Title: # Collatz 3x+1 Conjecture Proved!

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Abstract

This manuscript presents a very simple and general approach for deciding any *Collatz 3x+1-type problem* — that is, finding all the *cycles* (*if any*) of *integer sequences* recursively defined by some *branching-function* — through a *computable generalization* of the valid *solution-values* to all the *cycle-equations* of their presupposed *periodic subsequences*.

By straightforward application of the easy method developed, the *Collatz 3x+1 conjecture* in the positive integers domain is proved while its inherent *computational complexity* and *undecidability* in the negative integers domain is explained.

Collatz 3x+1 Conjecture Proved!

I. *Introduction*

The *standard* *Collatz problem* **[1]** asks if, or the *Collatz conjecture* **[2]** claims that, iterating

$$x_n = \begin{cases} \dfrac{x_{n-1}}{2} & \text{if } x_{n-1} \text{ is even} \\ \\ 3x_{n-1} + 1 & \text{if } x_{n-1} \text{ is odd} \end{cases}$$

always return to 1 for every starting positive integer x_0. In **[3]**, *Marc Chamberland* exclaims: "*The 3x+1 problem is perhaps today's most enigmatic unsolved mathematical problem — it can be explained to a child who has learned how to divide by 2 and multiply by 3; yet, there are relatively few strong results toward solving it. Paul Erdös was correct when he stated:* '*Mathematics is not ready for such problems*' **[4, 5]**."

The authoritative *up-to-date* reference to the *Collatz 3x+1 problem* is **[6]** — it is categorically declared that (*as of* 8 August 2006): "*At present, the 3x+1 conjecture remains unsolved.*". This "*enigma*" is a source of recreational mathematical diversions **[7, 8, 9, 10, 11, 12, 13, 14]** and it has been subjected to numerous advanced number theory and theoretical computer science computational methods for its resolution — such as extending the domain to *rational numbers* **[15, 16, 17, 18]** or to the *real number line* **[19, 20, 21]** or the entire *complex plane* **[22, 23]**; *ergodic theory* or *Markov chains* **[5, 24, 25, 26, 27, 28, 29, 30, 31]**; *finite automata* **[32]**; *cellular automata* **[33, 34, 35]**; *graph theory* **[36, 37]**; *functional equations* **[38, 39, 40]**; *difference equations* **[41, 42, 43]**; *dynamical systems* **[44, 45, 46]**; *conjugacy map* **[47]**; *combinatorics* **[48]**; *Diophantine approximation* **[49, 50]**; *binomial representation* **[51]**; *probabilistic model* **[52, 53]**; *2-adic integers* **[54, 55]**; *cybernetics* **[56]**; "*tuple-sets*" **[57]**; "*sequence vectors*" **[58]**; "*predecessor trees*" **[59]**; "*3x+1 trees*" **[60]**; *etc*. In reality, the *Collatz 3x+1 problem* is studied for its *computational complexity* **[61]** and is even deemed to be "*unsolvable*" **[62, 63, 64, 65, 66, 67]** — which might have prompted *Bryan Thwaites* **[68, 69]** (*as well as others* **[1, 2, 5]**) to offer prize money (*up to* £1,000.00) for its solution.

II. *A Simple and General Approach to Collatz 3x+1-type Problems*

A very simple and general *solution-approach* for deciding many *Collatz 3x+1-type problems* — that is, finding (*if any*) all the *cycles* of integer sequences recursively defined by some *branching-function* — is presented in this manuscript. Every *Collatz-type problem* involves some arbitrary *branch* (*or path or trajectory or orbit*) that takes the form of an infinite sequence

$C_n = \langle n, m, l, k, \ldots, d, c, b, a, \ldots \rangle$

of *branch-points* where the starting number n ranges through all the elements of the specified domain **D** of a given *branching iteration function* (*with mutually exclusive sub-functions*)

$$f(n) = \begin{cases} \alpha(n) & \text{if } n \text{ satisfies condition A} \\ \beta(n) & \text{if } n \text{ satisfies condition B} \\ \vdots & \\ \zeta(n) & \text{if } n \text{ satisfies condition Z} \end{cases}$$

such that
$$\begin{aligned} m &= f(n); \\ l &= f(m) = f(f(n)) = f^2(n); \\ k &= f(l) = f(f(m)) = f(f(f(n))) = f^3(n); \\ &\vdots \\ c &= f(d) = f(f(e)) = f(f(f(f))) = \ldots = f^{u-1}(n); \\ b &= f(c) = f(f(d)) = f(f(f(e))) = \ldots = f^{u}(n); \\ a &= f(b) = f(f(c)) = f(f(f(d))) = \ldots = f^{u+1}(n). \end{aligned}$$

It is desired to find (*if any*) the *cycle* or *cyclic-subsequence* $C_i = \langle(i, h, g, \ldots, d, c, b)\rangle$ so that

$C_n = \langle n, m, l, \ldots, k, j, (i, h, g, \ldots, d, c, b)\rangle = \langle n, m, l, \ldots, k, j, C_i\rangle$

— that is, $a = i \in \{n, m, l, k, \ldots, d, c, b\}$ is the *first-duplicated-term* and the group of distinct iterates $i, h, g, \ldots, c$, and b (*in that fixed order*) repeats *ad infinitum*. A sequence that ultimately loops (*that is, has some periodic subsequence*) is said to be *convergent*; otherwise, it is held to be *divergent*. It is emphasized that any iterate k of some convergent or divergent sequence C_n corresponds to the *also-convergent* or the *also-divergent* subsequence C_k, respectively.

The typical *Collatz 3x+1 sequence branch* or *trajectory*, for arbitrary *starting number* $n \in \mathbf{D}$ —

$C_n = \langle n, m, l, k, \ldots, d, c, b, a, \ldots\rangle$

— is a *cycle* or has a *cyclic-subsequence* $C_i = \langle(i, h, g, \ldots, d, c, b)\rangle$ if and only if there is some *first-duplicated-term* a in C_n — that is, $a = f(b) = i \in \{n, m, l, \ldots, d, c, b\}$. It is emphasized that the elements of the set $\{n, m, l, \ldots, d, c, b\}$ are distinct by our definition of a (*therefore,* a *could only be equal to one of them*) — so every term of the cycle $C_i = (i, h, g, \ldots, d, c, b)$ is likewise different. It trivially immediately follows that there could only be one set of distinctive terms that form a cycle or a cyclic subsequence for each sequence C_n and it is not possible to have a cycle with larger length that includes a cycle with smaller length in any sequence C_n.

As in *elementary algebra* — where an arithmetic problem is solved by setting up some equation and finding all the valid *solution-values* for the unknown variables — the fact that C_n has a cycle can be readily established by seeking a valid *solution-value* to any one of these *cycle-equations*:

$a = b$ [*that is,* $f(b) = b$ *for length*-1 *cycle*];

or $a = c$ [*that is,* $f(b) = f^{-1}(b)$ *for length*-2 *cycle*];

or $a = d$ [*that is,* $f(b) = f^{-2}(b)$ *for length*-3 *cycle*];

⋮

or $a = n$ [*that is,* $f(b) = f^{u}(b)$ *for length*-$(u+1)$ *cycle with* $u \in \mathbf{N}^{+}$-{1}].

Since there are at least 2 *sub-functions* for a *general Collatz 3x+1-type function* f, the finding of valid *solution-values* for b may be *exponentially computationally complex*. In determining all the valid *solution-values* for b (*for every representative branch* C_n *of the iteration function* f), it is easier to evaluate $a = b$ first, next $a = c$, then $a = d$, …, and so on up to $a = n$. For some infinite domains, since $n \to \infty$, $a = n$ might remain elusive to solve and this would present some potentially divergent sequences. This scenario actually poses an *undecidability problem* — unless the existence of *length-*u cycles, $\forall u > z$ for some $z \in \mathbf{N}$, could be absolutely ruled out by some *computable generalization* then there would remain sequences whose convergence or divergence is *undecided*.

If there are s *sub-functions* of some given *Collatz 3x+1-type function* f, then every *branch-point* $f^{u}(b)$ [$\forall u \in \mathbf{N}^{+}$] has s^{u} *nodes* that are represented by respective expressions involving "*powers*" of f's *sub-functions* (*when viewed from the preceding terms*) or of f^{-1}'s *sub-functions* (*when seen from the succeeding iterates*) as well as s^{u+1} *cycle-equations*. If there is a *length*-(u+1) cycle then, in addition to the *branch-node* expressions that yield all of the *solution-values* for the trivial *length*-1 cycles, there would be u+1 *branch-node* expressions that actually yield each *cycle-term* as a valid respective *solution-value* for b (*in the appropriate domain*). The possibilities of loops with lengths greater than u+1 [$\forall u \in \mathbf{N}^{+}$] (*including perhaps divergent sequences*), exist whether or not the s^{u+1} *cycle-equations* for the *branch-point* $f^{u}(b)$ yield valid cycles with the length u+1. It would require some *computable generalization* reasoning like *mathematical induction* argument to surmount the *computational complexity* or *undecidability* issues innate with the determination of all the valid *cycle-lengths* of a *general* *Collatz 3x+1-type problem*.

For each member of the class of sequences with the *cyclic-subsequence* C_i = <(i, h, …, d, c, b)>, the presupposed *cycle-solution branch-nodes* expressions are delineated as follows:

$$\mathbf{a = i = f(b)} \quad \mathbf{h = f^{2}(b)} \quad \ldots \quad \mathbf{e = f^{-3}(b)} \quad \mathbf{d = f^{-2}(b)} \quad \mathbf{c = f^{-1}(b)} \quad \mathbf{b}$$

$$\left(\left\{ \begin{matrix} \alpha(b) \\ \vdots \\ \zeta(b) \end{matrix} \right. ,\; \begin{matrix} \left\{ \begin{matrix} \alpha^{2}(b) \\ \vdots \\ \zeta(\alpha(b)) \end{matrix} \right. \\ \\ \left\{ \begin{matrix} \alpha(\zeta(b)) \\ \vdots \\ \zeta^{2}(b) \end{matrix} \right. \end{matrix} ,\; \ldots ,\; \begin{matrix} \left. \begin{matrix} \alpha^{-3}(b) \\ \vdots \\ \zeta^{-1}(\alpha^{-2}(b)) \end{matrix} \right\} \\ \left. \begin{matrix} \alpha^{-1}(\zeta^{-1}(\alpha^{-1}(b))) \\ \vdots \\ \zeta^{-2}(\alpha^{-1}(b)) \end{matrix} \right\} \\ \left. \begin{matrix} \alpha^{-2}(\zeta^{-1}(b)) \\ \vdots \\ \zeta^{-1}(\alpha^{-1}(\zeta^{-1}(b))) \end{matrix} \right\} \\ \left. \begin{matrix} \alpha^{-1}(\zeta^{-2}(b)) \\ \vdots \\ \zeta^{-3}(b) \end{matrix} \right\} \end{matrix} ,\; \begin{matrix} \left. \begin{matrix} \alpha^{-2}(b) \\ \vdots \\ \zeta^{-1}(\alpha^{-1}(b)) \end{matrix} \right\} \\ \\ \left. \begin{matrix} \alpha^{-1}(\zeta^{-1}(b)) \\ \vdots \\ \zeta^{-2}(b) \end{matrix} \right\} \end{matrix} ,\; \left. \begin{matrix} \alpha^{-1}(b) \\ \vdots \\ \zeta^{-1}(b) \end{matrix} \right\} ,\; b \right)$$

It should not be surprising to find sequences with different starting numbers to converge to the same *cyclic-subsequence* $C_i = \langle(i, h, g, \ldots, d, c, b)\rangle$ since

$$C_z = \langle z, y, \ldots, o, C_n\rangle = \langle z, y, \ldots, o, n, C_m\rangle = \ldots = \langle z, y, \ldots, o, n, m, l, k, \ldots, j, C_i\rangle.$$

Some concern might be brought forward that our *cycle-equations* focus only on the "*middle*"- or "*end"-portion* terms — say, $\langle\ldots n, m, \ldots, c, b, a, \ldots\rangle$ — of a branch or trajectory and ignoring "*the fact*" that some sequence $C_z = \langle z, y, x, \ldots, q, p, o, \ldots\rangle$ might possibly have a term $x > 1$ among its initial iterates to be the *minimum* of all of its *branch-points* so that the sequence C_z might "*conceivably be headed to infinity*". It is simply reiterated that $b = f^u(z)$ [*for* $u \geq 0$ *with* $f^0(b) = b$] is a *fixed-positioned* iterate in each of the sequences C_z with *arbitrary finite* starting numbers z — that is, say for the *preferred* *Collatz 3x+1 sequences* (*defined in the next section*) in the domain of all the integers, b is the 1*st* term (*for* $u = 0$) in C_0 and C_{-1}; or the 2*nd* term (*for* $u = 1$) in C_{-2}, C_1, C_2; or the 3*rd* term (*for* $u = 2$) in C_{-10}, C_{-7}, C_{-5}, C_{-4}, C_4; or the 4*th* term (*for* $u = 3$) in C_{-20}, C_{-14}, C_{-8}, C_8; and so on *ad infinitum* — b is just the iterate whose *successor-term* $f(b) = a$ is the *first-duplicated-term* of a sequence C_z. Also, any starting number z is easily some "*middle*" term of other sequences since $f^{-u}(z)$ [$\forall u \in \mathbf{N^+}$] are readily available *predecessor-terms* of z — in particular, C_z is a subsequence of C_ω for $\omega = 2^e \cdot z$ [$\forall e \in \mathbf{N^+}$]. The convergence or divergence of *the sequence* C_z (*that is, where* z *is the starting number*) or *the subsequence* C_z (*that is, where* z *is a middle term*) does not change wherever z is positioned in a sequence.

A *trivial Collatz 3x+1-type problem* is one in which some characteristic of its branching iteration function or *branch-nodes* expressions can be simply *computably generalized* to all the sequences for the entire domain of the starting number — for instance, see *Example 3* in *section* V later — that *enables one to immediately render a sound conclusion identifying all its existing valid cycle-length values or the divergence of its sequences*. Rather than finding some *computable generalization* for the *non-periodic* terms of a given *Collatz 3x+1-type sequences*, it is easier to find one for the *branch-nodes* expressions of the *cycle-terms* — this is the very simple and general approach presented in this paper to decide any *Collatz 3x+1-type problem*. This straightforward method does not guarantee the full solution of every *non-trivial* *Collatz 3x+1-type problem* but it suffices to prove the *Collatz 3x+1 conjecture* in the positive integers domain.

The fundamental logic of our very simple and general approach is as follows:

- ► Any *Collatz 3x+1-type sequence* has the form C_n = <n, m, l, …, c, b, a, ...> which would satisfy a *cycle-equation* $a = i \in \{n, m, l, \ldots, d, c, b, \}$ — or $f(b) = f^u(b)$ — for exactly one $u \in \mathbf{N}$ if and only if it is convergent (*otherwise, it is divergent*).
- ► If it could be established by some *computable generalization* that there could not be valid *solution-values* for b to every *cycle-equation* $f(b) = f^u(b)$ [$\forall u \in \mathbf{N}$] then we can readily conclude that all of the sequences are divergent.
- ► If it could be ascertained that there are only a finite count of valid cycles [*that is, it can be established by some computable generalization that there cannot be valid solution-values for* b *to every cycle-equation* $f(b) = f^u(b)$, $\forall u > z$ *for some* $z \in \mathbf{N}$], then:
  - ● if each valid cycle has no *portal-cycle-term* [*that is, a cycle-term* g *with at least one of the* $f^{-1}(g)$ *values that is not also a cycle-term so that some other sequences could include the loop as a periodic subsequence through* g], then there are only a finite count of fully cyclic sequences (*that is, the first-duplicated-term is also the starting number, or* $f(b) = a = n$) and the other sequences, with respective starting number that is not some valid *cycle-term*, are all divergent sequences;
  - ● if there is at least one valid cycle with at least one *portal-cycle-term*, then every *non-fully-cyclic* sequence must include exactly 1 of the valid cycles with proper *portal-cycle-terms* — that is, the sequences are partitioned into the "*equivalence classes of cyclic subsequences*" — so there is no divergent sequence;
- ► If the "*largest valid cycle-length*" could not be found by some *computable generalization*, then the prospects of valid cycles with lengths longer than "*the largest known valid cycle*" or of divergent sequences in the appropriate domain could not be ruled out — that is, this is indeed an *undecidable problem*.

Applied to the *preferred Collatz 3x+1 sequences* C_n in the positive integers domain, its *general form* warrants that any C_n has *at least* the valid solution $b = 1$ for the *cycle-equation* $f(b) = f^1(b)$ — that is, the *length*-2 cycle (2, 1). Moreover, it is established by a *computable generalization* that any C_n could only have valid solutions for *at most length*-2 cycles only. Therefore, each C_n has *exactly* only the valid *cyclic-subsequence* <(2, 1)> and there are no "*divergent*" ones.

III. *Proof of Collatz 3x+1 Conjecture in the Positive Integers Domain*

To demonstrate our very simple and general approach — and at the same time prove the *Collatz 3x+1 conjecture* in the positive integers domain — we shall decide the *preferred* ("*favored in the mathematical literature*" **[3]**) *Collatz 3x+1 problem* defined by the 2-*branch* iteration function

$$f(n)=\begin{cases}\alpha(n)=\dfrac{n}{2} & \text{if } n \text{ is even}\\[2ex] \beta(n)=\dfrac{3n+1}{2} & \text{if } n \text{ is odd}\end{cases} \quad ; \quad f^{-1}(m)=\begin{cases}\alpha^{-1}(m)=2m\\[2ex] \beta^{-1}(m)=\dfrac{2m-1}{3}\end{cases} \begin{pmatrix}\text{no pre-conditions } \Rightarrow\\ \alpha^{-1}(\text{m}) \text{ always an integer;}\\ \beta^{-1}(\text{m}) \text{ valid if an integer}\end{pmatrix}$$

with the set **Z** of all the integers as domain. In the *standard Collatz 3x+1 sequence*, any odd natural number n iterates to $3n+1$ which then iterates to $\dfrac{3n+1}{2}$ — so, the division by 2 in the *odd* case of the *preferred Collatz 3x+1 sequences* avoids trivial *even* terms.

The claim that, for every starting integer n, the *preferred Collatz 3x+1 sequence* C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates is very important in the following *solution-analysis*. The generalization to all the elements of the specified domain is very simply invoked (*some rigorous proof by mathematical induction is not required*) from the arbitrariness of n in this very brief argument: Let n be an arbitrary positive or negative integer. Then, $f(n)$ — either $\alpha(n)=\dfrac{n}{2}$ (*for n even*) or $\beta(n)=\dfrac{3n+1}{2}$ (*for n odd*) — is also a positive or negative integer, respectively. Since n is arbitrary, the contention holds (*without much ado*) for C_n [$\forall n \in$ **Z**-{0}].

We now proceed to prove the *preferred Collatz 3x+1 conjecture* in the positive integers domain.

1. We first solve for any valid trivial *length*-1 cycle — that is, $a = f(b) = b$:

- ► $\alpha(b)=\dfrac{b}{2}=b$ yields $b = 0$ — the trivial solution C_0 = <(0)> = <0, 0, 0, …>;
- ► $\beta(b)=\dfrac{3b+1}{2}=b$ yields $b = -1$ — the trivial solution C_{-1} = <(-1)> = <-1, -1, -1, …>.

Therefore, there are 2 trivial *length*-1 cycles — C_0 = <(0)> and C_{-1} = <(-1)> — however, *they are not valid periodic sequences in the positive integers domain*.

2. The presupposed *cycle-solution branch-nodes* expressions are shown below:

a* = *i* = *f*(*b*) *h* = *f*²(*b*) . . . *e* = *f*³(*b*) *d* = *f*²(*b*) *c* = *f*⁻¹(*b*) *b

$$\left(\left\{ \begin{matrix} \frac{b}{2} \\ \\ \frac{3b+1}{2} \end{matrix} \right. ,\; \begin{matrix} \left\{ \begin{matrix} \frac{b}{2^2} \\ \frac{3b+2}{2^2} \end{matrix} \right. \\ \\ \left\{ \begin{matrix} \frac{3b+1}{2^2} \\ \frac{3^2 b+(3+2)}{2^2} \end{matrix} \right. \end{matrix} ,\; \ldots,\; \left. \begin{matrix} \left. \begin{matrix} \left. \begin{matrix} 2^3 b \\ \frac{2^3 b-1}{3} \end{matrix} \right\} \\ \left. \begin{matrix} \frac{2^3 b-2}{3} \\ \frac{2^3 b-(2+3)}{3^2} \end{matrix} \right\} \end{matrix} \right. \\ \\ \left. \begin{matrix} \left. \begin{matrix} \frac{2^3 b-2^2}{3} \\ \frac{2^3 b-(2^2+3)}{3^2} \end{matrix} \right\} \\ \left. \begin{matrix} \frac{2^3 b-(2^2+2\cdot 3)}{3^2} \\ \frac{2^3 b-(2^2+2\cdot 3+3^2)}{3^3} \end{matrix} \right\} \end{matrix} \right. \end{matrix} \right. ,\; \begin{matrix} \left. \begin{matrix} 2^2 b \\ \frac{2^2 b-1}{3} \end{matrix} \right\} \\ \\ \left. \begin{matrix} \frac{2^2 b-2}{3} \\ \frac{2^2 b-(2+3)}{3^2} \end{matrix} \right\} \end{matrix} ,\; \left. \begin{matrix} 2b \\ \\ \frac{2b-1}{3} \end{matrix} \right\} ,\; b \right)$$

3. For the 2 *branch-nodes* of the *length*-2 *cycle-equation* $a = c$ or $f(b) = f^{-1}(b)$, we evaluate if there is a valid *solution-value* for b when either $\alpha(b) = \frac{b}{2} = c$ or $\beta(b) = \frac{3b+1}{2} = c$.

For the *branch-node* $c = 2b$:

► $\alpha(b) = \frac{b}{2} = 2b$ yields $b = 0$ — the trivial solution C_0 = <(0)>;

► $\beta(b) = \frac{3b+1}{2} = 2b$ yields $b = \frac{1}{2^2 - 3} = 1$ — which is a valid solution in the positive integers domain corresponding to $a = 2$ and C_2 = <(2, 1)>.

For the *branch-node* $c = \frac{2b-1}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{2b-1}{3}$ yields $b = \frac{2}{2^2-3} = 2$ — which is a valid solution in the positive integers domain corresponding to $a = 1$ and $C_1 = \langle(1, 2)\rangle$;

► $\beta(b) = \frac{3b+1}{2} = \frac{2b-1}{3}$ yields $b = -1$ — the trivial solution $C_{-1} = \langle(-1)\rangle$ — it is stressed that $\alpha^{-1}(-1) = 2(-1) = -2$ provides a *portal* or *doorway* to the loop (-1) from many starting numbers $-n$ in the negative integers domain with

$C_{-n} = \langle -n, -m, -l, \ldots, -e, -4, -2, (-1)\rangle$ where $n, m, l, \ldots, e \in \mathbf{N}^{+}$-{1}.

We could express $C_1 = \langle(1, 2)\rangle = \langle 1, (2, 1)\rangle = \langle 1, C_2\rangle$. Other than the *cycle-term* 1, the *cycle-term* 2 also has a *predecessor-term* $\alpha^{-1}(2) = 2(2) = 4$ that is not a *cycle-term* — hence, 2 is the *portal-cycle-term* of the loop (2, 1) [*that is, many* (*perhaps, all*) *sequences with starting positive natural numbers other than* 1 *or* 2 *would include the cycle* (2, 1) *as a periodic subsequence by way of the cycle-term* 2].

4. That, in fact, *every preferred Collatz 3x+1 sequence in the positive integers domain has one of the following forms* —

$C_2 = \langle(2, 1)\rangle$;

$C_1 = \langle(1, 2)\rangle = \langle 1, (2, 1)\rangle = \langle 1, C_2\rangle$; *and*

$C_n = \langle n, m, l, \ldots, f, 8, 4, (2, 1)\rangle = \langle n, m, l, \ldots, f, 8, 4, C_2\rangle$ [$\forall n \in \mathbf{N}^{+}$-{1, 2}]

— immediately ensues from the following plain reasoning [$\forall u \in \mathbf{N}^{+}$-{1}]:

a. For any *length*-(u+1) *cyclic-subsequence* $\langle(i, h, g, \ldots, d, c, b)\rangle$, the *branch-node* equation $\alpha^{-u}(b) = \alpha(b)$ will always yield only the trivial *length*-1 *cycle-solution* $C_0 = \langle(0)\rangle$ while the *branch-node* equation $\beta^{-u}(b) = \beta(b)$ will always yield only the trivial *length*-1 *cycle-solution* $C_{-1} = \langle(-1)\rangle$.

b. In addition to the 2 *branch-nodes* expressions that produce the 2 trivial *length*-1 *cycles* C_0 = <(0)> and C_{-1} = <(-1)>, an additional *branch-node* expression must yield the valid *solution-value* $b_1 = b > 2$; another *branch-node* expression must yield the valid *solution-value* $b_2 = c > 2$; another *branch-node* expression must yield the valid *solution-value* $b_3 = d > 2$; …; and some different *branch-node* expression must yield the valid *solution-value* $b_{u+1} = i > 2$. The count $u+1$ of *cycle-terms* is finite — so, there is a *minimum-valued cycle-term* $b_{\min}$ [*which must be some odd natural number in the positive integers domain in order for the first-duplicated-cycle-term* $i = f(b_{\min})$ *to be greater than* $b_{\min}$] among the *cycle-terms* $\{i, h, g, …, d, c, b\}$ or $\{b_{u+1}, b_u, b_{u-1}, …, b_3, b_2, b_1\}$.

c. As could be readily generalized from the supposed *cycle-solution branch-nodes expressions* depicted in step 2 above, for the u-times applications of the inverse function f^{-1} (*that is, either* α^{-1} *or* β^{-1} *as appropriate*) on b, each of the $2^u - 2$ *nontrivial-length*-1-*nodes-expressions* of the *branch-point* $f^{-u}(b)$ has the form

$$\frac{2^u b - S}{3^v} \quad \text{where} \quad 0 < v \le u \quad \text{with} \quad u > 1;$$

$$S = \sum (2^p \cdot 3^q) > 0 \quad \text{with} \quad p, q \in \mathbf{N}.$$

Thus, for each such *branch-node* expression, the 2 *cycle-equations* $f^{-u}(b) = \alpha(b)$ and $f^{-u}(b) = \beta(b)$ [$\forall u > 1$ *or for all cycle-lengths greater than* 2] that have to be evaluated for valid *solution-values* for b are, respectively:

$$\frac{2^u b - S}{3^v} = \frac{b}{2} \quad \text{or} \quad b = \frac{2S}{2^{u+1} - 3^v} \qquad [1]$$

and

$$\frac{2^u b - S}{3^v} = \frac{3b+1}{2} \quad \text{or} \quad b = \frac{2S + 3^v}{2^{u+1} - 3^{v+1}} \qquad [2].$$

We emphasize that the numerator in the *right-hand* side of equation [1] is less than the numerator in the *right-hand* side of equation [2] while the denominator in the former is greater than the denominator in the latter — that is, equation [1] would indeed yield a smaller *solution-value* for b (*in the specified domain*) than equation [2]. In particular, this should be true for the *branch-node expression* of the respective *minimum-valued cycle-term* b_{min} [$\forall u > 1$ *or for all cycle-lengths greater than* 2] that is valid in the domain of discourse — but this implies that b_{min} should be an even integer (*equation* [1] *applies only to the even iterates*) so b_{min} cannot be a positive integer since, otherwise, $0 < \alpha(b_{min}) = \frac{b_{min}}{2} < b_{min}$ — contradicting b_{min}'s supposed minimality among its *all-positive-natural-number cycle-terms*. Any *preferred* *Collatz 3x+1 sequence* with *length-greater-than-*2 cyclic subsequence must, therefore, belong to the domain of negative integers.

d. Because there are no cycles other than (2, 1) in the domain of positive integers, *by the arbitrariness of b in all of the cycle-equations*, every *Collatz 3x+1 sequence* C_n must include the *length-*2 cycle (2, 1) as periodic subsequence — hence, there is no "*divergent*" *all-positive-integer-terms* C_n.

► The *form* of some infinite sequence of, say, binary digits prefixed by a fractional expansion point — for example, **.01011100011…** — readily establishes the fact that it (*as a real number point or as a subinterval of the unit interval*) indubitably lies between 0 and 1 or in the interval (0,1). Likewise, the *general form* of the *preferred* *Collatz 3x+1 sequence* C_n, (*by the very definition of the* *Collatz 3x+1 iteration function*) directly warrants that C_n always has *at least* the term $b = 1$ which satisfies the *length-*2 *cycle-equation* $f(b) = f^{-1}(b)$ in the positive integers domain.

The finite count of valid cyclic subsequences exactly partition all the *Collatz 3x+1-type sequences* into the "*equivalence classes of valid cyclic subsequences*" — that is, every *Collatz 3x+1-type sequence* includes precisely one of the available valid periodic subsequences.

Because we have categorically ruled out the possibility of any valid solution in the positive integers domain to the other *cycle-equations* with length different from 2 — that is, we have resolutely established that the *general form* of the *Collatz* 3*x*+1 *sequence* C_n yields valid solution of *at most* (*therefore, exactly*) $b = 1$ in the domain of positive integers for only the *cycle-equation* $f(b) = f^{-1}(b)$ — then, it trivially follows that each *preferred Collatz* 3*x*+1 *sequence* C_n includes the *cyclic-subsequence* <(2, 1)> and, thus, there is no "*divergent*" *all-positive-integer-terms* C_n.

► In related perspective, the claim of *no-divergent-all-positive-integer-terms Collatz 3x+1 sequence* is also supported by these *intuitive* reasoning (*but which are, unfortunately, very difficult to computably generalize*):

- The ongoing Internet online *Collatz 3x+1 project* **[70]** provides *experimental* evidence of the veracity of the *Collatz conjecture*.
- Since each current term is either divided by 2 or multiplied by 1.5, then succeeding terms eventually decreases so any *Collatz 3x+1 sequence* finally converges — a *probabilistic* argument **[2]**.
- The sequence $C_z = <z, y, x, w, \ldots>$ is convergent or "*divergent*" if and only if each subsequence C_y, C_x, C_w, … is convergent or "*divergent*", respectively. The *Collatz 3x+1 iteration function* ***f*** is *surjective* (*or, onto*) — that is, each positive integer always has (*at least*) one valid *predecessor-term*. In every convergent sequence $C_z = <z, y, \ldots, 4, (2, 1)>$, the *subsequent-to-z-terms* to the left of term 2 have corresponding convergent subsequences C_y, C_x, …, C_8, C_4 which also have their respective convergent *predecessor-terms* sequences, *ad infinitum*; thus, every positive integer appears at least once (*in addition to being the starting number of a Collatz 3x+1 sequence*) in this *inter-connections of convergent sequences* — a *realistic* argument.

IV. *Undecidability of Collatz Conjecture in Negative Integers Domain*

We continue analyzing the *preferred* Collatz 3x+1 problem in the domain of negative integers.

1. For the 4 *branch-nodes* of term $d = f^2(b)$, we evaluate if there is a valid solution for b when either $\alpha(b) = \frac{b}{2} = d$ or $\beta(b) = \frac{3b+1}{2} = d$.

For the *branch-node* $d = 2^2 b = 4b$:

► $\alpha(b) = \frac{b}{2} = 4b$ yields $b = 0$ — the trivial solution C_0 = <(0)>;

► $\beta(b) = \frac{3b+1}{2} = 4b$ yields $b = \frac{1}{5}$ — which is not a valid solution.

For the *branch-node* $d = \frac{2^2 b - 1}{3} = \frac{4b-1}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-1}{3}$ yields $b = \frac{2}{5}$ — which is not a valid solution;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-1}{3}$ yields $b = \frac{2+3}{2^3 - 3^2} = -5$ — which is a valid solution in the negative integers domain corresponding to C_{-7} = <(-7, -10, -5)>.

For the *branch-node* $d = \frac{2^2 b - 2}{3} = \frac{4b-2}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-2}{3}$ yields $b = \frac{4}{5}$ — which is not a valid solution;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-2}{3}$ yields $b = \frac{2^2+3}{2^3 - 3^2} = -7$ — which is a valid solution in the negative integers domain corresponding to C_{-10} = <(-10, -5, -7)>.

For the *branch-node* $d = \frac{2^2 b - (2+3)}{3^2} = \frac{4b-5}{9}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-5}{9}$ yields $b = \frac{2 \cdot 5}{2^3 - 3^2} = -10$ — which is a valid solution in the negative integers domain corresponding to C_{-5} = <(-5, -7, -10)>;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-5}{9}$ yields $b = -1$ — the trivial solution C_{-1} = <(-1)>.

Both -7 and -10 are *portal-cycle-terms* with $\alpha^{-1}(-7) = -14$ and $\alpha^{-1}(-10) = -20$. Thus, the *preferred* *Collatz 3x+1 sequences* with *length*-3 *cyclic-subsequences* are the following:

C_{-5} = <(-5, -7, -10)>;

C_{-7} = <(-7, -10, -5)>;

C_{-10} = <(-10, -5, -7)>;

C_{-n_1} = <$-n_1$, $-m_1$, $-l_1$, …, -14, (-7, -10, -5)> for many $n_1 \in \mathbf{N}^+$-{5, 7, 10};

C_{-n_2} = <$-n_2$, $-m_2$, $-l_2$, …, -20, (-10, -5, -7)> for many other $n_2 \in \mathbf{N}^+$-{5, 7,10}.

2. It is left to the reader to verify that there is no cycle or cyclic subsequence with length 4, 5, 6, 7, 8, 9, or 10 and that there are *length*-11 cycles or cyclic subsequences —

$$f(b) = f^{10}(b) \Leftrightarrow b \in \{-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34\}$$

— whose *branch-nodes* expressions are exhibited in Table 1. The *portal-cycle-terms* are -25, -34, -37, -55, -61, -82, -91, and -136. So, the *preferred* *Collatz 3x+1 sequences* in the negative integers domain with *length*-11 cyclic subsequences are the following:

C_{-17} = <(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)>

C_{-25} = <(-25, -37, -55, -82, -41, -61, -91, -136, -68, -34, -17)>

C_{-34} = <(-34, -17, -25, -37, -55, -82, -41, -61, -91, -136, -68)>

C_{-37} = <(-37, -55, -82, -41, -61, -91, -136, -68, -34, -17, -25)>

C_{-41} = <(-41, -61, -91, -136, -68, -34, -17, -25, -37, -55, -82)>

C_{-55} = <(-55, -82, -41, -61, -91, -136, -68, -34, -17, -25, -37)>

C_{-61} = <(-34, -17, -25, -37, -55, -82, -41, -61, -91, -136, -68)>

C_{-68} = <(-68, -34, -17, -25, -37, -55, -82, -41, -61, -91, -136)>

C_{-82} = <(-82, -41, -61, -91, -136, -68, -34, -17, -25, -37, -55)>

C_{-91} = <(-91, -136, -68, -34, -17, -25, -37, -55, -82, -41, -61)>

C_{-136} = <(-136, -68, -34, -17, -25, -37, -55, -82, -41, -61, -91)>

Table 1

***Branch-Nodes* Expressions for *Length*-11 Cycles or Cyclic Subsequences**

— <(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)> —

of many Preferred Collatz 3x+1 Sequences in the Negative Integers Domain

l $f^{10}(b)$	***k*** $f^{9}(b)$	***j*** $f^{8}(b)$	***i*** $f^{7}(b)$	***h*** $f^{6}(b)$	***g*** $f^{5}(b)$	***f*** $f^{4}(b)$	***e*** $f^{3}(b)$	***d*** $f^{2}(b)$	***c*** $f^{1}(b)$	***b***
$\frac{1024b-817}{729}$	$\frac{512b-287}{243}$	$\frac{256b-103}{81}$	$\frac{128b-38}{27}$	$\frac{64b-19}{27}$	$\frac{32b-5}{9}$	$\frac{16b-1}{3}$	$8b$	$4b$	$2b$	-17
$\frac{1024b-1373}{729}$	$\frac{512b-565}{243}$	$\frac{256b-242}{81}$	$\frac{128b-121}{81}$	$\frac{64b-47}{27}$	$\frac{32b-19}{9}$	$\frac{16b-8}{3}$	$\frac{8b-4}{3}$	$\frac{4b-2}{3}$	$\frac{2b-1}{3}$	-25
$\frac{1024b-2207}{729}$	$\frac{512b-982}{243}$	$\frac{256b-491}{243}$	$\frac{128b-205}{81}$	$\frac{64b-89}{27}$	$\frac{32b-40}{9}$	$\frac{16b-20}{9}$	$\frac{8b-10}{9}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-37
$\frac{1024b-3458}{729}$	$\frac{512b-1729}{729}$	$\frac{256b-743}{243}$	$\frac{128b-331}{81}$	$\frac{64b-152}{27}$	$\frac{32b-76}{27}$	$\frac{16b-38}{27}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-55
$\frac{1024b-5699}{2187}$	$\frac{512b-2485}{729}$	$\frac{256b-1121}{243}$	$\frac{128b-520}{81}$	$\frac{64b-260}{81}$	$\frac{32b-130}{81}$	$\frac{16b-65}{81}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-82
$\frac{1024b-2485}{729}$	$\frac{512b-1121}{243}$	$\frac{256b-520}{81}$	$\frac{128b-260}{81}$	$\frac{64b-130}{81}$	$\frac{32b-65}{81}$	$\frac{16b-19}{27}$	$\frac{8b-5}{9}$	$\frac{4b-1}{3}$	$2b$	-41
$\frac{1024b-3875}{729}$	$\frac{512b-1816}{243}$	$\frac{256b-908}{243}$	$\frac{128b-454}{243}$	$\frac{64b-227}{243}$	$\frac{32b-73}{81}$	$\frac{16b-23}{27}$	$\frac{8b-7}{9}$	$\frac{4b-2}{3}$	$\frac{2b-1}{3}$	-61
$\frac{1024b-5960}{729}$	$\frac{512b-2980}{729}$	$\frac{256b-1490}{729}$	$\frac{128b-745}{729}$	$\frac{64b-251}{243}$	$\frac{32b-85}{81}$	$\frac{16b-29}{27}$	$\frac{8b-10}{9}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-91
$\frac{1024b-9452}{2187}$	$\frac{512b-4726}{2187}$	$\frac{256b-2363}{2187}$	$\frac{128b-817}{729}$	$\frac{64b-287}{243}$	$\frac{32b-103}{81}$	$\frac{16b-38}{27}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-136
$\frac{1024b-4726}{2187}$	$\frac{512b-2363}{2187}$	$\frac{256b-817}{729}$	$\frac{128b-287}{243}$	$\frac{64b-103}{81}$	$\frac{32b-38}{27}$	$\frac{16b-19}{27}$	$\frac{8b-5}{9}$	$\frac{4b-1}{3}$	$2b$	-68
$\frac{1024b-2363}{2187}$	$\frac{512b-817}{729}$	$\frac{256b-287}{243}$	$\frac{128b-103}{81}$	$\frac{64b-38}{27}$	$\frac{32b-19}{27}$	$\frac{16b-5}{9}$	$\frac{8b-1}{3}$	$4b$	$2b$	-34

C_{-n_1} = <-n_1, -m_1, -l_1, …, -50, (-25,-37,-55,-82,-41,-61,-91,-136,-68,-34,-17)>

for many $n_1 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_2} = <-n_2, -m_2, -l_2, …, -23, (-34,-17,-25,-37,-55,-82,-41,-61,-91,-136,-68)>

for many other $n_2 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_3} = <-n_3, -m_3, -l_3, …, -74, (-37,-55,-82,-41,-61,-91,-136,-68,-34,-17,-25)>

for many other $n_3 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_4} = <-n_4, -m_4, -l_4, …, -110, (-55,-82,-41,-61,-91,-136,-68,-34,-17,-25,-37)>

for many other $n_4 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_5} = <-n_5, -m_5, -l_5, …, -122, (-61,-91,-136,-68,-34,-17,-25,-37,-55,-82,-41)>

for many other $n_5 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_6} = <-n_6, -m_6, -l_6, …, -164, (-82,-41,-61,-91,-136,-68,-34,-17,-25,-37,-55)>

for many other $n_6 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_7} = <-n_7, -m_7, -l_7, …, -182, (-91,-136,-68,-34,-17,-25,-37,-55,-82,-41,-61)>

for many other $n_7 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_8} = <-n_8, -m_8, -l_8, …, -272, (-136,-68,-34,-17,-25,-37,-55,-82,-41,-61,-91)>

for many other $n_8 \in$ **N⁺-**{17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136}.

3. Without some *computable generalization* that identifies all the valid cycles, or which establishes that there are no more valid loops with lengths greater than 11 (*nor divergent sequences*), for the *preferred* *Collatz 3x+1 sequences* in the negative integers domain, this problem would remain *undecidable* inasmuch as it is *exponentially computationally complex* — that is, for arbitrary *branch-point* $f^{-u}(b)$, there are 2^{u+1} *cycle-equations* to be evaluated for valid *solution-values* for *b* and there will not be adequate computing space and time resources to solve all of these *cycle-equations* at some stage, say for $u > 100$ [*in fact, theoretical physicists have computed* **[71]** *that there are less than* 10^{78} ($\sim 2^{260}$) *particles in the entire observable universe while the age of the universe after the deemed Big Bang is less than* 10^{40} ($\sim 2^{133}$) *atomic units or* 10^{18} ($\sim 2^{60}$) *seconds*].

V. *More Examples of Collatz 3x+1-type Problems*

The capable readers could just write their own simple computer programs to easily automate the evaluation of *solution-values* for b in the *general* *Collatz 3x+1 sequences' cycles-equations* (*which are merely linear equations in* 1 *variable*). The interested readers could solve (*and verify for themselves the reliability of the simple and general method demonstrated in this manuscript*) the following additional examples of *Collatz 3x+1-type problems* taken from the references listed at the ending pages of this paper.

Example 1. The *standard* *Collatz 3x+1 sequences* **[1, 2, 5]** are defined for positive integers:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if} \quad n \equiv_2 0 \\ 3n+1 & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n (*in the domain of all integers*) has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates. These are known:

- 1 trivial *length*-1 cycle — (0);
- 1 *length*-2 cycle — (-2, -1); 1 *length*-5 cycle — (-5, -14, -7, -20, -10); 1 *length*-18 cycle — (-17, -50, -25, -74, -37, -110, -55, -164, -82, -41, -122, -61, -182, -91, -272, -136, -68, -34) — in the negative integers domain; *and*
- 1 *length*-3 cycle — (4, 2, 1) — with *all-positive-integers* terms.

The same very simple argument for the *preferred* *Collatz 3x+1 sequences* in section III above readily justifies the conclusion that each *standard* *Collatz 3x+1 sequence* in the domain of positive integers has one of the following forms:

$C_4 = <(4, 2, 1)>$;

$C_1 = <1, (4, 2, 1)> = <1, C_4>$;

$C_2 = <2, 1, (4, 2, 1)> = <2, 1, C_4>$;

$C_n = <n, m, \ldots, 16, 8, (4, 2, 1)> = <n, m, \ldots, 8, C_4)>; \forall n \in \mathbf{N^+}\text{-}\{1, 2, 4\}$.

Example 2. The *original* Collatz *sequences* **[5, 64]** are defined for positive integers:

$$f(n)=\begin{cases}\dfrac{2n}{3} & \text{if} \quad n\equiv_3 0\\[2ex] \dfrac{4n-1}{3} & \text{if} \quad n\equiv_3 1\\[2ex] \dfrac{4n+1}{3} & \text{if} \quad n\equiv_3 2\end{cases};\qquad f(-n)=\begin{cases}-\dfrac{2n}{3} & \text{if} \quad n\equiv_3 0\\[2ex] -\dfrac{4n-1}{3} & \text{if} \quad n\equiv_3 1\\[2ex] -\dfrac{4n+1}{3} & \text{if} \quad n\equiv_3 2\end{cases}$$

It is easily verified that any *original* Collatz *sequence* C_n, $\forall n \in \mathbf{Z}$, has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates and that they are *symmetric* with respect to the positive integers and negative integers domain.

In the domain of all integers, it is known that there are:

- ► 3 trivial *length*-1 cycles — (0), (1), and (-1);
- ► 2 *length*-2 cycles — (2, 3) and (-2, -3);
- ► 2 *length*-5 cycles — (4, 5, 7, 9, 6) and (-4, -5, -7, -9, -6); *and*
- ► 2 *length*-12 cycles — (44, 59, 79, 105, 70, 93, 62, 83, 111, 74, 99, 66) and (-44, -59, -79, -105, -70, -93, -62, -83, -111, -74, -99, -66).

Without some *computable generalization* that would rule out the prospect of valid *length-greater-than*-12 cycles, the complete determination of all the valid *cycle-lengths* of the *original* Collatz *sequences* is a *computationally exponentially complex* process and is an *undecidable* problem. Specifically, whether or not

C_8 = <8, 11, 15, 10, 13, 17, 23, 31, 41, 55, 73, 97, ...>

eventually converges to some periodic subsequence would remain to be actually an *undecidable* problem until somebody establishes it otherwise — which is very unlikely since the probability of its being a divergent sequence is higher because there are 2 *sub-functions* that increases, and only 1 *sub-function* that decreases, *successor-term* values.

Example 3. The following *Collatz 3x+1-type sequences* **[5]** are defined for all integers:

$$f(n) = \begin{cases} \dfrac{3n}{2} & \text{if} \quad n \equiv_2 0 \\ \\ \dfrac{3n+1}{2} & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* terms. There are 2 trivial *length*-1 cycles — (0) and (-1). Because each *sub-function* is either monotonic increasing (*for positive integer starting numbers*) or monotonic decreasing (*for negative integer less than* -1 *starting numbers*), then there are no cycles or cyclic subsequences with lengths greater than 1 — that is, each *nontrivial-length*-1 sequence is indeed divergent.

Example 4. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 5 cycles with starting numbers 0, 1, 13, 17, and -1:

$$f(n) = \begin{cases} \alpha(n) = \dfrac{n}{2} & \text{if} \quad n \equiv_2 0 \\ \\ \beta(n) = \dfrac{5n+1}{2} & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates. In the *all-integers* domain, these are known:

- ► 1 trivial *length*-1 cycle — (0);
- ► 1 *length*-5 cycle — (1, 3, 8, 4, 2); as well as 2 *length*-7 cycles — (13, 33, 83, 208, 104, 52, 26) and (17, 43, 108, 54, 27, 68, 34) — in the positive integers domain; *and*
- ► 1 *length*-2 cycle — (-2, -1) — in the negative integers domain.

Following the same basic analysis as with the *Collatz 3x+1 sequences*:

- ► The trivial *length*-1 cycle (0) [*that is, for* $u = 0$] as well as the *length*-2 cycle (-2, -1) [*that is, for* $u = 1$] are easily computed.
- ► For $u > 1$, each of the $2^u - 2$ *nontrivial-length*-1 *nodes-expressions* of the *branch-point* $\boldsymbol{f}^u(b)$ has the form

$$\frac{2^u b - S}{5^v} \quad \text{where } 0 < v \le u, \quad S = \sum (2^p \cdot 5^q) > 0 \text{ with } p, q \in \mathbf{N}.$$

The *solution-values* for $b_{\min}$ are obtained from either

$$b = \frac{2S}{2^{u+1} - 5^v} \quad [1] \qquad or \qquad b = \frac{2S + 5^v}{2^{u+1} - 5^{v+1}} \quad [2].$$

If the domain is restricted only to the negative integers, then there will be a contradiction that equation [2] should yield $b_{\min} < 0$ (*that is, with greater absolute value*) and the fact that it should be applied to odd integers only. Thus, there are no valid cycles other than (-2, -1) in the negative integers domain — hence, every *all-negative-integers-terms* sequence must include the *cyclic-subsequence* <(-2, -1)>.

The possibilities exist in the positive integers domain for sequences having loops with lengths greater than 7 (*including divergent ones*).

Example 5. The following *Collatz 3x+1-type sequences* **[1, 24]** (*defined for all integers*) are claimed to have 3 cycles with starting numbers 0, -1, and -2:

$$f(n) = \begin{cases} \alpha(n) = 2n & \text{if} \quad n \equiv_3 0 \\ \beta(n) = \dfrac{7n+2}{3} & \text{if} \quad n \equiv_3 1 \\ \gamma(n) = \dfrac{n-2}{3} & \text{if} \quad n \equiv_3 2 \end{cases}.$$

As Example 2.7 in **[24]**, *Keith Matthews* offers a prize of $100.00 (*Australian*) [*also advertised in* **[1]** *but with a different* $\alpha(n) = 7n + 3$] for a valid proof of his (*not altogether correct*) conjecture that *all divergent trajectories starting in the congruence classes* $n \equiv_3 \pm 1$ *appear to eventually enter the zero residue class mod* 3 and *if* $f^k(n) \equiv_3 \pm 1$, $\forall k \geq 0$*, then the trajectory must eventually enter one of the cycles* (-1) *or* (-2, -4).

It is easily verified that any sequence C_n has either *all-0* or *all-negative-integers* or *all-positive-integers* terms except for $n = \gamma^{-u}(2)$, $\forall u \in \mathbf{N^+}$. There are 2 trivial *length*-1 cycles — (0) and (-1) — but there are no sequences having either <(0)> or <(-1)> as cyclic subsequence.

If $n \in \mathbf{Z}$-{0} and $n \equiv_3 0$, or $n = 3p$ for some $p \in \mathbf{Z}$-{0}, then

$$f(n) = \alpha(n) = 2n = 2{\bullet}3p \equiv_3 0$$

and, consequently,

$$C_{3p} = \langle 3p,\ 2{\bullet}3p,\ 2^2{\bullet}3p,\ 2^3{\bullet}3p,\ \ldots\rangle$$ (*which are divergent sequences*).

All the *nontrivial-length*-1-*branch-node-expressions* for *length*-2 *cycle-equations* $f^{-1}(b) = f(b)$ yield only valid *solution-values* for the sole *length*-2 cycle (-4, -2) with -4 as the *portal-cycle-term*. The possibilities for cycles with lengths greater than 2 in the negative integers domain could not be ruled out — so, only for many starting negative integers $-n \notin \{-1, -2, -4\}$ could we claim C_{-n} to converge to C_{-4}:

$$C_{-n} = \langle -n, -m, -l, \ldots, -f, -e, -10, (-4, -2)\rangle$$

$$= \langle -n, -m, -l, \ldots, -f, -e, -10, C_{-4}\rangle$$ where all of $n, m, l, \ldots, f, e \equiv_3 \pm 1$.

Each *all-negative-integers* divergent sequence has one of the form:

$$C_{-3p} = \langle -3p,\ -2{\bullet}3p,\ -2^2{\bullet}3p,\ -2^3{\bullet}3p,\ \ldots\rangle, \forall p \in \mathbf{N^+}$$ *or*

$$C_{-n} = \langle -n, -m, -l, \ldots, -d, -c, -b, -2b, 2^2b, -2^3b, \ldots\rangle,$$

where $n \in \mathbf{N^+}$-{1, 2, 4} and all of $n, m, l, \ldots, d, c \equiv_3 \pm 1$ while $b \equiv_3 0$.

In the positive integers domain, we first note the following:

C_2 = <2, C_0> = <2, 0, 0, 0, …> —

so, $C_{\gamma^{-u}(2)}$ = <$\gamma^{-u}(2)$, $\gamma^{-u+1}(2)$, $\gamma^{-u+2}(2)$, …, $\gamma^{-2}(2)$, $\gamma^{-1}(2)$, C_2>, $\forall u \in \mathbf{N}^+$;

C_{3p} = <$3p$, $2\bullet 3p$, $2^2\bullet 3p$, $2^3\bullet 3p$, …>, $\forall p \in \mathbf{N}^+$;

C_{3p+2} = <$3p + 2$, C_p>, $\forall p \in \mathbf{N}^+$.

For arbitrarily large $n \in \mathbf{N}^+$, it could be established (*by directly going through each starting natural number from* 1 *to* n) that C_n converges to C_0 or diverges with C_{3z} for some $z \in \mathbf{N}^+$. Since $p < 3p+2$, the convergence of C_{3p+2} to C_0 or divergence of C_{3p+2} with C_{3z} immediately follows from the already known convergence of C_p to C_0 or divergence of C_p with C_{3z}.

For the case $n \in \mathbf{N}^+$, $n \equiv_3 1$ or $n = 3p+1$ for some $p \in \mathbf{N}$, it could be rigorously shown that it is impossible for C_{3p+1} not to eventually have a subsequence C_{3z} or C_{3z+2} for some $z \in \mathbf{N}^+$ — that is, there is no starting positive integer $n \equiv_3 1$ such that $f^u(n) \equiv_3 1$, $\forall u \in \mathbf{N}^+$ simply because:

$$C_{3p+1} = \left\langle 3p+1,\ 7p+3,\ \frac{7^2 p + 7\cdot 3 + 2}{3}, ... \right\rangle$$ implies that $p = 3q+1$ with $q < p$,

which upon substituting $3q+1$ for p,

$$C_{3^2 q+2^2} = \left\langle 3^2 q + 2^2,\ 7\cdot 3q + 2\cdot 5,\ 7^2 q + 2^3\cdot 3,\ \frac{7^3 q + 2^3\cdot 3\cdot 7 + 2}{3}, ... \right\rangle$$

implies that $q = 3r+1$ with $r < q$; . . . which upon substituting $3r+1$ for q . . . implies that $r = 3s+1$ with $s < r$; . . . which upon substituting $3s+1$ for r . . . implies that $s = 3t+1$ with $t < s$; . . . and so on *ad infinitum*, which is impossible by the *principle of no infinite descent* or by the *well-ordering principle of the natural numbers* (*that is, there is a smallest positive natural number* 1).

VI. *Conclusion*

In **[5]**, *Jeffrey Lagarias* made the following conclusion:

> "The difficulty of settling the 3x+1 problem seems connected to the fact that it is a deterministic process that simulates '*random*' behavior. We face this dilemma: On the one hand, to the extent that the problem has structure, we can analyze it — yet, it is precisely this structure that seems to prevent us from proving that it behaves '*randomly*'. On the other hand, to the extent that the problem is *structureless* and '*random*', we have nothing to analyze and, consequently, cannot rigorously prove anything. Of course, there remains the possibility that someone will find some hidden regularity in the 3x+1 problem that allows some of the conjectures about it to be settled. The existing general methods in *number theory* and *ergodic theory* do not seem to touch the 3x+1 problem; in this sense, it seems intractable at present."

The "*randomness of iterates*" alluded to *Collatz 3x+1-type sequences* in the above quote is brought on by the more than one branches of its iteration function (*on the whole, the bifurcation thwarts the computable generalizability of Collatz 3x+1-type sequences*) — therefore, it is inherently not a *bijective* (*or,* 1-*to*-1) function so that it is extremely difficult to computably generalize the same behavior of the iterates for all the sequences with infinite count of distinct starting numbers. However, very reminiscent of the quite notorious misleading "*Search for the Missing Dollar*" *puzzle* **[72],** the irregularity of the "*random*" terms of *Collatz 3x+1-type sequences* is truly *deceiving* (*one just have to review the numerous advanced mathematical methods and computational techniques applied in vain to solve the Collatz 3x+1 problem alluded to earlier in the introduction section*) because ***only the common cycle-terms of the related sequences are the aptly relevant iterates to take into worthwhile consideration in order to decide whether or not a given set of Collatz-type 3x+1 sequences have ending cyclic subsequences***. It is basically this failed realization that prolonged the truly injudicious "*undecidability*" view of the *Collatz 3x+1 conjecture* for the positive integers.

The truly successful applicability of our very simple and general approach to decide many *Collatz 3x+1-type problems* guarantees its own effective tenability as a *solution-method*. When viewed from its beginning *non-periodic* iterates, all the *Collatz 3x+1 sequences* in the positive integers domain defied every attempt of *computable generalization* — hence, the unjustified clamors of "*unsolvability*" of the very simply stated *Collatz 3x+1 conjecture*. However, a straightforward look at the ending periodic terms provides a *clear-cut* *computable generalization* that readily rules out the *existential-possibility* of an *all-positive-integer-terms* *Collatz 3x+1 sequence* with cyclic subsequence other than $C_2 = <(2, 1)>$ or which does not converge to C_2 — hence, quickly proving the *Collatz 3x+1 conjecture*.

The existence of a very simple proof of the *Collatz 3x+1 conjecture* has pervasive ramifications in the *theory of computation* — *computability theory* and *computational complexity theory*. It is purely the absence of some known appropriate *computable generalization* that exposes any *well-posed non-self-contradictory* mathematical problem to misbranding as "*computationally complex*" or "*undecidable*" or "*unsolvable*". In actuality, the *computability* concern and the "*computational complexity classes*" categorization do not properly apply to the mathematical problem but distinctly to each of its diverse proposed *solution-methods* in their suitable domains — that is, the *Collatz 3x+1 problem* is easily solvable (*by the very simple and general solution-technique that we have demonstrated in this paper*) in spite of its having numerous unsuccessful proposed *solution-approaches* (*please refer back to those mentioned in the introductory section*) or that the *brute-force solution-method* of evaluating individually its 2^{u+1} ($u \to \infty$) *cycle-equations* to find all its valid cyclic subsequences and to rule out "*divergent*" sequences is " *computationally exponentially complex*". Therefore, we have also truly affirmed that "*computational complexity classifications*" of mathematical problems are indeed untenable *ab initio* — the very simple and general *solution-approach* involving *computable generalization* that we just established does not require any computing resource at all (*whither* *Church-Turing thesis? **P** ≠ **NP**?*).

Likewise, the simple *solvability* of the *Collatz 3x+1 problem* in the *positive integers* domain and its *undecidability* in the *negative integers domain* have countless repercussions in resolving the "*modern crisis in the foundations of mathematics*" issues — in particular, in *David Hilbert*'s *entscheidungsproblem* ("*decision problem*") of mathematical logic as well as in the intrinsic (*that is, needs no* Hilbert-*sought* "*proof*") *consistency* but *incompleteness* of mathematics. **[73]**

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