Macroeconomics

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Contents

Preface

Chapter 1

Monday January 22, 2007 - Copied from the notes of Peter

1.1 Introduction: Dynamic Programming

1.1.1 Simple Robinson Crusoe example

One way to describe an individual's optimization problem on a dynamic context.

Consider Robinson Crusoe story for motivation. We have an individual that is infinitely live. There is one good, coconuts, which is used for consumption (denoted c when we are talking about consumption) and production (denoted k when we are talking about production). Utility is given by $u(c)$ and production by $f(k)$.

Our goal will be to maximize lifetime discounted utility (with discount β), where our objective function is $u(\vec{c}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$. So we must choose a feasible sequence $(c_1, c_2, \ldots)$. Our constraint is $0 \leq c_t + k_{t+1} \leq f(k_t), \forall t$ given k_0.

Let's assume $f(k)$ and $u(c)$ are increasing, we can substitute in for our objective and constraint functions to $\sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}$ and $k_{t+1} \leq f(k_t)$.

Remark 1.1.1. *We can try to solve using the calculus of variations, using the Euler equation we get* $u'[f(k_t) - k_{t+1}] = \beta u'[f(k_{t+1} - k_{t+2}] f'(k_{t+1}), \forall t$.

Question: given k_0*, how do you know that there exists a sequence* $\{k_t\}_{k=0}^{\infty}$ *solving this equation? For a finite sequence with a boundary that requires in the last round to have* $k = 0$ *we can use the Weierstrass theorem to show existance, but for infinite sequences, we do not know.*

1.1.2 A twist using randomness

Let Z be a random variable which represents a technology shock. Our new production function will be $Zf(k)$. Our problem uses the same objective function, but a new constraint. $u(\vec{c}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$. Our new constraint is $0 \leq k_{t+1} \leq Z_t f(k_t)$. Before, our solution was a sequence of numbers, but now with Z, our solution will be a sequence of functions. We will use dynamic programming to deal with optimization with and without uncertainty.

1.1.3 Derivation of a dynamic equation

We can expand our utility maximization problem to $v(\vec{c}) = \max \sum_{t=0}^{\infty} \beta^t u(c_t) = \max u(c) + \sum_{t=1}^{\infty} \beta^t u(c_t) = u(c) + v(k'$ where k' is correctlydefined. So our first look at a dynamic equation is $v(K) = \max u(c) + v(k')$.

Chapter 2

Friday January 26, 2007

2.1 Introduction

Study first six chapters in quite a bit of depth.

2.2 Math Concepts

Functional equation (FE), Bellman equation (BE) $v(k) = \max u(c) + \beta v(k')$ s.t. $0 \leq c + k' \leq f(k)$ Sequential problem (SP) $v(k) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$ st $0 \leq x_t + k_{t+1} \leq f(k_t) \forall t \geq 0$ $k_0 = k$

2.2.1 value equation

$v(p, w) = \max_{x \geq 0} u(x)$ a value equation st $px \leq w$

given k_0 we can calculate total utility based on utility functions and production. $v(k_0) = \max_{\vec{c},\vec{k}} u(c_0) + \beta u(c_1)$ - here we assume that we can additively separate utility (not necessarily true), constant discounted utility function over time, st $0 \leq c_0 + k_1 \leq f(k_0)$ $0 \leq c_1 + k_2 \leq f(k_1)$ $k_0 = \bar{k}$ But why stop at two periods, we now go to T periods:

$v(k_0) = \max_{\vec{c},\vec{k}} u(c_0) + \beta u(c_1) + \beta^2 u(c_2) \ldots + \beta^{T-1} u(c_{T-1})$ - here we assume that we can additively separate utility (not necessarily true), constant discounted utility function over time, st $0 \leq c_0 + k_1 \leq f(k_0)$ $0 \leq c_1 + k_2 \leq f(k_1)$ $k_0 = \bar{k}$

We assume $u' > 0, f' > 0, u'' > 0, f'' > 0$ For a continuous function on a compact domain the Weierstrass Theorem guarantees a solution.

Then we let T go to infinity. This gives us the sequential problem:

2.2.2 Sequential problem (SP)

$v(k) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$ st $0 \leq x_t + k_{t+1} \leq f(k_t) \forall t \geq 0$ $k_0 = k$

the constraint binds, so we can take out the condition and write: $v(k) = \max \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$

Our goal is to make a "'policy function"'.

In general in math, we use an axiomatic style. Here we want to know if these value functions exist, what are its properties (is it the same in the sequential problem and the FE/BE) (strictly increasing? strictly concave? differentiable?) (Chapter 4 of Stokey Lucas), is it unique?

2.3 Real Analysis

Definition 2.3.1. *A* Set *is a collection of objects.*

We can also have a set of functions (such as bounded functions).

Definition 2.3.2. *A* space *is a set of objects equipped with some general properties and structure*

We may be interested in a metric space, (S, ρ) where S is the set of all bounded rational functions, and ρ is some distance function.

Definition 2.3.3. *A* metric *is a function $S \times S \to \mathbb{R}$ with the properties that it is non-negative, symmetric, and satisfies the triangle inequality.*

Definition 2.3.4. fixed point

Definition 2.3.5. *A* correspondence *in $\mathbb{R}^2$ is an association between points in the domain and the range (like a function, but not necessarily that the points in the domain are unique for each point in the range).*

Two properties we may be interested are Upper-hemicontinuity (every sequence of points in the correspondence converges to a point in the correspondence) and Lower-hemicontinuity (every point in the correspondence has a sequence of points converging to it). So a continuous function is a continuous correspondence. A continuous correspondence may be a continuous function, or certain nice sets in $\mathbb{R}^2$.

So for the SP, our choice set (constraint) over which we will maximize will be a correspondence with an upper bound that is the production function.

Chapter 3

Monday, January 29, 2007

3.1 math review

We have a Bellman equation and first we want to know if there exists a value function that satisfies the equation and second we want to know the properties of such a solution. In order to answer the question we will define a mapping which maps a function to another function, and a fixed point of the mapping is to be a solution. The mapping we discussed is a mapping on the set of functions, which is a bit abstract. So today we will look at the math review.

So first we consider a set, $S \subset \mathbb{R}^l$. For us what it will be relevant to describe a sort of distance between any two points in a set. We will use the concept of a metric.

Definition 3.1.1. *A* metric *is a function $\rho : S \times S \to \mathbb{R}$ with the properties that it is non-negative, $\rho(x, y) \geq 0$, symmetric, $\rho(x, y) = \rho(y, x)$, and satisfies the triangle inequality,$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.*

A common metric is euclidean distance, $\rho_E(x, y) = \sqrt{\sum_{i=1}^{l} (x_i - y_i)^2}$. Another is $\rho_{max}(x, y) = \max_{1 \leq i \leq l} x_i - y_i$.

Definition 3.1.2. *A* space*, is a set of objects equipped with some general properties and structure*

We may be interested in a metric space, a space with a metric such as, (S, ρ) where S is the set of all bounded rational functions, and ρ is some distance function. Once we have a metric space we can discuss convergence and continuity.

Definition 3.1.3. *A sequence, $\{x_i\} \subset S$,* converges *to x, $x_i \to x$, if $\forall \epsilon > 0, \exists N_\epsilon$ s.t. $\rho(x_n, y_n) < \epsilon$ for $n > N_\epsilon$.*

Definition 3.1.4. *A sequence , $\{x_i\} \subset S$, is called a* Cauchy sequence *if* $\forall \epsilon > 0, \exists N_\epsilon s.t. \rho(x_n, x_m) < \epsilon$ *for* $n, m > N_\epsilon$.

Question: does every Cauchy sequence converge?

Definition 3.1.5. *The metric space, (S, ρ) is* complete *if every Cauchy sequence converges.*

Example 3.1.6.

- $(\mathbb{R}, \rho_E)$ *is complete.*
- $((0,1), \rho_E)$ *is not complete. Proof: let* $x_n = \frac{1}{n+2}$. *So* $\{x_n\}$ *os Cauchy, but does not converge to a point in our set* $(0,1)$.
- $([0,1], \rho_E)$ *is complete. Are all closed sets complete? A closed subspace of a complete space is complete.*
- $(\{0,1,2\}, \rho_E)$ *is complete.*

Definition 3.1.7. *A mapping $T : S \to S$ is a* contraction mapping *on a metric space, (S, ρ), if $\exists 0 \geq \beta < 1$ such that $\rho(Tx, Ty) \leq \beta\rho(x, y) \forall x, y \in S$. Sometimes we write $T(x)$ instead of Tx.*

This means that any two points in our set, S, are mapped such that after the mapping the distance between the points shrinks.

Example 3.1.8. $Tx = .9x$ *is a contraction mapping on* $[(0,1], \rho_E)$.

Now we state the *contraction mapping theorem.*

Theorem 3.1.9 (Contraction Mapping Theorem). *If (S, ρ) is complete and $T : s \to S$ is a contraction mapping, then $\exists! x^*$ with $Tx^* = x^*$.*

We will prove this theorem for a general metric space later on. However, we must remember that it is necessary for this proof that the space be complete.

Let us now look at a criteria to verify that a mapping is a contraction mapping.

Theorem 3.1.10 (Blackwell's conditions). *For $S \subset \mathbb{R}^l$ and $\rho = \rho_E$. Let $T : S \to S$ satisfy the following two conditions:*

i. (M, monotonic condition) $\forall x = (x_1, x_2, \ldots, x_l) \in S$ *and* $y = (y_1, y_2, \ldots, y_l) \in S$, *and* $T = (T_1, T_2, \ldots, T_l$, *if* $x_i \geq y_i \Leftrightarrow x \geq y$ *then* $T_i x \geq T_i y \Leftrightarrow Tx \geq Ty$.

ii. (D, discount condition) $\forall x = (x_1, x_2, \ldots, x_l) \in S$, *for* $\vec{\underline{a}} = (a, a, \ldots, a)$, $T_i(x_1 + a, x_2 + a, \ldots, x_l + a) \leq T_i(x_1, x_2, \ldots, x_l) + \beta a \forall i \Leftrightarrow T(x + \underline{a}) \leq TX + \beta\underline{a}$

Then T is a contraction mapping.

Chapter 4

Wednesday, January 31, 2007

4.1 Review of last lecture

Consider a metric space (S, ρ) where $S \subset \mathbb{R}$. We introduced last time the concepts of convergence, Cauchy sequence, and complete metric space. We also discussed two theorems, but first let us consider the example we mentioned last time.

Example 4.1.1. *Claim* $([0,1], \rho_E)$ *is complete. The proof is by definition, we pick an arbitrary Cauchy sequence and prove that this sequence converges. Proof: Let* $\{x_n\} \subset S$ *be an arbitrary Cauchy sequence.* $\{x_n\}$ *is a Cauchy sequence, in* $(\mathbb{R}, \rho_E)$*. We have a fact from last time that* $(\mathbb{R}, \rho_E)$ *is a complete metric space. Thus* $\{x_n\}$ *converges in* $\mathbb{R}$*, let the limit point of the sequence be* x*. Now because* $(0,1)$ *is a closed subset of* $\mathbb{R}$*,* $\{x_N\}$*, it must be the case that* $x \in [0,1]$*. Therefore* (S, ρ_E) *is complete.*

Remember our two theorems from last time.

Theorem 4.1.2 (Contraction Mapping Theorem)**.** *If* (S, ρ) *is complete and* $T : s \to S$ *is a contraction mapping, then* $\exists! x^*$ *with* $Tx^* = x^*$*.*

Let us now look at a two sufficient conditions to help us make a judgment about whether a mapping is a contraction mapping.

Theorem 4.1.3 (Blackwell's Conditions)**.** *For* $S \subset \mathbb{R}^l$ *and* $\rho = \rho_E$*. Let* $T : S \to S$ *satisfy the following two conditions:*

i. (M, monotonic condition) $\forall x = (x_1, x_2, \ldots, x_l) \in S$ *and* $y = (y_1, y_2, \ldots, y_l) \in S$*, and* $T = (T_1, T_2, \ldots, T_l$*, if* $x_i \geq y_i \Leftrightarrow x \geq y$ *then* $T_i x \geq T_i y \Leftrightarrow Tx \geq Ty$*.*

ii. (D, discount condition) $\forall x = (x_1, x_2, \ldots, x_l) \in S$*, for* $\underline{\vec{a}} = (a, a, \ldots, a)$*,* $T_i(x_1 + a, x_2 + a, \ldots, x_l + a) \leq T_i(x_1, x_2, \ldots, x_l) + \beta a \forall i \Leftrightarrow T(x + \underline{a}) \leq TX + \beta \underline{a}$

Then T is a contraction mapping.

Remember our definition of a contraction mapping.

Definition 4.1.4. *A mapping $T : S \to S$ is a* contraction mapping *on a metric space, (S, ρ), if $\exists o \geq \beta < 1$ such that $\rho(tx, Ty) \leq \beta\rho(x, y) \forall x, y \in S$. Sometimes we write $T(x)$ instead of TX.*

4.2 Another look at one sector growth model

We will go back to that model and make some changes on the conditions and see if we can apply these theorems.

The individual will again try to maximize the same utility function with the same technology:

$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$ such that $0 \leq c_t + k_{t+1} \leq f(k_t), k_0 = k \in A$ is given, $k_{t+1} \in A$.

But we will restrict the capital holding possibility set to $A = \{0, 1, 2, \ldots, K\}$. We have now a sort of Bellman Equation:

$v(k) = \max U(c) + \beta v(k')$ such that $0 \leq c + k' \leq f(k), k' \in A$.

Which will in some way simplify our problem. Question: Given this Bellman Equation, is there any function v which satisfies this equation. The trick we will use is to define a mapping given by: consider the mapping T, and a vector $w = (w(0), w(1), \ldots, w(K))$, such that $Tw = (Tw(0), Tw(1), \ldots, Tw(K))$, and $Tw(k) = \max U(C) + \beta w(k')$ with $0 \leq c + k' \leq f(k), k' \in A$. w is an association with a level of capital with a future value of having such a level?. If we find a fixed point w^* of the mapping T, then w^* is the solution of the Bellman Equation. So the economics interpretation of this system is that you want to know, given a k, what is the value of having that k over your future.

Example 4.2.1. *Let $K = 1, U(c) = 2\sqrt{c}, f(k) = 2k$. Also, $w = (w(0), w(1)) = (0, 1) \in \mathbb{R}^2$. What is $Tw = (Tw(0), Tw(1)) \in \mathbb{R}^2$. $Tw(0) = max\sqrt{c} + \beta w(k')$ such that $0 \leq c + k' \leq f(0) = 0, k' \in \{0, 1\}$. So optimal $(c, k') = (0, 0) \Rightarrow Tw(0) = 0$.*

To solve for optimal $Tw(1)$, consider $Tw(1) = max\sqrt{c} + \beta w(k')$ such that $0 \leq c + k' \leq f(1), k' \in \{0, 1\}$. Substitute the constrain into the objective function to get $Tw(1) = \max \sqrt{2 - k'} + \beta w(k')$. If $k' = 0$, then $Tw(1) = \sqrt{2} + \beta w(0) = \sqrt{2}$. If $k' = 1$, then $Tw(1) = \sqrt{1} + \beta w(0) = 1 + \beta$

We need to check if the mapping T satisfies the monotonic and discounting conditions? It does, and we may come back to this. So by the second theorem, T

is a contraction mapping. Thus by the first theorem, $\exists! w^*$ with $w^* = Tw^*$. Let us check the monotonicity and discount conditions. In order to verify monotonicity, we need to verify that for arbitrary $w \in \mathbb{R}^2, v \in \mathbb{R}^2$ such that $v \geq w$ component-wise, we have $Tv \geq Tw$ componentwise. In order to verify the discounting condition, we will use the β given in the problem, arbitrary $w, \underline{a} = (a, a) \in \mathbb{R}^2$. We need to verify $Tw + \underline{a} \leq Tw + \beta\underline{a}$ componentwise. These verifications are in the second homework.

Chapter 5

Friday, February 2, 2007

5.1 Real Analysis

5.1.1 Applying the contraction mapping theorem

In lecture on Wednesday professor Zhu discussed the following problem: $w(k) = max\sqrt{c} + \beta w(k')$ such that $0 \leq c + k' \leq f(k), k' \in \{0, 1, \ldots, k\}$. We have two ways to prove that a function is a contraction mapping, the definition and Blackwell's conditions. Remember that the choice of metric influences the applicability of Blackwell's conditions.

$Tx = .5x$ is a contraction mapping. This is easy to show in the space of real numbers using Blackwell's conditions.

In general the domain will be the set of positive reals, but in this example, we have a discrete, finite domain.

This lecture is a mess so far. Russel is going over the question we worked on with Zhu, but he is being so general and there isn't much to get out of it.

5.1.2 Space Versus Set

We are used to talking about spaces such as $S = \mathbb{R}$ or $s = [0, 1]$. When we define convergence, convergence of a sequence is relative to the space. so on the space $S = (0, 1]$, the sequence $\{1/n\}$ is not a convergent sequence.

Definition 5.1.1. *A* Cauchy sequence *is a sequence,* $\{x_i\} \in S$*, such that* $\forall \epsilon > 0, \exists N_\epsilon s.t. \rho(x_n, x_m) < \epsilon$ *for* $n, m > N_\epsilon$*.*

This is sometimes visualized as two points chasing each other and getting closer and closer. But it is also useful to think about the idea that if you take an ϵ, and find you N_ϵ and point x_n ($n \geq N_\epsilon$), all points after x_n are within a distance of ϵ of x_n. This notion also helps to illustrate the property that Cauchy sequences are bounded (since our N_ϵ is finite $\forall \epsilon > 0$.

5.1.3 More results for future reference

In what follows, we regard ρ_E as the metric of S as the metric is not specified.

Definition 5.1.2. *For metric space* (S, ρ)*, a sequence* $\{x_n\}$ *is* convergent *to* $x \in S$ *if* $\forall \epsilon > 0 \exists N_\epsilon \in \ltimes$ *such that* $\rho(x_n, x) < \epsilon \forall n \geq N_\epsilon$.

Theorem 5.1.3. *A convergent sequence in* (S, ρ) *has a unique limit. That is, if* $x_n \to x$ *and* $x_n \to y$*, then* $x = y$.

Theorem 5.1.4. *Let* $\{x_n\}$*,* $\{y_n\}$*, and* $\{z_n\}$ *be sequences of real numbers. If* $x_n \to x$ *and* $y_n \to y$ *and* $z_n \to z$ *and* $z_n = x_n = y_n$*, then* $z = x = y$.

Theorem 5.1.5. *Closed intervals (*$[a, b]$ *with* $a \leq b$ *or* $(-\infty, a]$ *or* $[b, +\infty)$*) are closed.*

Theorem 5.1.6. *Weierstrass Theorem: Let* $f : X \subset \mathbb{R}^l \to \mathbb{R}$ *be continuous. If* X *is compact, then both* $\max_{x \in X} f(x)$ *and* $\min_{x \in X} f(x)$ *exist.*

R4: Let f : X . Rl . Rm be continuous. If X is compact, then f (X) = f (x) : x . Rl . Rm is compact. R6: Let xn . (S, .). Then xn converges if and only if all of its subsequences converge. R7: Let X . Rl be compact and let xn . X. Then there is a subsequence of xn that converges to some x . X. R8: Let X . Rl be compact and let X1 be a closed subset of Rl. If X1 . X, then it is compact. R9: Let (S, .) be a complete metric space and let S1 be a closed subset of S. Then (S1, .) is a complete metric space. R10: Let xn be a sequence of real numbers. If xn is decreasing (that is, xn+1 = xn all n) and bounded below, then xn converges and limn.8 xn = infxn. R11: Let xn be a sequence of real numbers. If xn is increasing (that is, xn+1 = xn all n) and bounded above, then xn converges and limn.8 xn = supxn. R12: Let xn be a sequence of real numbers. Then lim inf xn = lim sup xn, and if equal then the sequence converges. R13: Let xn be a sequence of real numbers. If the sequence converges, then it is bounded. (That is, there exists some M = —xn— all n.) R14: Let f : X . R . R be strictly concave and differentiable. Then f (x) ¿ f(y) . y 6= x . f 0(x) = 0. R15: Let f, g : X . Rl . R be continuous. Let h : X . Rl . R be defined by h(x) =maxf (x), g(x). Then h is continuous. R16: If S is a compact subset of (Rl, .E) and X is a closed subset of (S, .E), then X is a closed subset of (Rl, .E).

Chapter 6

Monday, February 5, 2007

6.1 Review

Example 6.1.1. *We have already introduced the Bellman equation which says* $\forall k \in \mathbb{R}_+$, $v(k) = \max U(c) + \beta v(k')$ *such that* $0 \leq c + k' \leq f(k), c \geq 0, k' \geq 0$. *To solve, we consider* T *the set of real valued functions defined on* $\mathbb{R}_+$, *with* $\forall k \in \mathbb{R}_+, TW(k) = \max u(c) + \beta w(k')$ *such that* $0 \leq c + k' \leq f(k), c \geq 0, k' \geq 0$. *A fixed point* $w^* = Tw^*$ *is a solution to the Bellman equation.*

We have also discussed the topics of Cauchy sequences, convergence, completeness, contraction mapping, a contraction mapping theorem, and Blackwell's sufficient conditions. One way to simplify the problem is to restrict our initial capital holdings. So we have the same Belman equation, but we write it:

Example 6.1.2. $\forall k \in \{0, 1, \ldots, K\}$, *such that* $v(k) = \max U(c) + \beta v(k')$ *such that* $0 \leq c + k' \leq f(k), c \geq 0, k' \in \{0, 1, \ldots, K\}$. *To solve, we consider* T *the set of real valued functions defined on* $\mathbb{R}_+$, *with* $\forall k \in \{0, 1, \ldots, K\}, TW(k) = \max u(c) + \beta w(k')$ *such that* $0 \leq c + k' \leq f(k), c \geq 0, k' \in \{0, 1, \ldots, K\}$. *A fixed point* $w^* = Tw^*$ *is a solution to the Bellman equation.*

Notice that the mapping is still on $k + 1$-Euclidean space, although k ranges only from 0 to k. What we will do today is generalize our mathematical concepts to infinite dimensional spaces so we can allow for more goods. At this point, we are at Chapter 3.1 in the textbook.

6.2 Generalized Math Concepts

Let S be an arbitrary set, let $\rho : S \times S \to \mathbb{R}_+$ be an arbitrary metric for our set S. Remember, this implies that ρ satisfies non-negativity, $\rho(x, y) \geq 0$, symmetry, $\rho(x, y) = \rho(y, x)$, and the triangle inequality,$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. So (S, ρ) is a metric space, and for this metric we can discuss convergence.

Definition 6.2.1. *A sequence, $\{x_i\} \subset S$,* converges *to $x \in S$, $x_i \to x$, if $\forall \epsilon > 0, \exists N_\epsilon$ s.t. $\rho(x_n, y_n) < \epsilon$ for $n > N_\epsilon$.*

Definition 6.2.2. *A sequence , $\{x_i\} \subset S$, is called a* Cauchy sequence *if $\forall \epsilon > 0, \exists N_\epsilon s.t. \rho(x_n, x_m) < \epsilon$ for $n, m > N_\epsilon$.*

Definition 6.2.3. *The metric space, (S, ρ) is* complete *if every Cauchy sequence converges.*

Definition 6.2.4. *A mapping $T : S \to S$ is a* contraction mapping *on a metric space, (S, ρ), if $\exists 0 \geq \beta < 1$ such that $\rho(Tx, Ty) \leq \beta\rho(x, y) \forall x, y \in S$. Sometimes we write $T(x)$ instead of Tx.*

6.2.1 Functional Analysis

Let $X \subset \mathbb{R}^l$. We write $B(X)$ as the set of bounded real valued functions on X. We write $C(X)$ as the set of continuous bounded real valued functions on X. Notice that $C(X) \subset B(X)$. So if $X = \{0, 1\}$, then $B(X)$ is the set of functions which have domain equal to $\{0, 1\}$ and a range $\mathbb{R}$. $f \in B(X), f : \{0, 1\} \to \mathbb{R}, f = (f(0), f(1)) \in \mathbb{R}$. In this case, $B(X) = C(X)$.

So now we have a set (or two) $B(X)$ (or $C(X)$). In the generally defined $X \subset \mathbb{R}^l$, we can get space if we define a metric. Let $f, g \in B(X)$. The metric we introduce is: $\rho_{\text{sup}}(f, g) = \sup_{x\in X} |f(x) - g(x)| = \|f - g\|$. We leave it as an exercise to show that ρ_{sup} is a metric. In the finite dimensional space, $\rho_{\text{sup}}(f, g) = \sup_{x\in X} |f(x) - g(x)| = \max_{x\in X} |f(x) - g(x)| = \rho_{max}(f, g)$.

Theorem 6.2.5 (3.1 in textbook)**.** *Let $X \subset \mathbb{R}^l$. Then $(C(X), \rho_{sup})$ is a complete metric space.*

Proof. Pick an arbitrary Cauchy sequence $\{f_n\}$ in $C(X)$. We need to show $\{f_n\}$ converges to some $f \in C(X)$. First we will construct a candidate function f. $\forall x \in X$. Define $f(x) = \lim f_n(x)$. Note: $f(x) \in \mathbb{R}$ and $f_n(x) \in \mathbb{R}$. Question: Is this limit well defined? Yes, we will go back to this next lecture. $f_n \to f \forall \epsilon > 0, \exists N_\epsilon$ such that $\rho_{\text{sup}}(f_n, f) < \epsilon$ for $n > N_\epsilon$. $f \in C(X), f : X \to \mathbb{R}$ is continuous and bounded (showing boundedness is in exercises). □

Chapter 7

Wednesday, February 7, 2007

7.1 Introduction: Dynamic Programming

To begin with, let us continue the proof from last time.

Theorem 7.1.1 (3.1 in textbook)**.** *Let* $X \subset \mathbb{R}^l$*. Then* $(C(X), \rho_{sup})$ *is a complete metric space.*

Proof. Pick an arbitrary Cauchy sequence $\{f_n\}$ in $C(X)$. We need to show $\{f_n\}$ converges to some $f \in C(X)$. First we will construct a candidate function f. $\forall x \in X$. Define $f(x) = \lim f_n(x)$. Note: $f(x) \in \mathbb{R}$ and $f_n(x) \in \mathbb{R}$. Question: Is this limit well defined? Yes. We know $\{f_n\}$ is Cauchy in $(C(X), \rho_{sup})$, and we need to show that it is Cauchy in $(\mathbb{R}, \rho_E)$. This is true since $\rho_{sup}(f,g) = \sup_{x\in X} \rho(f(x), g(x)) = sup_{x\in X}|f(x) - g(x)|$.

$f_n \to f \forall \epsilon > 0, \exists N_\epsilon$ such that $\rho_{\sup}(f_n, f) < \epsilon$ for $n > N_\epsilon$. $f \in C(X), f : X \to \mathbb{R}$ is continuous and bounded (showing boundedness is in exercises). □

The discussion goes on to prove in detail various aspects of the proof, which I will skip.

7.2 Haiku for the day

I hurt my finger, ring finger on my right hand; a dislocation.

Theorem 7.2.1. *Let* $(S < \rho)$ *be a complete metric space. Let* $T : S \to S$ *be a contraction mapping. Then* T *has a unique fixed point.*

Proof. Uniqueness: Let $Tw = w, TV = v$ be two arbitrary fixed points. We will show that they are equal. $\rho(w, v) = \rho(Tw, Tv) \leq \beta\rho(w, v)$ for $\beta \in [0, 1)$, since T is a contraction mapping. Therefore $\rho(w, v) = 0 \Rightarrow w = v$.

Existence: Start with any arbitrary $w_0 \in S$. $w_i = Tw_{i-1}$. Construct a sequence $\{w_n\}$ in S. This sequence is a Cauchy sequence. Let us verify this is Cauchy: We know, $\rho(w_1, w_2) = \rho(Tw_0, Tw_1) \leq \beta\rho(w_0, w_1)$. Likewise, $\rho(w_2, w_3) = \rho(Tw_1, Tw_2) \leq \beta\rho(w_1, w_2) \leq \beta^2\rho(w_0, w_1)$. In general, $\rho(w_n, w_{n+1}) \leq \beta^n\rho(w_0, w_1)$ Now, by the triangle inequality, $\rho(w_n, w_m) \leq \rho(w_n, w_{n+1} + \ldots + \rho(w_{m-1}, w_m) = \beta^n d(1 + \beta + \beta^2 + \ldots + \beta^k) \leq \beta^n d(\frac{1}{1-\beta})$ for some k, d. Therefore $\{w_n\}$ is a Cauchy sequence.

(S, ρ) is complete so our sequence converges, say $w_n \rightarrow w$. $Tw = w$ (need to verify this). $\square$

Chapter 8

Friday February 9, 2007

Chicago was nice. I missed class.

Chapter 9

Monday, February 12, 2007

9.1 Extension of Blackwell's Conditions

Last time we were working on showing existance, uniqueness, and the property of being Cauchy for a certain sequence of elements. We had finished shiwng Cauchy and that our space was complete (which implied existance of a limit of any Cauchy sequence). We copy our work here and finish showing uniqueness.

Theorem 9.1.1. *Let $(S < \rho)$ be a complete metric space. Let $T : S \to S$ be a contraction mapping. Then T has a unique fixed point.*

Proof. Uniqueness: Let $Tw = w, TV = v$ be two arbitrary fixed points. We will show that they are equal. $\rho(w, v) = \rho(Tw, Tv) \leq \beta\rho(w, v)$ for $\beta \in [0, 1)$, since T is a contraction mapping. Therefore $\rho(w, v) = 0 \Rightarrow w = v$.

Existence: Start with any arbitrary $w_0 \in S$. $w_i = Tw_{i-1}$. Construct a sequence $\{w_n\}$ in S. This sequence is a Cauchy sequence. Let us verify this is Cauchy: We know, $\rho(w_1, w_2) = \rho(Tw_0, Tw_1) \leq \beta\rho(w_0, w_1)$. Likewise, $\rho(w_2, w_3) = \rho(Tw_1, Tw_2) \leq \beta\rho(w_1, w_2) \leq \beta^2\rho(w_0, w_1)$. In general, $\rho(w_n, w_{n+1}) \leq \beta^n\rho(w_0, w_1)$ Now, by the triangle inequality, $\rho(w_n, w_m) \leq \rho(w_n, w_{n+1} + \ldots + \rho(w_{m-1}, w_m) = \beta^n d(1 + \beta + \beta^2 + \ldots + \beta^k) \leq \beta^n d(\frac{1}{1-\beta})$ for some k, d. Therefore $\{w_n\}$ is a Cauchy sequence.

(S, ρ) is complete so our sequence converges, say $w_n \to w$. Now we will show uniqueness. We will show $w^* = Tw^* \Leftrightarrow \rho(Tw^*, w^*) = 0$. In order to show this, it is sufficient to show $\rho(Tw^*, w^*) < \epsilon$ for arbitrary $\epsilon < 0$.

Now fix $\epsilon > 0$. By the triangle inequality, we have, $\rho(tw^*, w^*) \leq \rho(Tw^*, w_{n+1}) + \rho(w_{n+1}, w^*)$. But $\rho(Tw^*, w_{n+1}) + \rho(w_{n+1}, w^*) = \rho(Tw^*, Tw_n) + \rho(w_{n+1}, w^*) \leq \beta\rho(w^*, w_n) + \rho(w_{n+1}, w^*)$, since T is a contraction mapping. Finally, we have: $\beta\rho(w^*, w_n) + \rho(w_{n+1}, w^*) < \epsilon$ for sufficiently large n because of

□

The question in our mind, we are in $(B(X), \rho_{sup})$. By theorem 3.1, this space is complete (as is $(C(X), \rho_{sup})$). Now we will discuss a generalized version of Blackwell's sufficientcy conditions to see if a mapping in this space is a contraction mapping.

Theorem 9.1.2 (Theorem 3.3 an extension of Blackwell's Conditions)**.** *For $S = B(X)$ or $S = C(X)$, and $\rho = \rho_{sup}$. Let $T : S \to S$ satisfy the following two conditions:*

i. (M, monotonic condition) If $v, w \in S$ such that $v \geq w$ (i.e. $v(x) \geq w(x) \forall x \in X$), then $Tv \geq Tw$.

ii. (D, discount condition) $\exists \beta \in [0,1)$ such that $\forall v \in S, \forall \underline{a} \in S$ (so $\underline{a}(x) = a \forall x \in X$), we have $T(v + \underline{a}) \leq Tv + \beta\underline{a}$ ($\Leftrightarrow T(v + \underline{a})(x) \leq Tv(x) + \beta\underline{a} \forall x \in X$)

Then T is a contraction mapping.

Proof. We will sketch the proof here, although the complete proof is left for homework. Choose the β in condition (D). We need to show $\rho_{sup}(Tv, Tw) \leq \beta\rho_{sup}(v, w) \forall v, w \in S$. So fix $v, q \in S$. Let $a = sup_{x \in X}|v(x) - w(x)|$. Fix $x \in X$. By monotonicity, $Tw(x) \leq T(v + a)(x)$. By discounting, $T(v + a)(x) \leq Tv(x) + \beta a$. Likewise, $Tw(x) \leq T(w + a)(x) \leq Tw(x) + \beta a$. The rest is up to the reader in homework. □

Example 9.1.3. *Let $X = [0,1]$. Monotonic just means that if we have functions the mapping preserves absolute order across the domain. Discounting means given two functions, the mapping brings them "'closer together"'.*

Remark 9.1.4 (authors note)**.** *There is some beautiful snow outside. It would be nice to watch the snow by a fireside with the people I love.*

Let $k \in X = \{0, 1, \ldots, k+1\}$. Let $T : B(X) \to B(X)$, $Tw(k) = \max U(c) + \beta w(k')$ such that $0 \leq c + k' \leq f(k), k' \in X$. The solution to this is in the homework. We want this so we can extend it to: Let $k \in X = \mathbb{R}_+$. Let $T : B(X) \to B(X)$, $Tw(k) = \max U(c) + \beta w(k')$ such that $0 \leq c + k' \leq f(k), k' \in X$. The same argument as before can be used to show that monotonicity and discounting hold in both cases. So by the theorem this is a contraction mapping, and by the contraction mapping theorem, do we have a fixed point for T? To be sure, we must show also that the $\max U(c)$ exists. In the discrete case, maximum is always well defined. But here we have a continuous case. Secondly, we must assure that our mapping T has range in $C(X)$.

Now we introduce a theorem to show the existence of certain maximums, which will likely be repeated in a general equilibrium theory course.

Theorem 9.1.5 (Theorem 3.6 in text). *Let $X \subset \mathbb{R}^l, Y \subset \mathbb{R}^l$, a function $f : X \times Y \to \mathbb{R}$, and a correspondence $\Gamma : X \to Y$. Let $h(x) = \max f(x, y)$ such that $y \in \Gamma(x)$. (In our example, we have $Tw(k) = \max U(f(k)+k')+\beta w(k')$ such that $\Gamma(k) = [0, f(k)]$.) Suppose f is continuous (with respect to ρ_E) and suppose Γ is also continuous (we will discuss continuity of a correspondence soon). Then*

- *i.* $\forall x \in X, h(x)$ *is well defined (our maximum is obtained),*
- *ii.* $G(x) = \arg\max f(x, y), y \in \Gamma(x)$ *(the set of maximizers)* $\forall x \in X, G(x)$ *is compact.*
- *iii.* $h : X \to \mathbb{R}$ *is continuous.*
- *iv.* $G : X \to Y$ *is upper-hemicontinuous.*

Definition 9.1.6. *T is* continuous *at $x \in X$ if T is upper-hemicontinuous at x and lower-hemicontinuous at x. T is continuous if it is continuous at each $x \in X$.*

Definition 9.1.7. *T is* upper-hemicontinuous *at $x \in X$ if for every $y \in \Gamma(x)$ and for every $x_n \to x$, there exists some sequence, $\{y_n\}$, with $y_n \in \Gamma(x_n) \forall n$, such that $y = \lim y_n$.*

Definition 9.1.8. *Suppose T is compact valued ($T(x)$ is compact for all $x \in X$), T is upper-hemicontinuous at x if for every $x_n \to x$, there exists some sequence, $\{y_n\}$ with $y_n \in \Gamma(x_n) \forall n$, such that there exists a convergent subsequence of $\{y_n\}$ whose limit is in $\Gamma(x)$.*

Example 9.1.9.

- *i.* $X = [0, 1], \Gamma(x) = \{0\}$ *if* $0 \leq x < 1$, $[0, 1]$ *if* $x = 1$. *Then Γ is not lower-hemicontinuous at $x = 1$, Choose $y = .5$, so $\{y_n\}$ converges to $.5$. But it is upper-hemicontinuous at $x = 1$. Fix $\{x_n\}$ such that $x_n \to 1$. Fix $y_n = 0 \in \Gamma(x_n)$. $\{y_n\} \subset [0, 1]$, a compact interval, so $\{y_n\}$ has a convergent subsequence.*
- *ii.* $X = [0, 1], \Gamma(x) = \{1\}$ *if* $0 \leq x < 1$, $[0, 1]$ *if* $x = 1$. *Then Γ is not upper-hemicontinuous at $x = 1$, Choose $x_n = 1 - \frac{1}{n}$, $y = .5$, so $\{y_n\}$ converges to $.5$. But Γ is lower-hemicontinuous at $x = 1$. So $y = 0 \in \Gamma(1)$. Pick a sequence $\{x_n\}$ such that $x_n \to 1$. Choose $y_n = 0 \in \Gamma(x_n)$. Clearly $y_n \to 0$.*

For graphical examples, see MWG p. 949-952.

Chapter 10

Wednesday, February 14, 2007

10.1 Examples regarding upper and lower hemicontinuity

Definition 10.1.1. *T is* continuous *at $x \in X$ if T is upper-hemicontinuous at x and lower-hemicontinuous at x. T is continuous if it is continuous at each $x \in X$.*

Definition 10.1.2. *T is* upper-hemicontinuous *at $x \in X$ if for every $y \in \Gamma(x)$ and for every $x_n \to x$, there exists some sequence, $\{y_n\}$, with $y_n \in \Gamma(x_n) \forall n$, such that $y = \lim y_n$.*

Definition 10.1.3. *Suppose T is compact valued ($T(x)$ is compact for all $x \in X$), T is upper-hemicontinuous at x if for every $x_n \to x$, there exists some sequence, $\{y_n\}$ with $y_n \in \Gamma(x_n) \forall n$, such that there exists a convergent subsequence of $\{y_n\}$ whose limit is in $\Gamma(x)$.*

Example 10.1.4.

i. $X = [0,1], \Gamma(x) = \{0\}$ if $0 \leq x < 1$, $[0,1]$ if $x = 1$. Then Γ is not lower-hemicontinuous at $x = 1$, Choose $y = .5$, so $\{y_n\}$ converges to $.5$

ii. $X = [0,1], \Gamma(x) = \{1\}$ if $0 \leq x < 1$, $[0,1]$ if $x = 1$. Then Γ is not upper-hemicontinuous at $x = 1$, Choose $x_n = 1 - \frac{1}{n}$, $y = .5$, so $\{y_n\}$ converges to $.5$

iii. $X = [0,1].\Gamma(x) = [0,x], \Gamma$ is continuous. We need to show Γ is upper-hemicontinuous and lower-hemicontinuous.

Proof. First we fix x. To show lower-hemicontinuity, we need to show that for a fixed $y \in \Gamma(x), x_n \to x$. We need to find some $y_n \in \Gamma(x_n)$ such that $y_n \to y$. So to find lower-hemicontinuity is not so bad, as we simply need to construct such a sequence. There are many ways, here is one: $x = 0$ so $y = 0$, so we just set $y_n = 0$, since $0 \in \Gamma(x) \forall x$. If $x \neq 0$, just let $y_n = \frac{x_n}{x} y$. We claim that this $y_n \in \Gamma(x_n)$ (true since $y_n \leq x$. To find upper-hemicontinuity, fix $x_n \to x, y_n \in \Gamma(x_n)$. We need to show $\{y_n\}$ has a convergent subsequence whose limit is some $y \in \Gamma(x)$. Two steps: Step 1, recognize that the sequence $\{y_n\} \in [0, 1]$. So $\{y_n\}$ has a convergent subsequence denoted by $\{y_{n_k}\}$. Step 2, we know $0 \leq y_{n_k} \leq x_{n_k}$. But the first step tells us that this sequence converges, so we take limits, $0 \leq \lim y_{n_k} \leq \lim x_{n_k} = x$. Therefore, $\lim y_{n_k} \in \Gamma(x)$. (By the way, if we replace $X = [0, 1]$ with $X = \mathbb{R}_+$, we don't have to change anything to get lower-hemicontinuity. To get upper-hemicontinuity, we still have compact valued, and we only need to change $\{y_n\} \subset [0, 1]$ to $\{y_n\} \subset [0, \Gamma(\sup\{x_n\})]$. $\square$

iv. *Γ is single valued, lower-hemicontinuous implies Γ is upper-hemicontinuous (and thus continuous). (Note: this works for upper-hemicontinuous single valued correspondences as well.)*

Proof. Fix $x \in X, x_n \to x, y_n \in \Gamma(x_n)$. We need to show $\{y_n\}$ has a convergent subsequence whose limit is $\hat{y} \in \Gamma(x) = \{\hat{y}\}$. Γ is single valued implies that every set $\Gamma(x_n)$ has a single element. Γ is lower-hemicontinuous implies $\exists \tilde{y_n} \in \Gamma(x_n)$ such that $\tilde{y_n} \to \hat{y}$. But since Γ is single valued, $\tilde{y_n} = \hat{y_n}$. So indeed, $\hat{y_n} \to \hat{y}$. $\square$

Now we have every tool we need to analyze the dynamic program. The contraction mapping theorem, Blackwell's sufficient conditions, now the theory of maximum, etc.

10.2 On the upcoming exam

i. Everything up to today may be in the exam.

ii. The points will be distributed

iii. Focus on the basics, what the theorems say, what the homework covered. In general there may be a couple tricky questions, but Zhu will focus on the basics, this is not a real analysis course.

iv. Theorem 3.4-3.5 is not in the exam. Everything in the notes may be in the exam. We won't worry about parts of chapter 3 not covered in the course.

Chapter 11

Friday, February 16, 2007

11.1 Monday's exam

Often these are a bit long. Advice: know the problem sets, know the concepts from the notes, watch out for quick easy counterexamples, and just know everything we've done so far.

11.2 Problem set 2

Umm, he goes over the solutions handed out in class.

11.3 Problem set 3

Again, he goes over the solutions handed out in class. I'm a bit miffed, as I have the same solutions as Russel in places where mine are marked wrong. I wonder who is grading the exam.

11.4 Correspondences and Theorem of the maximum

No time.

Bibliography

[1] L. Lamport. **LaTeX A Document Preparation System** Addison-Wesley, California 1986.