

2. Term structure models

In chapter 1, we have described a general equilibrium economy of the CIR (1985a) type and its probabilistic solution for contingent claims. In this chapter, we first develop a base model which “specialises” the general equilibrium framework introducing some simplifying assumptions. We show the restrictions which must be imposed on the model coefficients in order to write the model in terms of stochastic processes for which closed form solutions are known: translated square root, square root and elastic random walk. Then, in sections 2.2 and 2.3, we derive from the base model some well-known single-factor and multifactor specifications.

2.1. The base model

In this section we outline a model which represents the common basis for all the single- and multifactor affine general equilibrium models of the term structure. The model “specialises” the general equilibrium setting described in chapter 1 and makes use of the following assumptions.

There are a fixed number of identical individuals with rational expectations and time-additive logarithmic utility. The economy is competitive, with continuous trading and no transactions costs, and is characterised by the existence of markets for contingent claims and for instantaneous borrowing and lending at the riskless interest rate. In the economy a single physical good is produced, which may be allocated to consumption or investment. The investment opportunities consist of a stochastic production process, a set of contingent claims and a risk-free asset. In equilibrium, the rate of returns on these securities must be such that the wealth of individuals is totally invested in the production process.

The equilibrium of the economy is influenced by a vector of N unknown state variables, x , which is assumed to follow a multivariate stochastic process of the form:

$$dx = (\alpha + \Gamma x)dt + \Xi(x)dz \quad (2.1.)$$

where α is a N -vector and Γ a full $N \times N$ matrix. $\Xi(x)$ is a $N \times N$ diagonal matrix whose coefficients depend, in an unspecified way, by the state vector x . The N -vector z contains one-dimensional correlated Wiener processes:

$$E(dz, dz') = \Omega .$$

The state variables and time determine the investment in the physical production process. Denote by q the amount of the good invested in the production process:

$$q = q(x, t) .$$

As in CIR (1985b), we assume that the mean of the rate of return on the production process is linear in the state variables. Therefore, we may write the stochastic differential equation for the rate of return of the production process in the following form:

$$\frac{dq}{q} = \mu'xdt + \delta'\Xi(x)dz . \quad (2.2.)$$

Since, in equilibrium, the wealth of the individuals must be completely invested in the physical production process (see equation (1.7)), the diffusion process which describes the dynamics of wealth, W , is given by:

$$dW = W \frac{dq}{q} - Cdt = (W\mu'x - C)dt + W\delta'\Xi(x)dz$$

where C is the optimal consumption flow.

Given the dynamics of the production process and the logarithmic utility assumption, we know from CIR (1985a) and Breeden (1986) that the equilibrium expression of the instantaneous riskless interest rate, r , is:

$$r = E_t \left(\frac{dq}{q} \right) - Var_t \left(\frac{dq}{q} \right) = \mu'x - \delta'\Xi(x)\Omega\Xi(x)\delta . \quad (2.3.)$$

This expression says that in equilibrium the instantaneous real interest rate must be equal to the difference between expected rate of growth in output and the variance of the output growth rate. Such a relation follows from equation (1.8) of section 1.1 once the logarithmic utility assumption is enforced.

The equilibrium value of any contingent claim will depend on the vector of state variables and time. Denote by $P = P(x, t; T)$ the equilibrium price at time t of a unit discount bond maturing at time T , i.e., a default-free zero coupon bond which promises to pay one unit of the numeraire (good) at the maturity date T . Using Ito's Lemma, we get:

$$dP = \mu_p Pdt + \left(\frac{\partial P}{\partial x} \right)' \Xi(x)dz \quad (2.4.)$$

where ¹⁹ :

$$\mu_p \equiv \frac{1}{P} \left[\left(\frac{\partial P}{\partial x} \right)' (\alpha + \Gamma x) + \frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial x \partial x'} * (\Xi(x) \Omega \Xi(x)) \right].$$

Then, applying the no-arbitrage condition, we can obtain the partial differential equation which the price of the bond must satisfy.

The no-arbitrage condition is given by:

$$(\mu_p - r)P = \left(\frac{\partial P}{\partial x} \right)' \Phi(x) \quad (2.5.)$$

with:

$$\Phi(x) \equiv Cov \left(dx, \frac{dW}{W} \right) = \Xi(x) \Omega \Xi(x) \delta .$$

The no-arbitrage condition (2.5) is derived forming a portfolio which comprises investments in the production process, the zero coupon bond and the riskless asset, with proportions equal to ω_1 , ω_2 and $1 - \omega_1 - \omega_2$, respectively. The rate of return on this portfolio is:

$$\begin{aligned} \frac{d\Pi}{\Pi} &= \omega_1 \frac{dq}{q} + \omega_2 \frac{dP}{P} + (1 - \omega_1 - \omega_2) r dt \\ &= (r + \omega_1 (\mu'x - r) + \omega_2 (\mu_p - r)) dt + \left(\omega_1 \delta' \Xi(x) + \omega_2 \frac{1}{P} \left(\frac{\partial P}{\partial x} \right)' \Xi(x) \right) dz . \end{aligned}$$

The portfolio is riskless if the stochastic component of the rate of return is zero, that is:

¹⁹ The symbol * denotes an operation which implies that (i) each element of the first matrix is multiplied by the corresponding element of the second matrix (Hadamard product) and then (ii) the elements of the new matrix are summed.

$$\omega_1 \delta' \Xi(x) + \omega_2 \frac{1}{P} \left(\frac{\partial P}{\partial x} \right)' \Xi(x) = 0 . \quad (2.6.)$$

In absence of arbitrage opportunities a riskless portfolio must earn the riskless interest rate r . Therefore, we must have:

$$\omega_1 (\mu' x - r) + \omega_2 (\mu_p - r) = 0 .$$

Substituting from (2.3) and (2.6) above, we obtain the no-arbitrage condition (2.5). Using equation (2.4), we can observe that this is equivalent to:

$$\mu_p - r = \text{Cov} \left(\frac{dP}{P}, \frac{dW}{W} \right).$$

Substituting in the no-arbitrage condition (2.5) the expression for μ_p , we get the partial differential equation which represents the fundamental valuation equation for the price of a unit discount bond:

$$\frac{1}{2} \frac{\partial^2 P}{\partial x \partial x'} * (\Xi(x) \Omega \Xi(x)) + \left(\frac{\partial P}{\partial x} \right)' (\alpha - \Phi(x) + \Gamma x) + \frac{\partial P}{\partial t} - rP = 0 \quad (2.7.)$$

with boundary condition:

$$P(x, T; T) = 1 . \quad (2.8.)$$

The probabilistic solution of the Cauchy problem (2.7 - 2.8), which is also called the Feynman-Kac solution, is given by:

$$P(x, t; T) = \hat{E}_t \left\{ \exp \left[- \int_t^T r(u) du \right] \right\} . \quad (2.9.)$$

The expectation is taken with respect to the risk-adjusted probability measure. A straightforward application of the Girsanov theorem (see section 1.2) shows how to generate

the risk-adjusted probability measure from the following "correction" to the original Wiener processes:

$$d\hat{z} = dz + (\Xi(x))^{-1} \Phi(x) dt \quad (2.10.)$$

where \hat{z} denotes a Wiener process under the risk-adjusted probability measure.

Substituting in (2.1) above, we get the risk-adjusted process for the state variables:

$$dx = (\alpha - \Phi(x) + \Gamma x) dt + \Xi(x) d\hat{z} \quad (2.11.)$$

with:

$$E(d\hat{z}, d\hat{z}') = \Omega .$$

In order to solve the Cauchy problem (2.7 - 2.8) in terms of the state variables and the model parameters, some restrictions must be imposed on the form of the matrices in equation (2.11) and the correlation between the Wiener processes.

There are three stochastic processes, all frequently used in the literature, for which a closed form solution exists: square root, translated square root and elastic random walk.

The square root specifications (translated and untranslated) imply that the variance of each state variable is a linear combination of the variable itself and require the assumption that all the state variables must be mutually uncorrelated. Instead, the elastic random walk specification implies constant volatility but does not require to impose any restriction on the correlation between the state variables.

The following table summarises the restrictions which must be imposed on equation (2.11) in order to derive these three processes and the related closed form solutions for bond prices²⁰.

Γ	Ω	$\Xi(x)$	Stochastic process
Diagonal	Identity	$(\Xi_0 + \Xi_1 D_x)^{1/2}$	Translated square root
Diagonal	Identity	$(\Xi_1 D_x)^{1/2}$	Square root
No restrictions	No restrictions	$(\Xi_0)^{1/2}$	Elastic random walk

²⁰ More general specifications, such as, for example, square root processes with full matrices for Γ and $\Xi(x)$, are considered by Duffie and Kan (1996) and Dai and Singleton (1998). However, in these cases relatively strong restrictions must be imposed on the parameter values for a closed form solution to exist.

where Ξ_0 and Ξ_1 are $N \times N$ diagonal matrices containing the volatility coefficients, while D_x is a $N \times N$ diagonal matrix containing the state variables.

These restrictions imply that the risk premium coefficient $\Phi(x)$ and the real interest rate r take the form:

$\Phi(x)$	r	Stochastic process
$\Xi_0 \delta + D_\delta \Xi_1 x \equiv \Lambda_0 + \Lambda_1 x$	$\Lambda'_0 \Xi_0^{-1} \Lambda_0 + (\mu' - \iota' \Lambda_1 \Xi_1^{-1} \Lambda_1) x \equiv \varepsilon_0 + \varepsilon' x$	Transl. square root
$D_\delta \Xi_1 x \equiv \Lambda_1 x$	$(\mu' - \iota' \Lambda_1 \Xi_1^{-1} \Lambda_1) x \equiv \varepsilon' x$	Square root
$\Xi_0^{1/2} \Omega \Xi_0^{1/2} \delta \equiv \Lambda_0$	$-\Lambda'_0 (\Xi_0^{1/2} \Omega \Xi_0^{1/2})^{-1} \Lambda_0 + \mu' x \equiv \varepsilon_0 + \varepsilon' x$	Elastic random walk

where Λ_0 and Λ_1 are defined as a N -vector and a $N \times N$ diagonal matrix, respectively, containing the risk premium coefficients, while D_δ is a $N \times N$ diagonal matrix containing the coefficients of vector δ . The scalar ε_0 and the N vector ε in the table above are used to simplify the notation in what follows.

The restrictions imply that the underlying risk-adjusted processes for each state variable are of the following type:

dx_i	Stochastic process
$(\alpha_i - \lambda_{oi} + (\gamma_i - \lambda_{li}) x_i) dt + \sqrt{\xi_{oi} + \xi_{li} x_i} d\hat{z}_i$	Translated square root
$(\alpha_i + (\gamma_i - \lambda_{li}) x_i) dt + \sqrt{\xi_{li} x_i} d\hat{z}_i$	Square root
$\left(\alpha_i - \lambda_{oi} + \sum_{j=1}^N \gamma_{ij} x_j \right) dt + \sqrt{\xi_{oi}} d\hat{z}_i$	Elastic random walk

To sum up, in this section we have described an equilibrium economy assuming a general form for the equations determining the dynamics of the variables. We have shown that these dynamics can be “specialised” to three stochastic processes for which a closed form solution exists: square root, translated square root and elastic random walk.

In the following two sections, we exploit the features of these processes to derive closed form solutions for bond prices in both single-factor and multifactor models.

2.2. Single-factor models

In the context of the single-factor term structure theory, we examine three well-known models which differ for the type of stochastic differential equation governing the dynamics of the state variable: Vasicek (1977), CIR (1985b) and Pearson and Sun (1994).

2.2.a Gaussian process: Vasicek model

The Vasicek model assumes that the dynamics of the state variable x under the risk-adjusted probability measure is described by a gaussian stochastic process (elastic random walk) of the form:

$$dx = (\alpha - \lambda_0 + \gamma x)dt + \sqrt{\xi_0} d\hat{z} . \quad (2.12.)$$

In this case, the instantaneous interest rate r is affine in x :

$$r = \varepsilon_0 + \varepsilon_1 x \quad (2.13.)$$

so that we can write the risk-adjusted process of r as:

$$dr = \kappa(\theta - r)dt + \sigma d\hat{z} \quad (2.14.)$$

where we define: $\kappa \equiv -\gamma$, $\theta \equiv \frac{\varepsilon_1(\alpha - \lambda_0)}{\gamma} - \varepsilon_0$, $\sigma \equiv \varepsilon_1 \sqrt{\xi_0}$.

This stochastic differential equation allows for negative values of the interest rate and has constant variance. The distribution function is Normal:

$$E(r_u | r_t) = \theta + (r_t - \theta)e^{-\kappa(u-t)}, \quad u > t \quad (2.15.)$$

$$\text{Var}(r_u | r_t) = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa(u-t)}), \quad u > t . \quad (2.16.)$$

To sum up, the dynamics of r is characterised by the following properties:

- Mean reversion: the interest rate tends to revert towards its long-term mean θ . The speed of adjustment depends on the mean reversion coefficient κ .
- Constant variance (homoskedasticity), which implies constant risk premia.
- Distribution function of r is Normal and admits both positive and negative values.

As the instantaneous interest rate r is affine in the unknown state variable x , it can be used in place of it as the only factor summarising the uncertainty in the economy. Therefore, we can write the price at time t of a unit discount bond maturing at time T as a function of r and

time, $P = P(r, t; T)$. Applying Ito's Lemma and the no-arbitrage condition as in equations (2.4) and (2.5) above, we get the partial differential equation which must be solved in order to determine the equilibrium price of the zero coupon bond:

$$\frac{1}{2} P_{rr} \sigma^2 + P_r \kappa (\theta - r) + P_t - rP = 0 . \quad (2.17.)$$

where: $P_r \equiv \frac{\partial P}{\partial r}$, $P_{rr} \equiv \frac{\partial^2 P}{\partial r^2}$, $P_t \equiv \frac{\partial P}{\partial t}$.

The boundary condition is:

$$P(r, T; T) = 1 . \quad (2.18.)$$

The model has closed form solution:

$$P(r, t; T) = A(\tau) \exp[-B(\tau)r] \quad (2.19.)$$

where: $\tau \equiv T - t$ and

$$A(\tau) \equiv \exp\left(\frac{1}{2\kappa^2} \left[(2\kappa^2\theta - \sigma^2)(B(\tau) - \tau) - \frac{\sigma^2\kappa}{2} B^2(\tau) \right]\right)$$

$$B(\tau) \equiv \frac{1 - e^{-\kappa\tau}}{\kappa} .$$

From (2.19) we obtain an expression for the term structure:

$$Y(r, t; T) = -\frac{\log[P(r, \tau)]}{\tau} = -\frac{\log[A(\tau)]}{\tau} + \frac{B(\tau)}{\tau} r . \quad (2.20.)$$

The yield-to-maturity of the zero coupon bond is affine in the instantaneous interest rate r and tends to it as the time-to-maturity approaches zero. Instead, as the time-to-maturity approaches infinity, the yield-to-maturity tends to a constant value, independent of r :

$$\lim_{\tau \rightarrow \infty} Y(r, \tau) \equiv Y(\infty) = \theta - \frac{\sigma^2}{2\kappa^2} . \quad (2.21.)$$

The term structure can assume three shapes:

- Monotone increasing, when $r < \theta - (3\sigma^2 / 4\kappa^2) = Y(\infty) - (\sigma^2 / 4\kappa^2)$
- Monotone decreasing, when $r > \theta = Y(\infty) + (\sigma^2 / 2\kappa^2)$
- Humped, when $\theta - (3\sigma^2 / 4\kappa^2) < r < \theta$.

Figure 1 contains some term structures simulated for different values of r , given the model parameters. Figure 2 shows that the corresponding term structure volatility is constant over time, given the homoskedasticity assumption. In fact, the volatility of the yields does not depend on r :

$$Var\{dY\} = \left(\frac{B(\tau)}{\tau} \right)^2 \sigma^2 . \quad (2.22.)$$

2.2.b Square-root process: Cox-Ingersoll-Ross model

In the CIR (1985b) model it is assumed that the unknown state variable x follows a square root mean reverting process:

$$dx = (\alpha + (\gamma - \lambda_1)x)dt + \sqrt{\xi_1 x} d\hat{z} . \quad (2.23.)$$

As seen above, this specification implies that the instantaneous interest rate is proportional to x . Then, the risk-adjusted process for r is given by:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}d\hat{z} \quad (2.24.)$$

where we define: $\kappa \equiv -(\gamma - \lambda_1)$, $\theta \equiv \frac{\varepsilon_1 \alpha}{\gamma - \lambda_1}$, $\sigma \equiv \sqrt{\varepsilon_1 \xi_1}$.

This process allows only for non-negative values of the interest rate and is such that, when the interest rate reaches zero, it subsequently becomes positive. The variance of the interest rate is proportional to its level.

The distribution function is the non-central chi-square with:

$$E(r_u | r_t) = \theta + (r_t - \theta)e^{-\kappa(u-t)}, \quad u > t \quad (2.25.)$$

$$\text{Var}(r_u | r_t) = \frac{\sigma^2}{\kappa} \left[r_t (e^{-\kappa(u-t)} - e^{-2\kappa(u-t)}) + \frac{\theta}{2} (1 - e^{-\kappa(u-t)})^2 \right], \quad u > t. \quad (2.26.)$$

From (2.25) it follows that, as the mean reversion coefficient κ increases, the conditional mean of r tends to the long-term mean θ and the variance decreases. Instead, when mean reversion is very low, the conditional mean of r tends to coincide with the current value r_t and the conditional variance tends to a value which is proportional to the time interval: $\sigma^2 r_t (u-t)$. The steady state density function is a gamma distribution, with mean value θ and variance $\theta\sigma^2 / 2\kappa$.

To sum up, the stochastic process describing the dynamics of r is characterised by the following properties:

- Mean reversion: the interest rate tends to revert towards its long-term mean θ . The speed of adjustment depends on the mean reversion coefficient κ .
- Variance and risk premia proportional to the level of r : particular form of heteroskedasticity.
- Non-central chi-square distribution function which admits only non-negative values.

As r is proportional to x , then r itself can be used as the state variable governing the dynamics of the economy. Following the same steps as in equations (2.4) and (2.5) above, we can derive the partial differential equation for the equilibrium price of a unit discount bond, $P = P(r, t; T)$:

$$\frac{1}{2} P_{rr} \sigma^2 r + P_r \kappa (\theta - r) + P_t - rP = 0. \quad (2.27.)$$

where: $P_r \equiv \frac{\partial P}{\partial r}$, $P_{rr} \equiv \frac{\partial^2 P}{\partial r^2}$, $P_t \equiv \frac{\partial P}{\partial t}$.

This equation is solved subject to the following boundary condition:

$$P(r, T; T) = 1. \quad (2.28.)$$

The problem has a closed form solution, which allows us to write the equilibrium price of the zero coupon bond in terms of the model parameters:

$$P(r, t; T) = A(\tau) \exp[-B(\tau)r] \quad (2.29.)$$

where: $\tau \equiv T - t$ and

$$A(\tau) \equiv \left(\frac{2\zeta e^{(\kappa+\zeta)\tau/2}}{(\kappa+\zeta)(e^{\zeta\tau} - 1) + 2\zeta} \right)^{\frac{2\kappa\theta}{\sigma^2}}$$

$$B(\tau) \equiv \frac{2(e^{\zeta\tau} - 1)}{(\kappa+\zeta)(e^{\zeta\tau} - 1) + 2\zeta}$$

$$\zeta \equiv \sqrt{\kappa^2 + 2\sigma^2} .$$

From the closed form solution (2.29) we derive an expression for the yield-to-maturity on the zero coupon bond and, therefore, for the term structure:

$$Y(r, t; T) = -\frac{\log[P(r, \tau)]}{\tau} = -\frac{\log[A(\tau)]}{\tau} + \frac{B(\tau)}{\tau} r . \quad (2.30.)$$

The yield-to-maturity is affine in the instantaneous interest rate r and converges to it as the time-to-maturity of the zero coupon bond approaches zero. As the time-to-maturity tends to infinity, the yield-to-maturity approaches a constant value:

$$\lim_{\tau \rightarrow \infty} Y(r, \tau) \equiv Y(\infty) = \frac{2\kappa\theta}{\kappa + \zeta} . \quad (2.31.)$$

The term structure can assume three shapes:

- Monotone increasing, when $r < Y(\infty)$
- Monotone decreasing, when $r > \theta$
- Humped, when $Y(\infty) < r < \theta$.

Figure 3 shows some term structures generated by the model for different values of the instantaneous interest rate r , given the model parameters. Figure 4 shows the corresponding term structure volatility. This is obtained from (2.30) and is proportional to the level of r :

$$\text{Var}\{dY\} = \left(\frac{B(\tau)}{\tau} \right)^2 \sigma^2 r . \quad (2.32.)$$

2.2.c Translated square-root processes: Pearson-Sun model

Pearson and Sun (1994) model assumes that the risk-adjusted stochastic process describing the dynamics of the unknown state variable x follows a translated square root mean reverting process:

$$dx = (\alpha - \lambda_0 + (\gamma - \lambda_1)x)dt + \sqrt{\xi_0 + \xi_1 x}d\hat{z} . \quad (2.33.)$$

As with this specification the instantaneous interest rate is affine in x , we have:

$$dr = \kappa(\theta - r)dt + \sqrt{\sigma_0^2 + \sigma_1^2 r}d\hat{z} \quad (2.34.)$$

where: $\kappa \equiv -(\gamma - \lambda_1)$, $\theta \equiv \frac{\varepsilon_1(\alpha - \lambda_0)}{\gamma - \lambda_1} - \varepsilon_0$, $\sigma_0 \equiv \sqrt{\varepsilon_1(\varepsilon_1\xi_0 + \varepsilon_0\xi_1)}$, $\sigma_1 \equiv \sqrt{\varepsilon_1\xi_1}$.

This process has the following properties:

- Mean reversion: the interest rate tends to revert towards its long-term mean θ . The speed of adjustment depends on the mean reversion coefficient κ .
- Variance affine function of the level of r : heteroskedasticity.
- Translated non-central chi-square distribution function, which admits also negative values.

The translated square root process seems to combine the features of the processes used by CIR and Vasicek. The Pearson-Sun process (2.34) can also be derived in the following way, which allows us to better understand how the “translation” works.

Let the instantaneous interest rate r be given by the sum of two components, a negative constant and a stochastic process of the square root type:

$$r = \bar{r} + r_1 \quad (2.35.)$$

where:

$$dr_1 = \kappa(\theta_1 - r_1)dt + \sigma_1\sqrt{r_1}d\hat{z} \quad (2.36.)$$

$$\bar{r} = -\frac{\sigma_0^2}{\sigma_1^2} \quad (2.37.)$$

and all the parameters are positive. The dynamics of r is given by:

$$\begin{aligned} dr = dr_1 &= \kappa(\theta_1 - (r - \bar{r}))dt + \sigma_1 \sqrt{r - \bar{r}} d\hat{z} = \\ &= \kappa \left[\left(\theta_1 - \frac{\sigma_0^2}{\sigma_1^2} \right) - r \right] dt + \sqrt{\sigma_1^2 \left(r + \frac{\sigma_0^2}{\sigma_1^2} \right)} d\hat{z} . \end{aligned} \quad (2.38.)$$

Simplifying and placing $\theta \equiv \theta_1 - (\sigma_0^2 / \sigma_1^2)$, we obtain (2.34).

The constant \bar{r} corresponds to a translation to the right of the vertical axis in the non-central chi-square distribution function of r .

As in the case of the Vasicek and CIR models, the instantaneous interest rate r can be used as the single factor which summarises the uncertainty in the economy. In fact, r is affine in x and so can substitute it as state variable underlying the dynamics of the economy. Therefore, we can write the price at time t of a unit discount bond maturing at time T as a function of r and time, $P = P(r, t; T)$, and apply Ito's Lemma and the no-arbitrage condition as in equations (2.4) and (2.5) above. We get the partial differential equation which must be solved in order to determine the equilibrium price of the zero coupon bond:

$$\frac{1}{2} P_{rr} (\sigma_0^2 + \sigma_1^2 r) + P_r \kappa (\theta - r) + P_t - rP = 0 \quad (2.39.)$$

with boundary condition:

$$P(r, T; T) = 1 . \quad (2.40.)$$

where: $P_r \equiv \frac{\partial P}{\partial r}$, $P_{rr} \equiv \frac{\partial^2 P}{\partial r^2}$, $P_t \equiv \frac{\partial P}{\partial t}$.

The problem has closed form solution:

$$P(r, t; T) = A(\tau) \exp[-B(\tau)r] \quad (2.41.)$$

where: $\tau \equiv T - t$ and

$$A(\tau) \equiv \left(\frac{2\zeta e^{(\kappa+\zeta)\tau/2}}{(\kappa+\zeta)(e^{\zeta\tau}-1)+2\zeta} \right)^{2\frac{\kappa(\theta+(\sigma_0^2/\sigma_1^2))}{\sigma_1^2}} \cdot e^{-\frac{\sigma_0^2}{\sigma_1^2}B(\tau)-\tau}$$

$$B(\tau) \equiv \frac{2(e^{\zeta\tau}-1)}{(\kappa+\zeta)(e^{\zeta\tau}-1)+2\zeta}$$

$$\zeta \equiv \sqrt{\kappa^2 + 2\sigma_1^2} .$$

By inspection, we can easily verify that:

- a) if $\sigma_0 = 0$, then the coefficients of (2.41) are the same as in the CIR model;
- b) if $\sigma_1 = 0$, then the coefficients of (2.41) are the same as in the Vasicek model.

From the closed form solution we derive an expression for the term structure, where the yields are affine in r :

$$Y(r, t; T) = -\frac{\log[P(r, \tau)]}{\tau} = -\frac{\log[A(\tau)]}{\tau} + \frac{B(\tau)}{\tau} r . \quad (2.42.)$$

As time-to-maturity approaches infinity, yield-to-maturity tends to a constant value:

$$\lim_{\tau \rightarrow \infty} Y(r, \tau) \equiv Y(\infty) = \frac{2\sigma_1^2 \kappa \theta + \sigma_0^2 (\kappa - \zeta)}{\sigma_1^2 (\kappa + \zeta)} . \quad (2.43.)$$

As in the Vasicek and CIR case, the term structure can assume only three different shapes:

- Monotone increasing, when $r < Y(\infty)$
- Monotone decreasing, when $r > \theta$
- Humped, when $Y(\infty) < r < \theta$.

Figure 5 shows some term structures obtained for different values of r , given the model parameters, whereas figure 6 contains the corresponding term structure volatility. In this case, the variance of the yields is affine in r :

$$\text{Var}\{dY\} = \left(\frac{B(\tau)}{\tau} \right)^2 (\sigma_0^2 + \sigma_1^2 r) . \quad (2.44.)$$

Comparing figure 5 with figures 1 and 3, we can observe that the term structures generated by the Pearson-Sun, Vasicek and CIR models look quite similar. This is due to the fact that single-factor models admit only three different shapes for the term structure. The models can instead produce substantially different term structure volatilities.

A digression about the specification of the stochastic process of r

The restrictions a) and b) above could be tested by estimating the Pearson-Sun specification, as it encompasses both the CIR and the Vasicek model. A more general specification (which, however, does not have closed form solution for the price of a unit discount bond) has been tested by Chan-Karolyi-Longstaff-Sanders (CKLS, 1992):

$$dr = (\alpha_0 + \alpha_1 r)dt + (\sigma_0^2 + \sigma_1^2 r)^{\delta} dz .$$

Estimating this model, CKLS can test some of the specifications which have been proposed in the literature to describe the dynamics of the instantaneous interest rate:

	α_0	α_1	σ_0	σ_1	δ	closed form solution
CIR	*	*		*	0.5	*
Vasicek	*	*	*		1	*
Pearson-Sun	*	*	*	*	0.5	*
Merton	*		*		1	*
Brennan-Schwartz	*	*		*	1	
CKLS	*	*	*	*	free	

2.3. Multifactor models

Single-factor term structure models undergo two main limitations:

- 1) They account only for a limited number of shapes of the term structure. In fact, as we have seen in the previous section, the CIR, Vasicek and Pearson and Sun models admit only three different shapes: monotone increasing, monotone decreasing and humped. Instead, the observed term structure can assume a variety of shapes, including a trough, a shape which occurs when short-term and long-term yields are higher than intermediate maturity yields.
- 2) They imply that yields of any maturity are perfectly correlated. To see this, consider the CIR model and take two zero coupon bond with maturities τ_1 and τ_2 , respectively. We obtain:

$$dY(\tau_1) = \frac{B(\tau_1)}{\tau_1} dr = \frac{B(\tau_1)}{\tau_1} [\kappa(\theta - r)dt + \sigma\sqrt{r}dz] \quad (2.45.)$$

$$dY(\tau_2) = \frac{B(\tau_2)}{\tau_2} dr = \frac{B(\tau_2)}{\tau_2} [\kappa(\theta - r)dt + \sigma\sqrt{r}dz] \quad (2.46.)$$

$$Corr\{dY(\tau_1), dY(\tau_2)\} = \frac{(B(\tau_1)/\tau_1)(B(\tau_2)/\tau_2)\sigma^2 r}{\sqrt{(B(\tau_1)\sigma\sqrt{r}/\tau_1)(B(\tau_2)\sigma\sqrt{r}/\tau_2)}} = 1, \quad \forall \tau_1, \tau_2. \quad (2.47.)$$

The following table shows that this assumption is too restrictive, as actual correlations between yields generally are significantly lower than one. The table contains the correlation between daily changes in Italian lira swap rates of different maturities for the period January 1994 - September 1995.

Correlation between lira denominated swap rates

	2 years	3 years	5 years
3 years	0.97		
5 years	0.90	0.95	
10 years	0.85	0.92	0.97

These two limitations can be overcome by a multifactor model.

The literature on the term structure of interest rates comprises a variety of multifactor models, where different variables are used as state variables: short rate, long rate, spread between short and long rate, expected inflation rate, forward rates, short rate volatility, and so on. Moreover, all these models differ for the type of stochastic processes used in modelling the dynamics of the state variables.

In this section, we derive some of these models: the Longstaff and Schwartz (1992) model, which uses as state variables the instantaneous interest rate and its volatility following square root processes; the Langetieg (1980) model, based on N unknown factors whose dynamics is governed by an elastic random walk process; the Duffie and Kan (1996) model, where the state variables are a set of observed yields which can follow either gaussian or square root type processes; the Balduzzi, Das, Foresi and Sundaram (BDFS, 1998) model, which extends the Vasicek model allowing for a stochastic long-term mean.

In the multifactor framework, the choice of the stochastic process has relevant implications for the existence of a closed form solution, as the state variables can be correlated. The gaussian-type specification admits a closed form solution both in the case of correlated and uncorrelated state variables. A closed form solution for the square root and translated square root specification can always be obtained when the state variables are uncorrelated. When they are correlated, instead, some (relatively strong) restrictions must be imposed on model coefficients in order to get a closed form solution²¹.

²¹ See Duffie and Kan (1996) and Dai and Singleton (1998).

2.3.a Interest rate volatility: Longstaff-Schwartz model

Assume that the dynamics of the economy is affected by two unknown state variables, x_1 and x_2 , which follow risk-adjusted uncorrelated square root mean reverting processes of the type:

$$dx_i = (\alpha_i + (\gamma_i - \lambda_{1i})x_i)dt + \sqrt{\xi_{1i}x_i}d\hat{z}_i, \quad i = 1,2 \quad (2.48.)$$

From the general equilibrium framework, it follows that the instantaneous interest rate r is a linear combination of the two unknown state variables:

$$r = \varepsilon_1 x_1 + \varepsilon_2 x_2. \quad (2.49.)$$

Now, define:

$$y_i \equiv \varepsilon_i x_i, \quad i = 1,2$$

so that we have:

$$r = y_1 + y_2. \quad (2.50.)$$

For this new set of state variables, we get:

$$dy_i = \kappa_i(\theta_i - y_i)dt + \sigma_i \sqrt{y_i}d\hat{z}_i, \quad i = 1,2 \quad (2.51.)$$

where we define: $\kappa_i \equiv -(\gamma_i - \lambda_{1i})$, $\theta_i \equiv \frac{\varepsilon_i \alpha_i}{\gamma_i - \lambda_{1i}}$, $\sigma_i \equiv \sqrt{\varepsilon_i \xi_{1i}}$, $i = 1,2$.

From equation (2.50) we can calculate the volatility of the instantaneous interest rate:

$$V \equiv \text{Var}(dr) = \text{Var}(dy_1) + \text{Var}(dy_2) = \sigma_1^2 y_1 + \sigma_2^2 y_2 \quad (2.52.)$$

Putting together equation (2.50) and (2.52), we obtain a two-equation system:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sigma_1^2 & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} r \\ V \end{pmatrix} = \begin{pmatrix} \frac{\sigma_2^2 r - V}{\sigma_2^2 - \sigma_1^2} \\ \frac{V - \sigma_1^2 r}{\sigma_2^2 - \sigma_1^2} \end{pmatrix} \quad (2.53.)$$

This expression implies that: $y_1 \geq 0$ if $\sigma_2^2 \geq \sigma_1^2$ and $\sigma_2^2 r \geq V$; $y_2 \geq 0$ if $\sigma_2^2 \geq \sigma_1^2$ and $V \geq \sigma_1^2 r$. Therefore, in order to have both y_1 and y_2 non-negative (implication of the square-root specification), the following relation must hold:

$$\sigma_2^2 \geq \frac{V}{r} \geq \sigma_1^2. \quad (2.54.)$$

Using (2.53), we can derive the stochastic processes of r and V :

$$\begin{aligned} dr = & \left[(\kappa_1 \theta_1 + \kappa_2 \theta_2) + \frac{\kappa_2 \sigma_1^2 - \kappa_1 \sigma_2^2}{\sigma_2^2 - \sigma_1^2} r + \frac{\kappa_1 - \kappa_2}{\sigma_2^2 - \sigma_1^2} V \right] dt + \\ & + \sqrt{\frac{\sigma_1^2 (\sigma_2^2 r - V)}{\sigma_2^2 - \sigma_1^2}} d\hat{z}_1 + \sqrt{\frac{\sigma_2^2 (V - \sigma_1^2 r)}{\sigma_2^2 - \sigma_1^2}} d\hat{z}_2 \end{aligned} \quad (2.55.)$$

$$\begin{aligned} dV = & \left[(\kappa_1 \theta_1 \sigma_1^2 + \kappa_2 \theta_2 \sigma_2^2) + \sigma_1^2 \sigma_2^2 \frac{\kappa_2 - \kappa_1}{\sigma_2^2 - \sigma_1^2} r + \frac{\kappa_1 \sigma_1^2 - \kappa_2 \sigma_2^2}{\sigma_2^2 - \sigma_1^2} V \right] dt + \\ & + \sigma_1^2 \sqrt{\frac{\sigma_1^2 (\sigma_2^2 r - V)}{\sigma_2^2 - \sigma_1^2}} d\hat{z}_1 + \sigma_2^2 \sqrt{\frac{\sigma_2^2 (V - \sigma_1^2 r)}{\sigma_2^2 - \sigma_1^2}} d\hat{z}_2 \end{aligned} \quad (2.56.)$$

Note that the stochastic processes for r and V are correlated. Hence, although the model has been originally specified in terms of two uncorrelated unknown state variables, it could be transformed into a model with correlated state variables through a change of variables.

In the following, we determine the equilibrium price of a unit discount bond. We first derive the closed form solution as a function of y_1 and y_2 and then use (2.53) to express it in terms of the “new” state variables r and V .

Let $P = P(y_1, y_2, t; T)$ denote the price at time t of a unit discount bond maturing at time T . Applying Ito’s Lemma and the no-arbitrage condition as in equations (2.4) and (2.5) above, we obtain the following partial differential equation:

$$\frac{1}{2} \sum_{i=1,2} P_{y_i y_i} \sigma_i^2 y_i + \sum_{i=1,2} P_{y_i} \kappa_i (\theta_i - y_i) + P_t - rP = 0 \quad (2.57.)$$

with boundary condition:

$$P(y_1, y_2, T; T) = 1 . \quad (2.58.)$$

where: $P_{y_i} \equiv \frac{\partial P}{\partial y_i}$, $P_{y_i y_i} \equiv \frac{\partial^2 P}{\partial y_i^2}$, $P_t \equiv \frac{\partial P}{\partial t}$.

As y_1 and y_2 are uncorrelated, the solution for the price of the zero coupon bond is given by the product of two separate components:

$$P(y_1, y_2, t; T) = P(y_1, t; T) \cdot P(y_2, t; T) \quad (2.59.)$$

which is equivalent to the following Feynman-Kac solution:

$$P(y_1, y_2, t; T) = \hat{E}_t \left\{ \exp \left[- \int_t^T r(u) du \right] \right\} = \hat{E}_t \left\{ \exp \left[- \int_t^T y_1(u) du \right] \right\} \cdot \hat{E}_t \left\{ \exp \left[- \int_t^T y_2(u) du \right] \right\} . \quad (2.60.)$$

The problem has a closed form solution, which allows us to write the equilibrium price of the zero coupon bond in terms of the model parameters:

$$P(y_1, y_2, t; T) = A_1(\tau) \cdot A_2(\tau) \cdot \exp[-B_1(\tau)y_1 - B_2(\tau)y_2] \quad (2.61.)$$

where: $\tau \equiv T - t$ and

$$A_i(\tau) \equiv \left(\frac{2\zeta_i e^{(\kappa_i + \zeta_i)\tau/2}}{(\kappa_i + \zeta_i)(e^{\zeta_i\tau} - 1) + 2\zeta_i} \right)^{\frac{2\kappa_i\theta_i}{\sigma_i^2}}$$

$$B_i(\tau) \equiv \frac{2(e^{\zeta_i\tau} - 1)}{(\kappa_i + \zeta_i)(e^{\zeta_i\tau} - 1) + 2\zeta_i}$$

$$\zeta_i \equiv \sqrt{\kappa_i^2 + 2\sigma_i^2} , \quad i = 1, 2 .$$

Now, using relation (2.53), we can rewrite the equilibrium solution for the price of the zero coupon bond in terms of the “observables” r and V . We have:

$$P(r, V, t; T) = A(\tau) \exp[-B_r(\tau)r - B_V(\tau)V] \quad (2.62.)$$

where:

$$\begin{aligned} A(\tau) &\equiv A_1(\tau) \cdot A_2(\tau) \\ B_r(\tau) &\equiv \frac{\sigma_1^2 B_2(\tau) - \sigma_2^2 B_1(\tau)}{\sigma_2^2 - \sigma_1^2} \\ B_V(\tau) &\equiv \frac{B_1(\tau) - B_2(\tau)}{\sigma_2^2 - \sigma_1^2} . \end{aligned}$$

The term structure is given by:

$$Y(r, V, t; T) = -\frac{\log[P(r, V, \tau)]}{\tau} = -\frac{\log[A(\tau)]}{\tau} + \frac{B_r(\tau)}{\tau} r + \frac{B_V(\tau)}{\tau} V . \quad (2.63.)$$

The yields are affine in the instantaneous interest rate and its volatility. The form of the term structure depends on these two variables and can take a variety of shapes, including some which are precluded to single-factor models: monotone increasing, monotone decreasing, with a hump, a trough or both of them.

Figure 7 shows term structures obtained for different values of r and V , given the model parameters, whereas figure 8 contains the corresponding term structure volatility. Figure 7 illustrates the fact that, differently from the single-factor models, a multifactor model allows for a trough in the term structure.

The volatility of the yields is proportional to the state variables r and V :

$$\text{Var}\{dY\} = \frac{1}{\tau^2} \left\{ \begin{aligned} &[B_r^2(\tau) + B_V^2(\tau)[(\sigma_1^2 + \sigma_2^2)^2 - \sigma_1^2\sigma_2^2] + 2B_r(\tau)B_V(\tau)(\sigma_1^2 + \sigma_2^2)]V + \\ &[-\sigma_1^2\sigma_2^2[(\sigma_1^2 + \sigma_2^2)B_V^2(\tau) + 2B_r(\tau)B_V(\tau)]r \end{aligned} \right\} . \quad (2.64.)$$

As time-to-maturity approaches infinity, yields tend to a constant value, which does not depend on r and V :

$$\lim_{\tau \rightarrow \infty} Y(r, V, \tau) \equiv Y(\infty) = \sum_{i=1,2} \kappa_i \theta_i (\zeta_i - \kappa_i) . \quad (2.65.)$$

2.3.b Unknown factors: Langetieg model

The Langetieg (1980) model assumes that the dynamics of the economy is governed by N unknown state variables following an elastic random walk process:

$$dx_i = \left(\alpha_i - \lambda_{0i} + \sum_{j=1}^N \gamma_{ij} x_j \right) dt + \sqrt{\xi_{0i}} d\hat{z}_i , \quad i = 1, 2, \dots, N . \quad (2.66.)$$

In this case, the general equilibrium framework implies that the instantaneous interest rate r is an affine function of the factors. Then, we can define a new set of state variables:

$$y_i \equiv \varepsilon_{0i} + \varepsilon_i x_i , \quad i = 1, 2, \dots, N , \quad (2.67.)$$

with: $\sum_{i=1}^N \varepsilon_{0i} = \varepsilon_0$, such that:

$$r = \sum_{i=1}^N y_i . \quad (2.68.)$$

The stochastic process for this new set of state variables is:

$$dy_i = \left(\phi_i - \sum_{j=1}^N \kappa_{ij} y_j \right) dt + \sigma_i d\hat{z}_i , \quad i = 1, 2, \dots, N , \quad (2.69.)$$

where we define: $\kappa_{ij} \equiv -\frac{\gamma_{ij} \varepsilon_i}{\varepsilon_j}$, $\phi_i \equiv \varepsilon_i (\alpha_i - \lambda_{0i}) - \sum_{j=1}^N \frac{\varepsilon_i \varepsilon_{0j} \gamma_{ij}}{\varepsilon_j}$, $\sigma_i \equiv \varepsilon_i \sqrt{\xi_{0i}}$, $i = 1, \dots, N$.

In matrix form:

$$dy = K(\Theta - y)dt + \Sigma d\hat{z} \quad (2.70.)$$

where: $\Theta \equiv K^{-1}\Phi$, Φ being the vector containing the ϕ_i coefficients.

We allow for correlation among the state variables, so that:

$$E(d\hat{z}, d\hat{z}') = \Omega \neq I . \quad (2.71.)$$

Given this framework, we can determine the equilibrium price at time t of a unit discount bond maturing at time T , $P = P(y, t; T)$. We apply Ito's Lemma and the no-arbitrage condition as in (2.4) and (2.5) above and obtain the partial differential equation:

$$\frac{1}{2} P_{yy} * (\Sigma \Omega \Sigma) + P_y' K(\Theta - y) + P_t - rP = 0 \quad (2.72.)$$

with boundary condition:

$$P(y, T; T) = 1 \quad (2.73.)$$

where: $P_y \equiv \frac{\partial P}{\partial y}$, $P_{yy} \equiv \frac{\partial^2 P}{\partial y \partial y'}$, $P_t \equiv \frac{\partial P}{\partial t}$.

This is a Cauchy problem, which has a closed form solution of the type:

$$P(y, t; T) = A(\tau) \exp[-B(\tau)' y] \quad (2.74.)$$

where $\tau = T - t$ and ²²:

²² Matrix Ψ is a transformation of the correlation matrix Ω . To obtain it, first define with Z a matrix formed using as columns the eigenvectors of K and with U the diagonal matrix containing the eigenvalues of K and consider the following transformation:

$$\Omega^* = Z^{-1} \Sigma \Omega \Sigma (Z^{-1})' .$$

$$A(\tau) \equiv \exp \left(\begin{array}{l} -\mathbf{1}'(\Delta(\tau) - \mathbf{I}\tau)\Theta + \frac{1}{2}B(\tau)'\Psi B(\tau) + \\ -\frac{1}{2}\mathbf{1}'(\mathbf{K}^{-1}\Delta(\tau)\Psi + \Psi\Delta(\tau)'(\mathbf{K}^{-1})')\mathbf{1} + \frac{1}{2}\mathbf{1}'\mathbf{K}^{-1}\Sigma\Omega\Sigma(\mathbf{K}^{-1})'\mathbf{1}\tau \end{array} \right)$$

$$B(\tau) \equiv \Delta(\tau)'\mathbf{1}$$

$$\Delta(\tau) \equiv \mathbf{K}^{-1}(\exp[\mathbf{K}\tau] - \mathbf{I}) .$$

$\mathbf{1}$ is a N vector of ones.

The term structure is given by:

$$Y(y, t; T) = -\frac{\log[P(y, \tau)]}{\tau} = \frac{1}{\tau} \left(-\log[A(\tau)] + B(\tau)'y \right). \quad (2.75.)$$

As time-to-maturity approaches infinity, yields tend to a constant value:

$$\lim_{\tau \rightarrow \infty} Y(y, \tau) \equiv Y(\infty) = \mathbf{1}'\Theta - \frac{1}{2}\mathbf{1}'\mathbf{K}^{-1}\Sigma\Omega\Sigma(\mathbf{K}^{-1})'\mathbf{1} . \quad (2.76.)$$

In the case of uncorrelated univariate processes, i.e., \mathbf{K} diagonal and $\Omega = \mathbf{I}$, the term structure would be given by the super-position of N independent term structures:

$$Y(y, \tau) = \sum_{i=1}^N Y(y_i, \tau) = \frac{1}{\tau} \sum_{i=1}^N \left(-\log[A_i(\tau)] + B_i(\tau)y_i \right) \quad (2.77.)$$

where the coefficients have the same form as in Vasicek's model (2.19):

$$A_i(\tau) \equiv \exp \left(\frac{1}{2\kappa_i^2} \left[(2\kappa_i^2\theta_i - \sigma_i^2)(B_i(\tau) - \tau) - \frac{\sigma_i^2\kappa_i}{2}B_i^2(\tau) \right] \right)$$

Then divide each element of this matrix, ω_{ij}^* , by the sum of the corresponding eigenvalues of \mathbf{K} , i.e., $u_i + u_j$, $i, j = 1, \dots, N$. These elements form a new transformed matrix Ω^{**} which can be used to obtain:

$$\Psi = Z\Omega^{**}Z' .$$

$$B_i(\tau) \equiv \frac{1 - e^{-\kappa_i \tau}}{\kappa_i}, \quad i = 1, 2, \dots, N.$$

2.3.c Set of yields: Duffie-Kan model

Using an inversion methodology, from Langetieg model we can derive the Duffie and Kan (1996) specification, which uses as state variables a set of N yields. The inversion methodology consists of expressing the state variables as linear functions of (a subset of) yields.

The inversion follows from the affine properties of Langetieg model and may be applied along the lines suggested by Duffie and Kan (1996). Since yields are affine in the original unknown state variables, then, provided that we have more yields than state variables, we can express each state variable as a linear combination of a subset of yields. Hence, bond prices and the term structure may be rewritten in terms of this subset of yields which can be observed from the current term structure.

As an example, consider the closed form solution (2.75) with $N = 2$ and denote with \bar{Y}_1 and \bar{Y}_2 the yields included in the basis set and τ_1 and τ_2 their maturities. In this case, we have:

$$y_h = \frac{\sum_{j=1}^2 (-1)^{j-h+1} B_{3-h}(\tau_{3-j}) \left[\sum_{i=1}^2 \log A_i(\tau_j) + \tau_j \bar{Y}_j \right]}{B_h(\tau_{3-h}) B_{3-h}(\tau_h) - B_h(\tau_h) B_{3-h}(\tau_{3-h})}, \quad h = 1, 2. \quad (2.78.)$$

Any yield can then be expressed as an affine function of the yields contained in the basis set. In general, for any N , we get the following expression for the term structure:

$$Y(\bar{Y}_1, \dots, \bar{Y}_N, t; T) = \sum_{i=1}^N \left[-\frac{\log(A_i(\tau))}{\tau} + \frac{B_i(\tau)}{\tau} \left(\alpha_i(\tau_1, \dots, \tau_N) + \sum_{j=1}^N (\beta_{ij}(\tau_1, \dots, \tau_N) \bar{Y}_j) \right) \right] \quad (2.79.)$$

where the coefficients $\alpha_i(\tau_1, \dots, \tau_N)$ and $\beta_{ij}(\tau_1, \dots, \tau_N)$ are functions of the parameters $A_i(\tau_j)$ and $B_i(\tau_j)$, $i, j = 1, 2, \dots, N$, of equation (2.74) and the τ_j are the maturities of the bonds whose yields are included in the basis set.

The same inversion methodology might be applied to multifactor models specified in terms of N unknown state variables following square root or translated square root processes²³.

In addition, the methodology works also in deriving two-factor term structure models using as state variables a long-term yield and the spread between the short rate and the long-term yield, such as Nelson and Schaefer (1983), Schaefer and Schwartz (1984) and Brown and Schaefer (1994b). In fact, the spread between two yields is still a linear function of the state variables.

2.3.d Stochastic long-term mean: BDFS model

Take the single-factor Vasicek model and assume that the long-term mean of the instantaneous interest rate in equation (2.14) is stochastic and follows a gaussian process as well. Thus, we have:

$$dr = \kappa_r(\theta - r)dt + \sigma_r d\hat{z}_r \quad (2.80.)$$

$$d\theta = \kappa_\theta(\phi - \theta)dt + \sigma_\theta d\hat{z}_\theta \quad (2.81.)$$

According to Balduzzi, Das, Foresi and Sundaram (BDFS, 1998), the time-varying long-term mean can be interpreted as the target level of the short interest rate governed by the monetary authority. For example, the Federal funds rate in the U.S. case. Short-term interest rates tend to converge towards this monetary policy rate, with speed of adjustment which depends on the mean reversion coefficient κ_r .

Allowing for a stochastic long-term mean in the Vasicek specification, we get a two-factor model in which the state variables are the instantaneous interest rate r and its time-varying central tendency θ .

In matrix form:

$$dy = K(\Theta - y)dt + \Sigma d\hat{z} \quad (2.82.)$$

where:

$$K \equiv \begin{pmatrix} \kappa_r & -\kappa_r \\ 0 & \kappa_\theta \end{pmatrix}, \quad \Theta \equiv K^{-1} \begin{pmatrix} 0 \\ \kappa_\theta \phi \end{pmatrix}, \quad \Sigma \equiv \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_\theta \end{pmatrix}, \quad E(d\hat{z}, d\hat{z}') = \Omega \neq I.$$

²³ Some restrictions must be imposed in the case of correlated processes. On this point, see Duffie and Kan (1996) and Dai and Singleton (1998).

This specification corresponds to Langetieg model (2.70 - 2.71) in the case $N = 2$.

Given this framework, we can follow the same steps as in equations (2.72 – 2.75) and determine the equilibrium term structure. We get:

$$Y(r, \theta, t; T) = \frac{1}{\tau} (-\log[A(\tau)] + B_r(\tau)r + B_\theta(\tau)\theta) . \quad (2.83.)$$

The yields are affine in the two factors. The term structure can assume different shapes, such as monotone increasing, monotone decreasing, with a hump, a trough or both of them.