

Financial Mathematics

Otherwise known as

BOOKIE 101

Imagine you are a bookie for a specific horse race

- Say that there is only two horses in the race
 - Horse 1-Win 'R' Us
 - Horse 2-Y'll Lose
- Being a bookie you know that Horse 1 is fed power pellets and is being rode by Jocko the Small. That means that there is a 75% chance that Win 'R' Us will truly win.

Bookie and Odds

- You quote odds of 1-3 for Horse 1 and 3-1 for Horse 2.
- Odds of n-m mean:
 - A successful bet of \$m will be rewarded with \$n with the original \$m returned. The implied probability is $m/(m+n)$. So 1-3 odds means that if you bet \$3, then if Horse 1 wins you get \$4 back. And the probability is $3/4 = 75\%$.

Popular Sentiment

- Horse 1 - \$10,000 bet
- Horse 2 - \$5,000 bet
 - If Horse 1 wins you pay out $(\$10,000/3)+\$10,000$
 - Profit = $\$15,000 - \$13,333 = \$1,667$
 - If Horse 2 wins you pay out $(\$5,000/1)*3 + \$5,000$
 - Profit = $\$15,000 - \$20,000 = -\$5,000$
 - Expected Profit
 - $\$1,667*.75+(-\$5,000)*.25 = 0$

Set Odds based on Money Wagered

- Horse 1 - odds 1-2
- Horse 2 - odds 2-1
 - If Horse 1 wins you pay out $(\$10,000/2)+\$10,000$
 - Profit = $\$15,000 - \$15,000 = \$0$
 - If Horse 2 wins you pay out $(\$5,000/1)*2 + \$5,000$
 - Profit = $\$15,000 - \$15,000 = \$0$
 - Expected Profit
 - $\$0*.75+(0)*.25 = 0$

Set Odds based on Probabilities With Profit

- Horse 1 - odds 4-15
- Horse 2 - odds 13-5
 - If Horse 1 wins you pay out $(\$10,000/15)*4 + \$10,000$
 - Profit = $\$15,000 - \$12,666 = \$2,333$
 - If Horse 2 wins you pay out $(\$5,000/5)*13 + \$5,000$
 - Profit = $\$15,000 - \$18,000 = -\$3,000$
 - Expected Profit
 - $\$2,333*.75 + (-\$3,000)*.25 = \$1,000$

Set Odds based on Money Wagered with Profit

- Horse 1 - odds 2-5
- Horse 2 - odds 9-5
 - If Horse 1 wins you pay out $(\$10,000/5)*2 + \$10,000$
 - Profit = $\$15,000 - \$14,000 = \$1,000$
 - If Horse 2 wins you pay out $(\$5,000/5)*9 + \$5,000$
 - Profit = $\$15,000 - \$14,000 = \$1,000$
 - Expected Profit
 - $\$1,000*.75 + (\$1,000)*.25 = \$1,000$

Arbitrage 1

- Rod and Reels
 - Dealer A- \$50.00
 - Dealer B- \$45.00

Arbitrage 2

Any portfolio of possibilities

- Rods and Reels
 - Dealer A- \$50.00
 - Seller C- Rod - \$30, Reel- \$10
 - Repair Service D- Construct from parts \$2.

Arbitrage 3

Time Value of Money

- Rods and Reels
 - Dealer A- \$50.00
 - Catalog Supplier E - \$35 - 90 days till delivery

Profit- When should you take it?

- If you take your profit at time of delivery you won't have to borrow so much money
- If you take your profit when you borrow the money, you will need a larger loan.

Time Value of Money

- Because time is now a part of the problem we must take the time value of money into consideration.
- For simplicity we will assume that the interest rate is 3% for the 3 months. That is if you borrow \$1 today you would have to repay \$1.03 in 3 months
- All under the heading of Theory of Interest

Profit at time of delivery

- If you just wanted 1 rod and reel from the catalog, borrow \$35 from the bank. Order the rod and reel. After three months upon delivery you sell it to dealer A for \$55. Repay the bank \$36.05. You get a profit of \$18.95.

Profit at time of order

- Borrow \$55 dollars discounted from the bank. That is $\$55/1.03 = \53.40 .
- Order the rod and reel for \$35.
- You pocket $\$53.40 - \$35 = \$18.40$
- When the rod and reel arrives, sell it to dealer A. Take the \$55 and repay the bank.
- Note $\$18.40 * (1.03) = \18.95

Fair Market Price 1

- If everybody knows what everybody else's prices are.
- It doesn't cost any money to make the transaction.
- There are no taxes.
- Individuals don't have any limit on how many rods and reels they can buy or sell.
- No limit on the amount of money they can borrow.
- You can have fractional amounts of the commodity

Fair Market Price 2

Shorting the market

- You are allowed to short the market (Here you borrow the commodity from someone else and sell it. After some time you repurchase the commodity and give it back to the person you borrowed it from.)

Fair Market Price 3

- This will force the market to carry only one price because arbitrage will force the prices together.

Dealer A- Changing values Sold Forward

- Dealer A sells you a contract that allows you to buy a rod and reel from him or her in 90 days.
- The value of the rod and reel can change. Say that the value in 90 day is S_{90} .
- Say that today price is $S_0 = \$55$.
- What is the fair price that you the buyer of the contract should pay in ninety days?

Sold Forward

- Continuing to assume the 3%/quarter interest rate
- Answer $\$55 * (1.03) = \56.65

Sold Forward- Why?

- Consider Dealer A. They borrow \$55, buy the rod and reel, and hide it in a closet and wait. In ninety days they have to pay back the loan which is \$56.65 and give you the rod and reel. If they said anything less than \$56.65 in the original contract they would lose money. This implies that the forward price is bounded below by \$56.65.

Sold Forward-Why? Cont.

- If you the buyer agree to anymore than \$55.65 in the contract you will lose money. Here you short a rod and reel (borrow the rod and reel from a friend and sell it) and loan out the \$55. At the end of 3 months you collect \$55.65 from the loan. You buy the rod and reel from the dealer and return it to your friend. If you agree to pay dealer A more than \$55.65 you will lose money.

More on Shorting

- Shorting is not commonly thought of in our day to day lives. However it is used frequently in the stock and commodity markets. Here people borrow the stock from a dealer. Then they sell it to someone else. Then if the price of the stock goes down they buy it back cheaper than at the level they borrowed it at. They then replace the borrowed stock with the cheaper stock.

More on Shorting

- When people short stock or commodities they are assuming that the price or the value of the stock or commodities are going down. However, if the price goes up they will lose money, because they have to repurchase the stock or commodity at a higher price than what they sold it.

Market direction

- In a rising market you buy the commodity.
- In a falling market you short the commodity.

Call Options

- You would like to pay dealer A a price that will give you a right to buy a rod and reel for \$55 in three months. If the price of the rod and reel is above \$55, you then pay dealer A the \$55 and he/she has to sell you the rod and reel. If the price is below \$55, you will let the option expire. Here it is cheaper to buy the rod and reel at the lower price.

Options 2

- Here options allow you to place price protection on something you want to buy in the future.

Pigeon Hole Principle

- If you look like a pigeon, fly like a pigeon, act and eat like a pigeon, and especially if you smell like a pigeon then you must be a pigeon.

Pigeon Hole Principle & A Replicating Portfolio

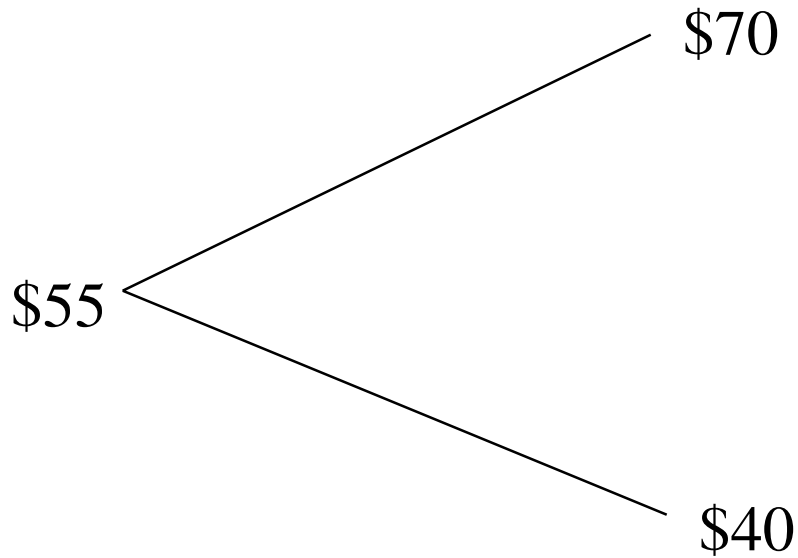
- If you can buy a bunch of things (portfolio) that have the same payments as something more exotic (replicating) then both must have the same value. Otherwise you could create an arbitrage and force the prices to finally be the same.

RODtions

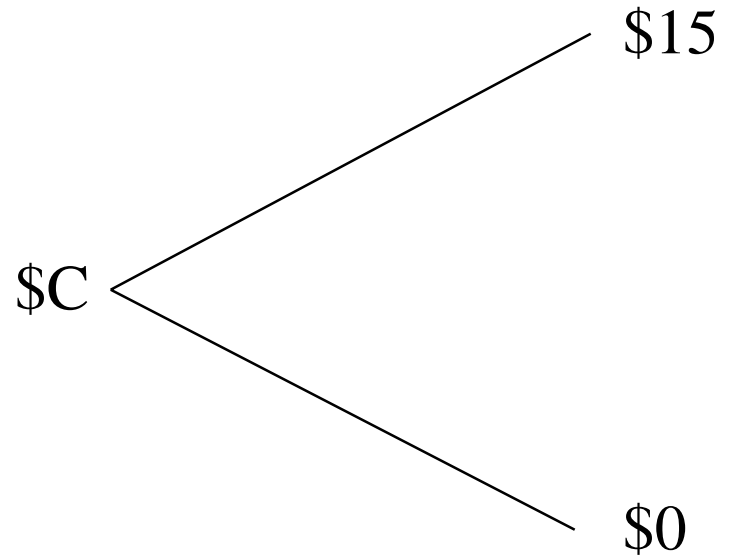
- Next we will price a very simple option.
- First we assume that the price of the rod and reel will be either \$70 or \$40 in 3 months. Say it is currently at \$55.
- You want to buy a option from Dealer A. This option is the right to buy the rod and reel for \$55 in 3 months. This is called the strike price.
- How much should you pay A for this option?
- Assume the 3%/quarter interest rate again

Replicating Portfolio

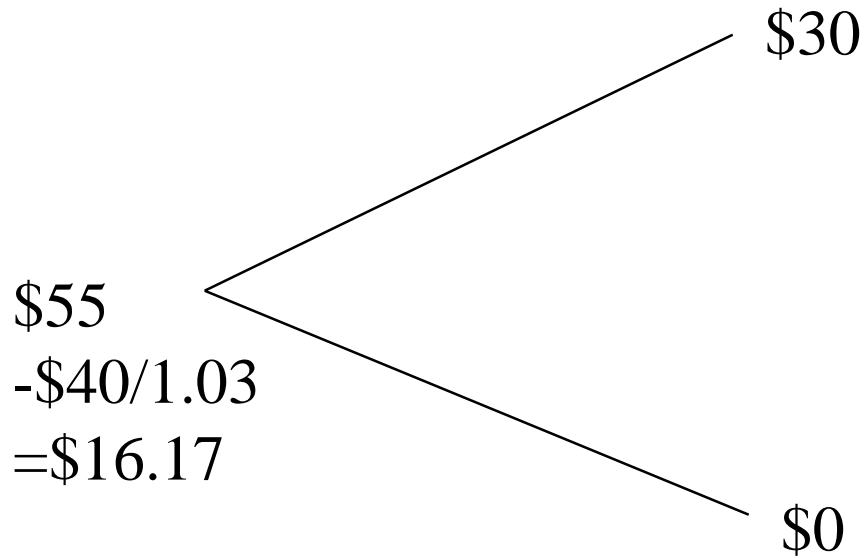
- We will either borrow or lend money
- We will either short or buy (long) the underlying commodity.
- We will consider fractional amounts of the commodity (We could have a portion of a rod and reel)



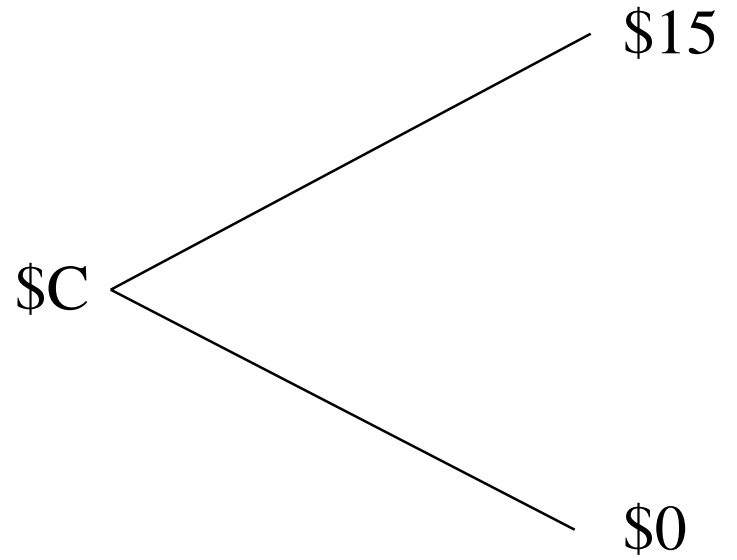
Rod and Reel Price



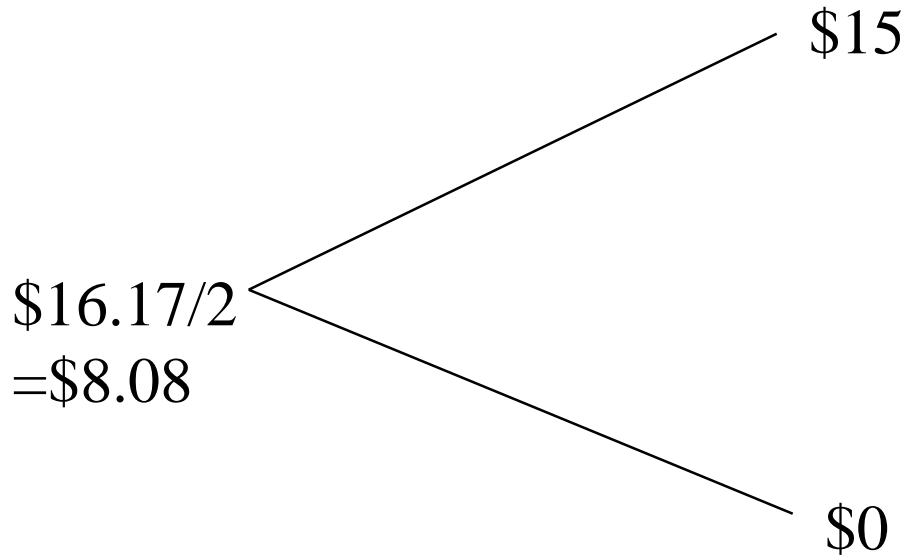
RODtion price



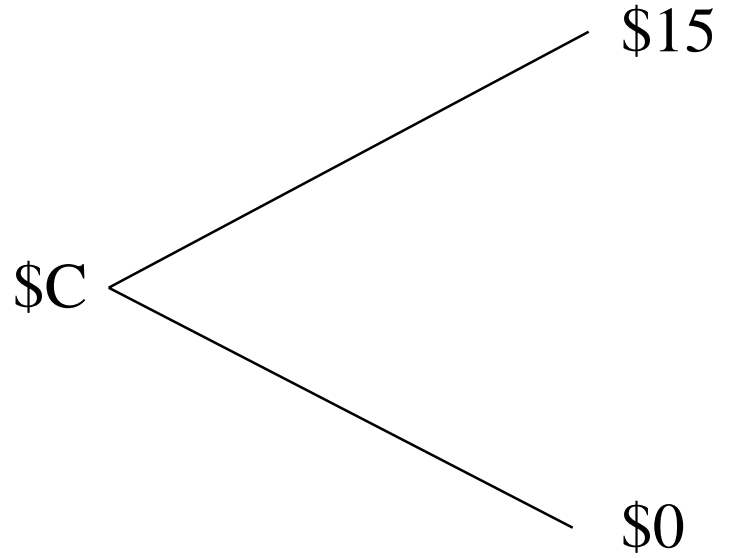
Rod and Reel Price



RODtion price



Rod and Reel Price



RODtion price

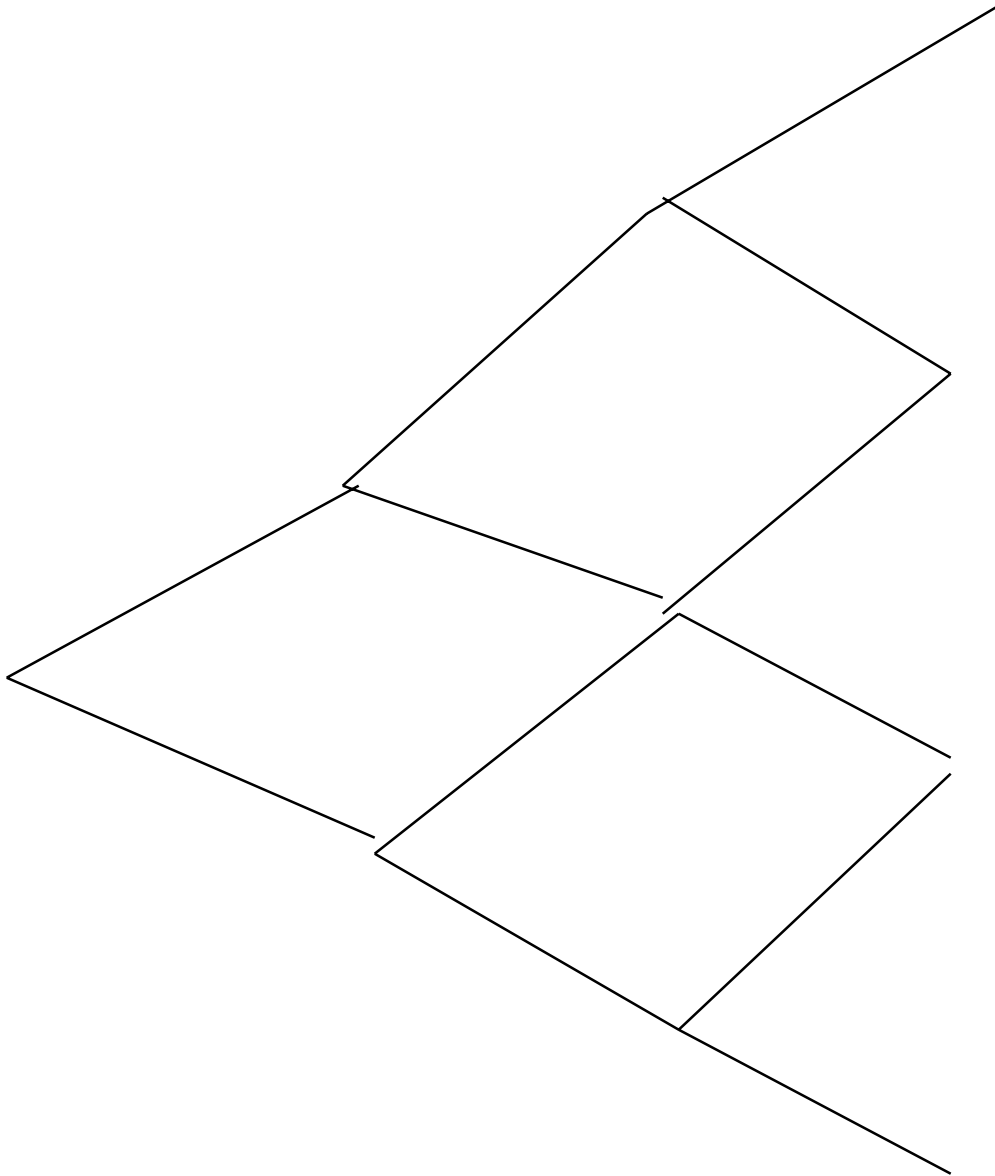
$$\$C = \$8.08$$

RODtion

- The fair price that you should pay dealer A for this option is \$8.08.

Make it more complicated

- Here the rod and reel either is \$70 or \$40.
The actual future price could be any price.
The way you find out the option price is to
make the tree more complex



More complex tree

- As you give the tree more nodes, you must make your assumptions about interest more complex.
- If you take the tree to the limit, you will obtain the Black-Scholes formula. The 1997 Economics Nobel Prize was awarded because of the following equation:

Black-Scholes Formula

$$C = S_0 \Phi\left(\frac{\ln(S_0/k) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - k e^{-rT} \Phi\left(\frac{\ln(S_0/k) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

How about profit?

Actual Probabilities	75%	25%	
Money Wagered	\$10,000	\$5,000	
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Quoted odds	4-15	13-5	
Probability	79%	28%	Total 107%
Profit if wins	\$2,333	-\$3,000	Expected profit = \$1,000
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Quoted odds	2-5	9-5	
Probability	71%	36%	Total 107%
Profit if wins	\$1,000	\$1,000	Expected profit = \$1,000