag{11}$$

$Quant_\theta(Y_t|\{X_{t1}, X_{t2}, \cdots, X_{tk}\})$ denotes the conditional quantile of Y_t, which is conditional on $\{X_{t1}, X_{t2}, \cdots, X_{tk}\}$, the regression vector. The distribution $F_{u\theta}$ of u_{θ_t}, the error term is not specified. Formula 10 implies that $Quant_\theta(u_{\theta_t}|\{x_{t1}, x_{t2}, \cdots, x_{tk}\}) = 0$ for a specific component vector $\{x_{t1}, x_{t2}, \cdots, x_{tk}\}$.

Let's look at this from the perspective of the residual or error term. Assume that there is a model

$$\hat{Y}_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}, \tag{12}$$

which tries to predict certain behavior of the responses Y_t that is some linear combination of the elements of the design matrix. The residuals $u_t = Y_t - \hat{Y}_t$ then are a measurement of how well the model relates to the actual responses. The difference between QR and OLSR is that instead of the fact that $E[u_t] = 0$ one assumes that $Quant_\theta(u_t) = 0$. This leads to the relationship

$$Quant_\theta(u_t) = Quant_\theta(Y_t - \hat{Y}_t) = 0. \tag{13}$$

The determination of the $\{\beta_i\}$ that allows this relationship to hold will produce the necessary model. However, because the determination of a quantile requires sorting, the quantile operator is not linear. Hence

$$Quant_\theta(Y_t - \hat{Y}_t) \neq Quant_\theta(Y_t) - Quant_\theta(\hat{Y}_t). \tag{14}$$

Koenker and Basset [13] made the following observation: Let Y be a random variable with distribution F. Let $\{y_t : t = 1, \cdots, T\}$ be a random sample on Y. The θth sample quantile for $0 < \theta < 1$ is defined to be any solution of the following minimization problem:

$$min_{b \in \mathbb{R}} \left[\sum_{t \in \{t : y_t \geq b\}} \theta |y_t - b| + \sum_{t \in \{t : y_t < b\}} (1 - \theta) |y_t - b| \right]. \tag{15}$$

From the above Koenker and Basset are able to generalize the l_1 regression estimator from the median to all quantiles $0 < \theta < 1$, by finding the $\{\beta_i\}$ that minimizes the following:

$$\sum_{t=1}^{T} \rho_\theta(Y_t - (\beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk})), \tag{16}$$

where $0 < \theta < 1$ and

$$\rho_\theta(u) = \begin{cases} \theta u & \text{when } u \geq 0, \\ (1 - \theta) u & \text{when } u < 0. \end{cases} \tag{17}$$

Buchinsky [6] discusses that under certain regularity conditions the consistency and asymptotic normality of $\hat{\beta}_\theta = (\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k)$ (which is the estimator of $\beta_\theta = (\beta_0, \beta_1, \cdots, \beta_k)$) that

$$\sqrt{n}((\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k) - (\beta_0, \beta_1, \cdots, \beta_k)) \xrightarrow{L} N((0, 0, \cdots, 0), \Lambda_\theta) \tag{18}$$

$$\sqrt{n}(\hat{\beta}_\theta - \beta_\theta) \xrightarrow{L} N(\mathbf{0}, \Lambda_\theta) \tag{19}$$

where

$$\Lambda_\theta = \theta(1 - \theta)(E[f_{u\theta}(\mathbf{0}|\mathbf{x_t})\mathbf{x_t}\mathbf{x_t'}])^{-1} E[\mathbf{x_t}\mathbf{x_t'}](E[f_{u\theta}(\mathbf{0}|\mathbf{x_i})\mathbf{x_i}\mathbf{x_i'}])^{-1}. \tag{20}$$

If the density of the error term u_θ at $\mathbf{0}$ is independent of x_t then formula 20 simplifies to

$$\Lambda_\theta = \frac{\theta(1 - \theta) E[\mathbf{x_t}\mathbf{x_t'}]^{-1}}{f_{u\theta}^2(0)}, \tag{21}$$

which corresponds to the result in Koenker and Basset [13].

For any y, if $f_{u\theta}(y|x)$ is independent of x, then the only difference between the quantile regression parameters β_θ for all quantiles θ is in the intercept β_0.

The major advantage of using quantile regression is the ability to relate the behavior of a specific quantile of the responses to the design matrix X of predictors. The partial derivative of the conditional quantile of y_t with respect to a specific regressor k is $\partial Quant_\theta(y_t|\mathbf{x_t})/\partial X_{tk}$. Interpret this derivative as the change in the θth conditional quantile due to the change in the kth element of x. Note: when the kth element of x changes this does not imply that when the y_t in a specific quantile θ changes that it will remain in the θth quantile.

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