Title: **Collatz 3x+1 Conjecture Proved!**

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Abstract

This manuscript presents a very simple and general approach for deciding any *Collatz 3x+1-type problem* — that is, finding all the cycles (*if any*) of integer sequences recursively defined by some *branching-function*. By straightforward application of the easy method developed, the *Collatz 3x+1 conjecture* in the positive integers domain is proved while its inherent *computational complexity* and *undecidability* in the negative integers domain is explained.

***The Collatz 3x+1 syndrome* — the unwarranted *unsolvability-hyping* of *Collatz 3x+1-type problems* — is revealed, defused, and related to the prevalent *flimflams in the foundations of mathematics* that are the harbingers of the "*modern crisis in mathematics*"**
[*for examples, the failure to distinguish the* **determinate sets** *from the* **non-computably-generalizable collections** (*which are not true* **sets**) *of objects with regard to* **1-1** *correspondence, cardinality|countability, size-comparison, power set, etc.; the unjustified cries of* "**unsolvable**" *mathematical problems due to the use of the untenable* ***reduction-to-self-contradiction* "*proof*"** *or* ***Cantor's anti-diagonal argument;*** *and the mislabeling of* "**computational complexity classes**" *to the problems but which are actually merely applicable to their proposed solution-methods*].

Collatz 3x+1 Conjecture Proved!

I. *Introduction*

The *standard* *Collatz problem* **[1]** asks if, or the *Collatz conjecture* **[2]** claims that, iterating

$$x_n = \begin{cases} \dfrac{x_{n-1}}{2} & \text{if } x_{n-1} \text{ is even} \\ \\ 3x_{n-1} + 1 & \text{if } x_{n-1} \text{ is odd} \end{cases}$$

always return to 1 for every starting positive integer x_0. In **[3]**, *Marc Chamberland* exclaims: "*The 3x+1 problem is perhaps today's most enigmatic unsolved mathematical problem — it can be explained to a child who has learned how to divide by 2 and multiply by 3; yet, there are relatively few strong results toward solving it. Paul Erdös was correct when he stated:* '*Mathematics is not ready for such problems*'." **[4, 5]**

The authoritative *up-to-date* reference to the *Collatz 3x+1 problem* is **[6]** — it is categorically declared that (*as of* 8 August 2006): "*At present, the 3x+1 conjecture remains unsolved.*". This "*enigma*" is a source of recreational mathematical diversions **[7, 8, 9, 10, 11, 12, 13, 14]** and it has been subjected to numerous advanced number theory and theoretical computer science computational methods for its resolution — such as extending the domain to *rational numbers* **[15, 16, 17, 18]** or to the *real number line* **[19, 20, 21]** or the entire *complex plane* **[22, 23]**; *ergodic theory* or *Markov chains* **[5, 24, 25, 26, 27, 28, 29, 30, 31]**; *finite automata* **[32]**; *cellular automata* **[33, 34, 35]**; *graph theory* **[36, 37]**; *functional equations* **[38, 39, 40]**; *difference equations* **[41, 42, 43]**; *dynamical systems* **[44, 45, 46]**; *conjugacy map* **[47]**; *combinatorics* **[48]**; *probabilistic model* **[49, 50]**; *Diophantine approximation* **[51, 52]**; *binomial representation* **[53]**; *2-adic integers* **[54, 55]**; *cybernetics* **[56]**; "*tuple-sets*" **[57]**; "*sequence vectors*" **[58]**; "*predecessor trees*" **[59]**; "*3x+1 trees*" **[60]**; *etc*. In reality, the *Collatz 3x+1 problem* is studied for its *computational complexity* **[61]** and is even deemed to be "*unsolvable*" **[62, 63, 64, 65, 66, 67]** — which might have prompted *Bryan Thwaites* **[68, 69]** (*as well as others* **[1, 2, 5]**) to offer prize money (*up to* £1,000.00) for its solution.

The simple proof of the *Collatz 3x+1 conjecture* presented in this paper is summarized as follows:

- ► Each *preferred* *Collatz 3x+1 sequence* takes the form of a distinct infinite sequence

  $$C_n = \langle n, f(n), f^2(n), f^3(n), \ldots\rangle = \langle n, m, l, k, \ldots, d, c, b, a, \ldots\rangle$$

  of *branch-points* where the starting number n ranges through all the positive integers for the branching iteration function $f(n) = \frac{n}{2}$ if n is even or $f(n) = \frac{3n+1}{2}$ if n is odd.

- ► A *Collatz 3x+1 sequence* C_n is *cyclic* or has a *cyclic-subsequence* $C_i = \langle(i, h, \ldots, c, b)\rangle$ if and only if C_n has a *first-duplicated-term* $f(b) = a = i \in \{n, m, l, \ldots, d, c, b\}$ and the collection of distinct iterates $i, h, \ldots, c$, and b (*in that fixed order*) repeats *ad infinitum*. This simply means that some iterate of C_n satisfy exactly just one of the *cycle-equations* $a = b$ or $f(b) = b$, $a = c$ or $f(b) = f^{1}(b)$, …, $a = m$ or $f(b) = f^{u+1}(b)$, $a = n$ or $f(b) = f^{u}(b)$ — that is, all the *Collatz 3x+1 sequences* in the positive integers domain are partitioned into the "*equivalence classes of cyclic-subsequences*".

- ► It is affirmed from the *general form* of the *preferred* Collatz sequence *cycle-equations* that it has *exactly* 1 valid *length*-2 *cyclic-subsequence* $\langle(2, 1)\rangle$ in the positive integers domain — that is, any periodic subsequence with length other than 2 actually belongs to the negative integers domain. Just as the *form* of an infinite sequence of binary digits prefixed by a fractional expansion point establishes the fact that it (*as some real number point or an interval*) lies between 0 and 1, the *form* of the *preferred* Collatz sequence guarantees that it always has exactly the iterate $b = 1$ that satisfies $a = c$ or $f(b) = f^{1}(b)$ in the positive integers domain. The author contends that a complete rigorous proof of the *Collatz 3x+1 conjecture* has been established — that is, every *preferred* Collatz sequence includes the *cyclic-subsequence* $\langle(2, 1)\rangle$ and there are no divergent ones.

- ► To alleviate skeptics' objection that the concern about the *existential-possibility* of some "*divergent*" *Collatz 3x+1 sequences* has not been suitably answered by the preceding argument, the convergence of each C_{2p+1} ($\forall p \in \mathbf{N}^{+}$) sequence to the *cyclic-subsequence* C_1 is affirmed from the *convergence-by-Collatz-3x+1-Christmas-tree-construction* of "8x+5" ($\forall x \in \mathbf{N}$) *base sequence* ("*trunk*") $T(1)$ and its *extension sequences* ("*branches*").

II. *Model Statement of a Collatz 3x+1-type Problem*

A very simple and general approach for deciding many *Collatz 3x+1-type problems* — that is, finding (*if any*) all the *cycles* of integer sequences recursively defined by some *branching-function* — is presented in this manuscript. As depicted in Figure 1, every *Collatz-type problem* involves some arbitrary *branch* (*or path or trajectory or orbit*) that takes the form of an infinite sequence

$C_n = \langle n, m, l, k, \ldots, d, c, b, a, \ldots \rangle$

of *branch-points* where the starting number n ranges through all the elements of the specified domain **D** of a given *branching iteration function* (*with mutually exclusive sub-functions*)

$$f(n) = \begin{cases} \alpha(n) & \text{if } n \text{ satisfies condition A} \\ \beta(n) & \text{if } n \text{ satisfies condition B} \\ \vdots & \\ \zeta(n) & \text{if } n \text{ satisfies condition Z} \end{cases}$$

such that $m = f(n)$;

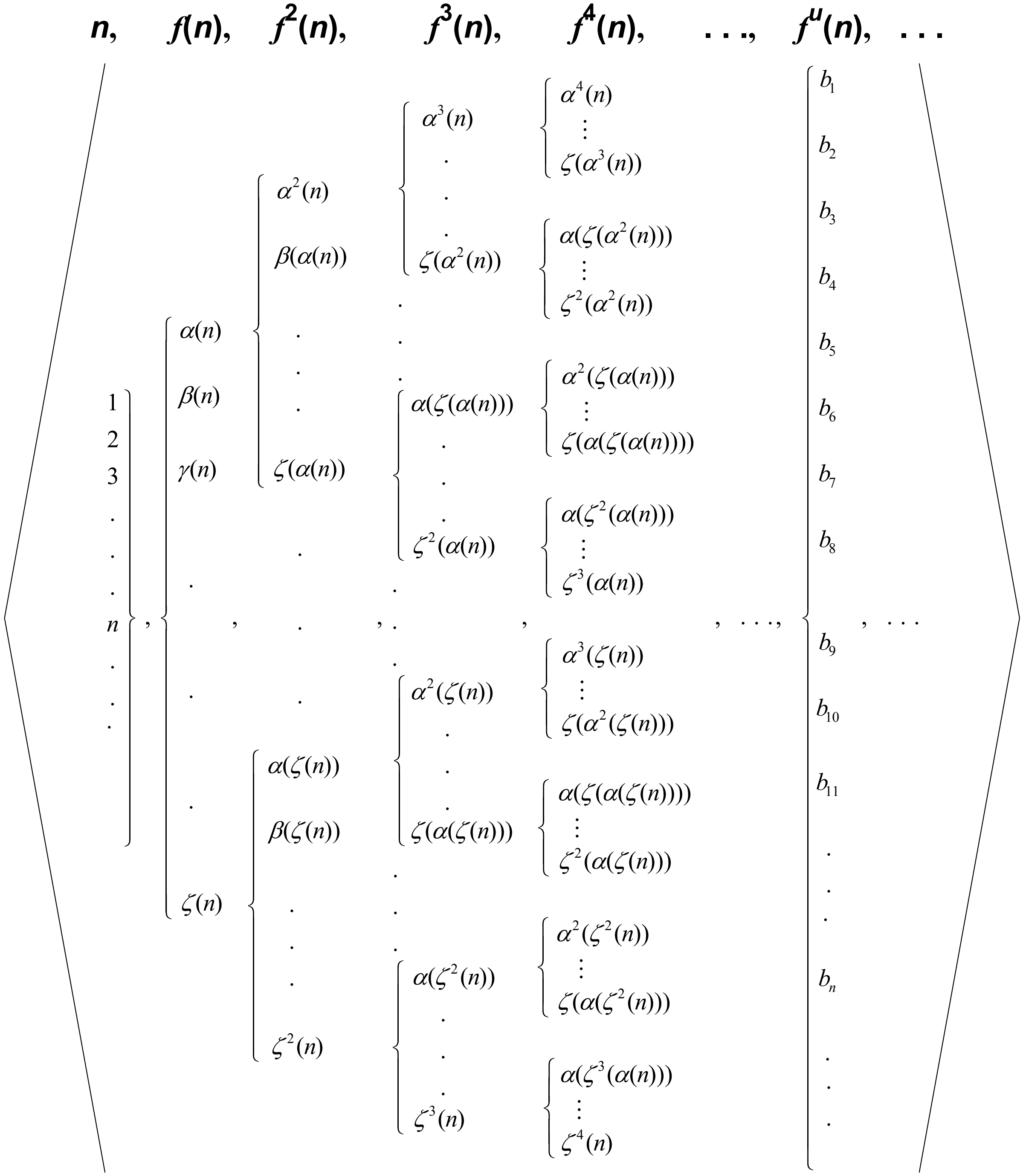
$$\begin{aligned} l &= f(m) = f(f(n)) = f^2(n); \\ k &= f(l) = f(f(m)) = f(f(f(n))) = f^3(n); \\ &\vdots \\ c &= f(d) = f(f(e)) = f(f(f(f))) = \ldots = f^{u-1}(n); \\ b &= f(c) = f(f(d)) = f(f(f(e))) = \ldots = f^{u}(n); \\ a &= f(b) = f(f(c)) = f(f(f(d))) = \ldots = f^{u+1}(n). \end{aligned}$$

It is desired to find (*if any*) the *cycle* or *cyclic-subsequence* $C_i = \langle (i, h, g, \ldots, d, c, b) \rangle$ so that

$C_n = \langle n, m, l, \ldots, k, j, (i, h, g, \ldots, d, c, b) \rangle = \langle n, m, l, \ldots, k, j, C_i \rangle$

— that is, $a = i \in \{n, m, l, k, \ldots, d, c, b\}$ is the *first duplicated term* and the group of distinct iterates $i, h, g, \ldots, c$, and b (*in that fixed order*) repeats *ad infinitum.* A sequence that ultimately loops (*that is, has some periodic subsequence*) is said to be *convergent*; otherwise, it is held to be *divergent*. It is emphasized that any iterate k of some convergent or divergent sequence C_n corresponds to the *also-convergent* or the *also-divergent* subsequence C_k, respectively.

Figure 1



Starting from some arbitrary number n in the domain **D** of a branching function $\boldsymbol{f}$ that defines some set of *Collatz 3x+1-type sequences* C_n, any finite count u ($u \to \infty$) of iterations of n would always yield the term $\boldsymbol{f}^u(n) = b_n$ in the domain **D** — for examples:

$C_1 = \langle 1, \zeta(1), \gamma(\zeta(1)), \alpha(\gamma(\zeta(1))), \zeta(\alpha(\gamma(\zeta(1)))), \beta(\zeta(\alpha(\gamma(\zeta(1))))), \mu(\beta(\zeta(\alpha(\gamma(\zeta(1)))))), \ldots, b_1, \ldots\rangle$;

$C_2 = \langle 2, \beta(2), \zeta(\beta(2)), \zeta^2(\beta(2)), \alpha(\zeta^2(\beta(2))), \delta(\alpha(\zeta^2(\beta(2)))), \delta^2(\alpha(\zeta^2(\beta(2)))), \chi(\delta^2(\alpha(\zeta^2(\beta(2))))), \ldots, b_2, \ldots\rangle$;

⋮

$C_n = \langle n, \alpha(n), \zeta(\alpha(n)), \delta(\zeta(\alpha(n))), \zeta(\delta(\zeta(\alpha(n)))), \beta(\zeta(\delta(\zeta(\alpha(n))))), \varphi(\beta(\zeta(\delta(\zeta(\alpha(n)))))), \ldots, b_n, \ldots\rangle$.

All of these expressions is simply generalized [*with* $a = \boldsymbol{f}(b)$] as an infinite sequence of *branch-points*:

$C_n = \langle n, m, l, k, \ldots, d, c, b, a, \ldots\rangle$.

For example, the *standard* *Collatz 3x+1 conjecture* claims that each sequence C_n, $\forall n \in \mathbf{N}^+$ (*the set of all positive natural numbers*), defined by the 2-*branch* iteration function

$$f(n) = \begin{cases} \alpha(n) = \dfrac{n}{2} & \text{if } n \text{ is even} \\ \\ \beta(n) = 3n+1 & \text{if } n \text{ is odd} \end{cases},$$

has the form

$$C_n = \langle n, m, l, k, \ldots, f, e, (4, 2, 1)\rangle = \langle n, m, l, k, \ldots, f, e, C_4\rangle.$$

As a matter of fact, the first 5 *standard* *Collatz 3x+1 sequences* satisfy the conjecture:

$C_1 = \langle 1, 4, 2, 1, \ldots\rangle = \langle (1, 4, 2)\rangle$ [*that is,* $b = 2$, $a = 1$ *and the cycle is* (1, 4, 2)]
$= \langle 1, (4, 2, 1)\rangle$
$= \langle 1, C_4\rangle$;

$C_2 = \langle 2, 1, 4, 2, \ldots\rangle = \langle (2, 1, 4)\rangle$ [*that is,* $b = 4$, $a = 2$ *and the cycle is* (2, 1, 4)]
$= \langle 2, 1, (4, 2, 1)\rangle$
$= \langle 2, 1, C_4\rangle$;

$C_3 = \langle 3, 10, 5, 16, 8, 4, 2, 1, 4, \ldots\rangle$ [*that is,* $b = 1$, $a = 4$ *and the cycle is* (4, 2, 1)]
$= \langle 3, 10, 5, 16, 8, (4, 2, 1)\rangle$
$= \langle 3, 10, 5, 16, 8, C_4\rangle$;

$C_4 = \langle 4, 2, 1, 4, \ldots)\rangle$ [*that is,* $b = 1$, $a = 4$ *and the cycle is* (4, 2, 1)]
$= \langle (4, 2, 1)\rangle$;

$C_5 = \langle 5, 16, 8, 4, 2, 1, 4, \ldots\rangle$ [*that is,* $b = 1$, $a = 4$ *and the cycle is* (4, 2, 1)]
$= \langle 5, 16, 8, (4, 2, 1)\rangle$
$= \langle 5, 16, 8, C_4\rangle$.

An ongoing Internet online 3x+1 project **[70]** reports that the *Collatz 3x+1 conjecture* has been computably verified (*as of* May 2006) to hold up to n = 484,549,993,128,097,215. Of course, the conjecture would also hold for $2^e \cdot n$, $\forall e \in \mathbf{N}^+$ — that is, for any *power-of-*2 multiple of *n*.

III. *A Simple and General Approach to Collatz 3x+1-type Problems*

The typical *Collatz sequence branch* or *trajectory*, for an arbitrary *starting number* $n \in \mathbf{D}$ —

$C_n = \langle n, m, l, k, \ldots, d, c, b, a, \ldots\rangle$

— is a *cycle* or has a *cyclic-subsequence* $C_i = \langle(i, h, g, \ldots, d, c, b)\rangle$ if and only if there is some *first-duplicated term* a in C_n — that is, $a = f(b) = i \in \{n, m, l, ..., d, c, b\}$. It is emphasized that the elements of the set $\{n, m, l, ..., d, c, b\}$ are distinct by our definition of a (*therefore, a could only be equal to one of them*) — so every term of the cycle $C_i = (i, h, g, \ldots, d, c, b)$ is likewise different. It trivially immediately follows that there could only be one set of distinctive terms that form a cycle or a cyclic subsequence for each sequence C_n and it is not possible to have a cycle with larger length that includes a cycle with smaller length in any sequence C_n.

As in *elementary algebra* — where an arithmetic problem is solved by setting up some equation and finding all the valid *solution-values* for the unknown variables — the fact that C_n has a cycle can be readily established by seeking a valid *solution-value* to any one of these *cycle-equations*:

	$a = b$	[*that is,* $f(b) = b$	*for length*-1 *cycle*];
or	$a = c$	[*that is,* $f(b) = f^{-1}(b)$	*for length*-2 *cycle*];
or	$a = d$	[*that is,* $f(b) = f^{-2}(b)$	*for length*-3 *cycle*];
	$\vdots$		
or	$a = n$	[*that is,* $f(b) = f^{u}(b)$	*for length*-$(u+1)$ *cycle with* $u \in \mathbf{N}^{+} - \{1\}$].

Since there are at least 2 *sub-functions* for a *general Collatz 3x+1-type function* f, the finding of valid *solution-values* for b may be *exponentially computationally complex*. In determining all the valid *solution-values* for b (*for every representative branch* C_n *of the iteration function* f), it is easier to evaluate $a = b$ first, next $a = c$, then $a = d$, …, and so on up to $a = n$. For some infinite domains, since $n \to \infty$, $a = n$ might remain elusive to solve and this would present some potentially divergent sequences. This scenario actually poses an *undecidability problem* — unless the existence of *length*-u cycles, $\forall u > z$ for some $z \in \mathbf{N}$, could be absolutely ruled out by some *computable generalization* then there would remain sequences whose convergence or divergence is *undecided*.

If there are s *sub-functions* of some given *Collatz 3x+1-type function* f, then every *branch-point* $f^{u}(b)$ [$\forall u \in \mathbf{N}^{+}$] has s^{u} *nodes* that are represented by respective expressions involving "*powers*" of f's *sub-functions* (*when viewed from the preceding terms*) or f^{-1}'s *sub-functions* (*when seen from the succeeding iterates*) as well as s^{u+1} *cycle-equations*. If there is some *length*-(u+1) cycle then, in addition to the *branch-node* expressions that yield all of the *solution-values* for the trivial *length*-1 cycles, there would be u+1 *branch-node* expressions that actually yield each *cycle-term* as a valid respective *solution-value* for b (*in the appropriate domain*). The possibilities of loops with lengths greater than u+1, $\forall u \in \mathbf{N}^{+}$ (*including perhaps divergent sequences*), exist whether or not the s^{u+1} *cycle-equations* for the *branch-point* $f^{u}(b)$ yield valid cycles with length u+1. It would require some *computable generalization* reasoning like *mathematical induction* argument to surmount the *computational complexity* or *undecidability* issues innate with the determination of all the valid *cycle-lengths* of a *general Collatz 3x+1-type problem*.

For each member of the set of sequences with the *cyclic-subsequence* C_i = <(i, h, …, d, c, b)>, the presupposed *cycle-solution branch-nodes* expressions are delineated as follows:

$a = i = f(b)$ $h = f^{2}(b)$ . . . $e = f^{-3}(b)$ $d = f^{-2}(b)$ $c = f^{-1}(b)$ b

$$\left(\left\{\begin{array}{c}\alpha(b)\ \left\{\begin{array}{c}\alpha^{2}(b)\\ \vdots\\ \zeta(\alpha(b))\end{array}\right.\\ \vdots\\ \zeta(b)\ \left\{\begin{array}{c}\alpha(\zeta(b))\\ \vdots\\ \zeta^{2}(b)\end{array}\right.\end{array}\right.,\ \ldots,\ \begin{array}{c}\left.\begin{array}{c}\alpha^{-3}(b)\\ \vdots\\ \zeta^{-1}(\alpha^{-2}(b))\end{array}\right\}\\ \left.\begin{array}{c}\alpha^{-1}(\zeta^{-1}(\alpha^{-1}(b)))\\ \vdots\\ \zeta^{-2}(\alpha^{-1}(b))\end{array}\right\}\\ \left.\begin{array}{c}\alpha^{-2}(\zeta^{-1}(b))\\ \vdots\\ \zeta^{-1}(\alpha^{-1}(\zeta^{-1}(b)))\end{array}\right\}\\ \left.\begin{array}{c}\alpha^{-1}(\zeta^{-2}(b))\\ \vdots\\ \zeta^{-3}(b)\end{array}\right\}\end{array},\ \begin{array}{c}\left.\begin{array}{c}\alpha^{-2}(b)\\ \vdots\\ \zeta^{-1}(\alpha^{-1}(b))\end{array}\right\}\\ \left.\begin{array}{c}\alpha^{-1}(\zeta^{-1}(b))\\ \vdots\\ \zeta^{-2}(b)\end{array}\right\}\end{array},\ \left.\begin{array}{c}\alpha^{-1}(b)\\ \vdots\\ \zeta^{-1}(b)\end{array}\right\},\ b\right)$$

It should not be surprising to find sequences with different starting numbers to converge to the same *cyclic-subsequence* C_i = <(i, h, g, …, d, c, b)> since

$$C_z = <z, y, \ldots, o, C_n> = < z, y, \ldots, o, n, C_m> = \ldots = <z, y, \ldots, o, n, m, l, k, \ldots, j, C_i>.$$

Some concern might be brought forward that our *cycle-equations* focus only on the "*middle*"- or "*end"-portion* terms — say, <… n, m, …, c, b ,a, …> — of a branch or trajectory and ignoring "*the fact*" that some sequence C_z = <z, y, x, …, q, p, o, …> might possibly have a term $x > 1$ among its initial iterates to be the *minimum* of all of its *branch-points* so that the sequence C_z might "*conceivably be headed to infinity*". It is simply reiterated that, as explained in Figure 1, $b = f^u(z)$ [*for* $u \geq 0$ *with* $f^0(b) = b$] is a *fixed-positioned* iterate in each of the sequences C_z with *arbitrary finite* starting numbers z — that is, say for the *preferred* *Collatz 3x+1 sequences* in the domain of all integers, b is the first term (*for* $u = 0$) in C_0 and C_{-1}; or the second term (*for* $u = 1$) in C_{-2}, C_1, C_2; or the third term (*for* $u = 2$) in C_{-10}, C_{-7}, C_{-5}, C_4; or the fourth term (*for* $u = 3$) in C_{-20}, C_{-14}, C_{-8}, C_8, and so on *ad infinitum* — b is just the iterate whose *successor-term* $f(b) = a$ is the *first-duplicated term* of a sequence C_z. Also, any starting number z is easily some "*middle*" term of other sequences since $f^{-u}(z)$, $\forall u \in \mathbf{N^+}$, are readily available *predecessor-terms* of z — in particular, C_z is a subsequence of C_ω for $\omega = 2^e \cdot z$ [$\forall e \in \mathbf{N^+}$]. The fixed convergence or "*divergence*" of *the sequence* C_z (*that is, where* z *is the starting number*) or of *the subsequence* C_z (*that is, where* z *is a middle term*) does not change wherever z is positioned in a sequence.

A *trivial Collatz 3x+1-type problem* is one in which some characteristic of its branching iteration function or *branch-nodes* expressions can be simply *computably generalized* to all the sequences for the entire domain of the starting number — for instance, see *Example 3* in *section* VII later — that *enables one to immediately render a sound conclusion identifying all its existing valid cycle-length values or the divergence of its sequences*. Rather than finding a computable generalization for the *non-periodic* terms of a given *Collatz 3x+1-type sequences*, it is easier to find one for the *branch-nodes* expressions of the *cycle-terms* — this is the very simple and general approach presented in this paper to decide any *Collatz 3x+1-type problem*. This straightforward method does not guarantee the full solution of every *non-trivial* *Collatz 3x+1-type problem* but it suffices to prove the *Collatz 3x+1 conjecture* in the positive integers domain.

The fundamental logic of our very simple and general approach is as follows:

- ► Any *Collatz 3x+1-type sequence* has the form $C_n = <n, m, \ldots, b, a, \ldots>$ [*see* *Figure 1*] which would satisfy a *cycle-equation* $a = i \in \{n, m, l, \ldots, d, c, b, \}$ — or $f(b) = f^u(b)$ — for exactly one $u \in \mathbf{N}$ if and only if it is convergent (*otherwise, it is divergent*).
- ► If it could be established by some *computable generalization* that there could not be valid *solution-values* for b to every *cycle-equation* $f(b) = f^u(b)$ [$\forall u \in \mathbf{N}$] then we can readily conclude that all of the sequences are divergent.
- ► If it could be ascertained that there are only a finite count of valid cycles [*that is, it can be established by some computable generalization that there cannot be valid solution-values for* b *to every cycle-equation* $f(b) = f^u(b)$, $\forall u > z$ *for some* $z \in \mathbf{N}$], then:
  - ● if each valid cycle has no *portal-cycle-term* [*that is, a cycle-term* g *with at least one of the* $f^1(g)$ *values that is not also a cycle-term so that some other sequences could include the loop as a periodic subsequence through* g], then there are only a finite count of fully cyclic sequences (*that is, the first duplicated term is also the starting number, or* $f(b) = a = n$) and the other sequences, with respective starting number that is not some valid *cycle-term*, are all divergent sequences;
  - ● if there is at least one valid cycle with at least one *portal-cycle-term*, then every *non-fully-cyclic* sequence must include exactly 1 of the valid cycles with proper *portal-cycle-terms* — that is, the sequences are partitioned into the "*equivalence classes of cyclic subsequences*" — so there is no divergent sequence;
- ► If the "*largest valid cycle-length*" could not be found by some *computable generalization*, then the prospects of valid cycles with lengths longer than "*the largest known valid cycle*" or of divergent sequences in the appropriate domain could not be ruled out — that is, this is indeed an *undecidable problem*.

Applied to the *preferred* *Collatz 3x+1 sequences* C_n in the positive integers domain, its *general form* warrants that any C_n has *at least* the valid solution $b = 1$ for the *cycle-equation* $f(b) = f^1(b)$ — that is, the *length*-2 cycle (2, 1). Moreover, it is established by a *computable generalization* that any C_n could only have valid solutions for *at most* *length*-2 cycles only. Hence, each C_n has *exactly* only the valid *cyclic-subsequence* $<(2, 1)>$ and there are no "*divergent*" ones.

IV. *Proof of Collatz 3x+1 Conjecture in the Positive Integers Domain*

To demonstrate our very simple and general approach — and at the same time prove the *Collatz 3x+1 conjecture* in the positive integers domain — we shall decide the *preferred* ("*favored in the mathematical literature*" **[3]**) *Collatz 3x+1 problem* defined by the 2-*branch* iteration function

$$f(n)=\begin{cases}\alpha(n)=\dfrac{n}{2} & \text{if } n \text{ is even}\\[2ex] \beta(n)=\dfrac{3n+1}{2} & \text{if } n \text{ is odd}\end{cases} \quad ; \quad f^{-1}(m)=\begin{cases}\alpha^{-1}(m)=2m\\[2ex] \beta^{-1}(m)=\dfrac{2m-1}{3}\end{cases} \left(\begin{array}{c}\text{no preconditions} \Rightarrow\\ \alpha^{-1}(m) \text{ always an integer;}\\ \beta^{-1}(m) \text{ valid if an integer}\end{array}\right)$$

with the set **Z** of all the integers as domain. In the *standard Collatz 3x+1 sequences*, any odd number *n* iterates to 3*n*+1 which then iterates to $\dfrac{3n+1}{2}$ — so, the division by 2 in the *odd* case of the *preferred Collatz 3x+1 sequences* avoids trivial *even* terms.

The claim that, for every starting integer *n*, the *preferred Collatz 3x+1 sequence* C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates is very important in the following *solution-analysis*. The generalization to all the elements of the specified domain is very simply invoked (*some rigorous proof by mathematical induction is not required*) from the arbitrariness of *n* in this very brief argument: Let *n* be an arbitrary positive or negative integer. Then, both $\alpha(n)=\dfrac{n}{2}$ (*for n even*) as well as $\beta(n)=\dfrac{3n+1}{2}$ (*for n odd*) are also positive or negative integers, respectively. Since *n* is arbitrary, the contention holds (*without much ado*) $\forall n \in \mathbf{Z} - \{0\}$.

We now proceed to prove the *preferred Collatz 3x+1 conjecture* in the positive integers domain.

1. We first solve for any valid trivial *length*-1 cycle — that is, $a = f(b) = b$:

   ► $\alpha(b)=\dfrac{b}{2}=b$ yields $b = 0$ — the trivial solution C_0 = <(0)> = <0, 0, 0, …>;

   ► $\beta(b)=\dfrac{3b+1}{2}=b$ yields $b = -1$ — the trivial solution C_{-1} = <(-1)> = <-1, -1, -1, …>.

   Therefore, there are 2 trivial *length*-1 cycles — C_0 = <(0)> and C_{-1} = <(-1)> — however, *they are not valid periodic sequences in the positive integers domain*.

2. The presupposed *cycle-solution branch-nodes* expressions are shown below:

a* = *i* = *f*(*b*)** ***h* = f^2(*b*)** **. . .** ***e* = f^3(*b*)** ***d* = f^2(*b*)** ***c* = f^1(*b*)** ***b

$$\left(\begin{cases}\dfrac{b}{2}\\[2ex]\dfrac{3b+1}{2}\end{cases},\quad\begin{matrix}\begin{cases}\dfrac{b}{2^2}\\[2ex]\dfrac{3b+2}{2^2}\end{cases}\\[4ex]\begin{cases}\dfrac{3b+1}{2^2}\\[2ex]\dfrac{3^2b+(3+2)}{2^2}\end{cases}\end{matrix}\quad \ldots,\quad \begin{matrix}\left.\begin{matrix}2^3b\\[1ex]\dfrac{2^3b-1}{3}\end{matrix}\right\}\\[3ex]\left.\begin{matrix}\dfrac{2^3b-2}{3}\\[1ex]\dfrac{2^3b-(2+3)}{3^2}\end{matrix}\right\}\\[3ex]\left.\begin{matrix}\dfrac{2^3b-2^2}{3}\\[1ex]\dfrac{2^3b-(2^2+3)}{3^2}\end{matrix}\right\}\\[3ex]\left.\begin{matrix}\dfrac{2^3b-(2^2+2\cdot 3)}{3^2}\\[1ex]\dfrac{2^3b-(2^2+2\cdot 3+3^2)}{3^3}\end{matrix}\right\}\end{matrix},\quad\begin{matrix}\left.\begin{matrix}2^2b\\[2ex]\dfrac{2^2b-1}{3}\end{matrix}\right\}\\[6ex]\left.\begin{matrix}\dfrac{2^2b-2}{3}\\[2ex]\dfrac{2^2b-(2+3)}{3^2}\end{matrix}\right\}\end{matrix},\quad\left.\begin{matrix}2b\\[6ex]\dfrac{2b-1}{3}\end{matrix}\right\},\ b\right)$$

3. For the 2 *branch-nodes* of the *length*-2 *cycle-equation* *a* = *c* or ***f(b)*** = ***f***$^{-1}$(*b*), we evaluate if there is a valid *solution-value* for *b* when either $\alpha(b)=\dfrac{b}{2}=c$ or $\beta(b)=\dfrac{3b+1}{2}=c$.

For the *branch-node* *c* = 2*b*:

► $\alpha(b)=\dfrac{b}{2}=2b$ yields *b* = 0 — the trivial solution C_0 = <(0)>;

► $\beta(b)=\dfrac{3b+1}{2}=2b$ yields $b=\dfrac{1}{2^2-3}=1$ — which is a valid solution in the positive integers domain corresponding to *a* = 2 and C_2 = <(2, 1)>.

For the *branch-node* $c = \frac{2b-1}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{2b-1}{3}$ yields $b = \frac{2}{2^2-3} = 2$ — which is a valid solution in the positive integers domain corresponding to $a = 1$ and C_1 = <(1, 2)>;

► $\beta(b) = \frac{3b+1}{2} = \frac{2b-1}{3}$ yields $b = -1$ — the trivial solution C_{-1} = <(-1)>

— it is stressed that $\alpha^{-1}(-1) = 2(-1) = -2$ provides a *portal* or *doorway* to the loop (-1) from many starting numbers -*n* in the negative integers domain with

C_{-n} = <-*n*, -*m*, -*l*, …, -*e*, -4, -2, (-1)> where *n*, *m*, *l*, …, $e \in \mathbf{N}^{+} - \{1\}$.

We could express C_1 = <(1, 2)> = <1, (2, 1)> = <1, C_2>. Other than the *cycle-term* 1, the *cycle-term* 2 also has a *predecessor-term* $\alpha^{-1}(2) = 2(2) = 4$ that is not a *cycle-term* — hence, 2 is the *portal-cycle-term* of the loop (2, 1) [*that is, many* (*perhaps, all*) *sequences with starting positive natural numbers other than* 1 *or* 2 *would include the cycle* (2, 1) *as a periodic subsequence by way of the cycle-term* 2].

4. That, in fact, *every preferred Collatz 3x+1 sequence in the positive integers domain has one of the following forms* —

C_2 = <(2, 1)>;

C_1 = <(1, 2)> = <1, (2, 1)> = <1, C_2>;

C_n = <*n*, *m*, *l*, …, *f*, 8, 4, (2, 1)> = <*n*, *m*, *l*, …, *f*, 8, 4, C_2)>; $\forall n \in \mathbf{N}^{+} - \{1, 2\}$

— immediately ensues from the following plain reasoning [$\forall u \in \mathbf{N}^{+} - \{1\}$]:

a. For any *length*-(*u*+1) *cyclic-subsequence* <(*i*, *h*, *g*, …, *d*, *c*, *b*)>, the *branch-node* equation $\alpha^{-u}(b) = \alpha(b)$ will always yield only the trivial *length*-1 *cycle-solution* C_0 =<(0)> while the *branch-node* equation $\beta^{-u}(b) = \beta(b)$ will always yield only the trivial *length*-1 *cycle-solution* C_{-1} =<(-1)>.

b. In addition to the 2 *branch-nodes* expressions that produce the 2 trivial *length*-1 *cyclic-subsequences* C_0 = <(0)> and C_{-1} = <(-1)>, an additional *branch-node* expression must yield the valid *solution-value* $b_1 = b > 2$; another *branch-node* expression must yield the valid *solution-value* $b_2 = c > 2$; another *branch-node* expression must yield the valid *solution-value* $b_3 = d > 2$; …; and a different *branch-node* expression must yield the valid *solution-value* $b_{u+1} = i > 2$. Since the count u+1 of *cycle-terms* is finite, then there is a *minimum-valued cycle-term* $b_{\min}$ [*which must be some odd natural number in the positive integers domain in order for the first-duplicated cycle-term* $i = f(b_{\min})$ *to be greater than* $b_{\min}$] among the *cycle-terms* ($i, h, g, \ldots, d, c, b$) or $\{b_{u+1}, b_u, b_{u-1}, \ldots, b_3, b_2, b_1\}$.

c. As could be readily observed from the presupposed *cycle-solution branch-nodes expressions* depicted in step 2 above, for the u-times applications of the inverse function f^{-1} (*that is, either* α^{-1} *or* β^{-1} *as appropriate*) on b, each one of the $2^u - 2$ *nontrivial-length*-1-*nodes-expressions* of the *branch-point* $f^{-u}(b)$ has the form

$$\frac{2^u b - S}{3^v} \quad \text{where} \quad 0 < v \le u \quad \text{with} \quad u > 1;$$

$$S = \sum (2^p \cdot 3^q) > 0 \quad \text{with} \quad p, q \in \mathbf{N}.$$

Thus, for each such *branch-node* expression, the 2 *cycle-equations* $f^{-u}(b) = \alpha(b)$ and $f^{-u}(b) = \beta(b)$ [$\forall u > 1$ *or for all cycle-lengths greater than* 2] that have to be evaluated for valid *solution-values* for b are, respectively:

$$\frac{2^u b - S}{3^v} = \frac{b}{2} \quad \text{or} \quad b = \frac{2S}{2^{u+1} - 3^v} \qquad [1]$$

and

$$\frac{2^u b - S}{3^v} = \frac{3b+1}{2} \quad \text{or} \quad b = \frac{2S + 3^v}{2^{u+1} - 3^{v+1}} \qquad [2].$$

We emphasize that the numerator in the *right-hand* side of equation [1] is less than the numerator in the *right-hand* side of equation [2] while the denominator in the former is greater than the denominator in the latter — that is, equation [1] would indeed yield a smaller *solution-value* for b (*in the specified domain*) than equation [2]. In particular, this should be true for the *branch-node expression* of the respective *minimum-valued cycle-term* $b_{\min}$ ($\forall u > 1$ *or for all cycle-lengths greater than* 2) that is valid in the domain of discourse — but this implies that $b_{\min}$ should be an even integer (*equation* [1] *applies only to the even iterates*) so $b_{\min}$ cannot be a positive integer since, otherwise, $0 < \alpha(b_{\min}) = \frac{b_{\min}}{2} < b_{\min}$ — contradicting $b_{\min}$'s supposed minimality among its *cycle-terms* in the domain of positive integers. Thus, any *preferred* *Collatz 3x+1 sequence* with *length*->-2 cyclic subsequence actually belongs to the domain of negative integers.

d. Because there are no cycles other than (2, 1) in the domain of positive integers, *by the arbitrariness of b in all of the cycle-equations*, every *Collatz 3x+1 sequence* C_n must include the *length*-2 cycle (2, 1) as periodic subsequence — therefore, there is no "*divergent*" *all-positive-integer-terms* C_n.

► The *form* of some infinite sequence of, say, binary digits prefixed by a fractional expansion point — for example, .01011100011… — readily establishes the fact that it (*as a real number point or as a subinterval of the unit interval*) indubitably lies between 0 and 1. Likewise, the *general form* of the *preferred* *Collatz 3x+1 sequence* C_n, (*by the very definition of the* *Collatz 3x+1 iteration function* f) directly warrants that C_n always has *at least* the term $b = 1$ which satisfies the *length*-2 *cycle-equation* $f(b) = f^{-1}(b)$ in the positive integers domain.

The finite count of valid cyclic subsequences directly partition all the *Collatz 3x+1-type sequences* into the "*equivalence classes of valid cyclic subsequences*" — that is, every *Collatz 3x+1-type sequence* includes exactly one of the finite count of valid periodic subsequences.

Because we have categorically ruled out the possibility of any valid solution in the positive integers domain to the other *cycle-equations* with length different from 2 — that is, we have resolutely established that the *general form* of the *Collatz* 3*x*+1 *sequence* C_n yields valid solution of *at most* (*therefore, exactly*) $b = 1$ in the domain of positive integers for only the *cycle-equation* $f(b) = f^{1}(b)$ — then, it trivially follows that each *preferred* *Collatz* 3*x*+1 *sequence* C_n includes the *cyclic-subsequence* <(2, 1)> and, thus, there is no "*divergent*" *all-positive-integer-terms* C_n.

► In related perspective, the claim of *no-divergent-all-positive-integer-terms Collatz 3x+1 sequence* is also supported by these *intuitive* reasoning (*but which are, unfortunately, quite difficult to computably generalize*):

- The ongoing Internet online *Collatz 3x+1 project* **[70]** provides *experimental* evidence of the veracity of the *Collatz conjecture*.
- Since each current term is either divided by 2 or multiplied by 1.5, then succeeding terms eventually decreases so any *Collatz 3x+1 sequence* finally converges — a *probabilistic* argument **[2]**.
- The sequence C_z = <*z*, *y*, *x*, *w*, …> is convergent or "*divergent*" if and only if each subsequence C_y, C_x, C_w, … is convergent or "*divergent*", respectively. The *Collatz 3x+1 iteration function* ***f*** is *surjective* (*or onto*) — that is, each positive integer always has (*at least*) one valid *predecessor-term*. In every convergent sequence C_z = <*z*, *y*, …, 4, (2, 1)>, the *subsequent-to-z-terms* to the left of term 2 have corresponding convergent subsequences C_y, C_x, …, C_8, C_4 which also have their respective convergent *predecessor-terms* sequences, *ad infinitum*; thus, every positive integer appears at least once (*in addition to being the starting number of a Collatz 3x+1 sequence*) in this *inter-connections of convergent sequences* — a *realistic* argument.

V. *No "Divergent" Collatz 3x+1 Sequence in Positive Integers Domain*

There are lingering doubts (*predominantly from Collatz 3x+1 problem solvable-skeptics currently in some state of "denial"*) that the simple argument presented in the previous section IV does not adequately eliminate the likelihood that certain *all-positive-integer-terms regular Collatz 3x+1 sequence* $C_z = \langle z, f(z), f^2(z), f^3(z), \ldots \rangle$ could possibly "*head off to infinity*" and not actually converge to the *cyclic-subsequence* $\langle(2, 1)\rangle$. This objection is plainly stated as $\exists\, z, u \in \mathbf{N}$ such that $n = f^u(z) > 1$ is the *minimum-valued* iterate among all of the distinct terms in C_z. Then:

- ► Term n must be some positive odd integer since, otherwise, the *successor-term* of n — $\alpha(n) = \frac{n}{2} < n$ — would contradict n's presupposed minimality;

- ► With n some odd natural number, the *successor-term* of n — $\beta(n) = \frac{3n+1}{2}$ — must also be an odd natural number because, otherwise, $\alpha(\beta(n)) = \frac{3n+1}{4} < n$ for $n > 1$;

  - ● The set of all odd natural numbers is comprised, $\forall p \in \mathbf{N}^+$, of the *union-set* of $O_0 = \{5, 9, 13, 17, \ldots, 4p+1, \ldots\}$ and $O_1 = \{3, 7, 11, 15, \ldots, 4p-1, \ldots\}$.
  - ● The *minimum-valued* term $n \notin O_0$ because if $n = 4p+1 \in O_0$, for $p \in \mathbf{N}^+$, then $\beta(n) = \beta(4p+1) = \frac{3(4p+1)+1}{2} = \frac{12p+4}{2} = 6p+2$ is even or $\alpha(\beta(n)) = \alpha(\beta(4p+1)) = \frac{6p+2}{2} = 3p+1 < 4p+1$.

  To prove the claim of *no-"divergent"-all-positive-integer-terms Collatz 3x+1 sequence*, we must verify that $n = 4p+1$ ($\forall p \in \mathbf{N}^+$) cannot actually be a term in whatever condition (*not just the restriction on its not being the minimum-valued iterate*) of a "*divergent*" sequence — in other words, that C_{4p+1} is convergent.

  Indeed, just demonstrating that each C_{4p+1} ($\forall p \in \mathbf{N}^+$) converges to C_1 is sufficient to fully prove the *Collatz 3x+1 conjecture* in the positive integers domain since *every* C_{4p-1} *sequence includes some* C_{4p+1} *subsequence* —

1. With $p = 2q+1$ for $q \in \mathbf{N}^+$, $\beta(4p-1) = 6p-1$ implies that $4p-1 \in O_1$ has a *successor-term* of the form $4x+1$ for some $x \in \mathbf{N}$ — that is,

$$\beta(4p-1) = \beta(8q+3) = \frac{3(8q+3)+1}{2} = 12q+5 = 4(3q+1)+1.$$

2. With $p = 2q$ for $q \in \mathbf{N}^+$,

$$4p-1 = 2^3q-1 \in O_1 \quad \text{and} \quad \beta^2(4p-1) = \beta^2(2^3q-1) = 3^2 \cdot 2q-1$$

implies that each $4p-1 = 2^3q-1 \in O_1$ with $q = 2r+1$ for $r \in \mathbf{N}^+$ has a *successor*^{2}*-term* of the form $4x+1$ for some $x \in \mathbf{N}$ — that is,

$$\beta^2(4p-1) = \beta^2(8q-1) = \beta^2(16r+7) = 36r+17 = 4(9r+4)+1.$$

In general, with $p = 2^\omega y$ for $\omega > 3$, $y \in \mathbf{N}^+$, $4p-1 = 2^{\omega+2}y - 1 \in O_1$ and

$$\beta^{\omega+1}(4p-1) = \beta^{\omega+1}(2^{\omega+2}y-1) = 3^{\omega+1} \cdot 2y - 1$$

implies that each $4p-1 = 2^{\omega+2}y - 1 \in O_1$ with $y = 2z+1$ for $z \in \mathbf{N}^+$ has a *successor*$^{\omega+1}$*-term* of the form $4x+1$ for $x \in \mathbf{N}$ — that is,

$$\beta^{\omega+1}(4p-1) = 3^{\omega+1} \cdot 2(2z+1) - 1 = 4\left(3^{\omega+1}z + \frac{2(3^{\omega+1}-1)}{4}\right) + 1.$$

► The *predecessor-term* of n must be a positive even integer — that is, $f^{-1}(n) = \alpha^{-1}(n) = 2n$ — since, otherwise, $f^{-1}(n) = \beta^{-1}(n) = \dfrac{2n-1}{3} < n$.

- With $n = 4p-1 \in O_1$ for $p \in \mathbf{N}^+$, $\beta^{-1}(4p-1) = \dfrac{2(4p-1)-1}{3} = \dfrac{8p-3}{3} < 4p-1$.

  Since $p \in \{3, 6, 9, 12, \ldots, 3q, \ldots\}$ for $q \in \mathbf{N}^+$ yields positive integer values for $\dfrac{8p-3}{3}$, then $n = 4p-1 \notin \{11, 23, 35, \ldots, 12q-1, \ldots\}$ — hence, $n \in O_2 \cup O_3$

  where $O_2 = \{3, 15, 27, 39, 51, 63, 75, 87, 99, \ldots, 12q+3, \ldots\}$

  and $O_3 = \{7, 19, 31, 43, 55, 67, 79, 91, 103, \ldots, 12q+7, \ldots\}$.

- With $n = 4p\text{-}1 = 12q+3 \cong 12r\text{-}9 \in O_2$ for $q \in \mathbf{N}$, $r \in \mathbf{N^+}$, $\beta^2(12r\text{-}9) = 27r - 19$. With $r = 2s\text{-}1$ for $s \in \mathbf{N^+}$, then $\alpha(\beta^2(12r\text{-}9)) = 27s\text{-}23$. With $s = 2t\text{-}1$ for $t \in \mathbf{N^+}$, then $\alpha^2(\beta^2(12r\text{-}9)) = 27t\text{-}25 < 12r\text{-}9$. Thus, $r \in \{1, 5, 9, 13, 17, \ldots, 4t\text{-}3, \ldots\}$ so that $12r\text{-}9 \in O_4 = \{3, 51, 99, 147, \ldots, 48u+3, \ldots\} \subset O_2$ for $u \in \mathbf{N}$. Hence, the *minimum-valued* term $n \notin O_4$. We could go on indefinitely to limit O_2 to any desired subset by eliminating its elements for which $f^k(12r\text{-}9)$ [*for* $k \in \mathbf{N^+}$] yield lower values than $12r\text{-}9$.
- With $n = 4p\text{-}1 = 12q+7 \cong 12r\text{-}5 \in O_3$ for $q \in \mathbf{N}$, $r \in \mathbf{N^+}$ —

  Going forward with the *preferred* *Collatz 3x+1 iteration*,

  $$\beta^2(12r-5) = \beta\left(\frac{3(12r-5)+1}{2}\right) = \beta(18r-7) = \frac{3(18r-7)+1}{2} = 27r-10$$

  With $r = 2s$ for $s \in \mathbf{N^+}$, then $\alpha(\beta^2(12r\text{-}9)) = 27s\text{-}5$. With $s = 2t\text{-}1$ for $t \in \mathbf{N^+}$, then $\alpha^2(\beta^2(12r\text{-}9)) = 27t\text{-}16 < 12r\text{-}5$. Thus, $r \in \{2, 6, 10, \ldots, 4t\text{-}2, \ldots\}$ so that $12r\text{-}5 \in O_5 = \{19, 67, 115, 163, \ldots, 48u+19, \ldots\} \subset O_3$ for $u \in \mathbf{N}$. Hence, the *minimum-valued* term $n \notin O_5$.

  Similarly, going backward with the *preferred* *Collatz 3x+1 iteration*,

  $$\beta^{-2}(\alpha^{-1}(12r-5)) = \frac{32r-15}{3} < 12r-5$$

  Inasmuch as $r \in \{3, 6, 9, \ldots, 3s, \ldots\}$ [*for* $s \in \mathbf{N^+}$] yields positive integer values for $\frac{32r-15}{3}$, then $12r\text{-}5 \in O_6 = \{31, 67, 103, \ldots, 36t+31, \ldots\} \subset O_3$ for $t \in \mathbf{N}$. Thus, the *minimum-valued* term $n \notin O_6$.

  We could go on indefinitely to limit O_3 to a desired subset by discarding its elements for which $f^k(12r\text{-}9)$ [*for* $k \in \mathbf{N^+}$] or $\beta^{-1}(\alpha^{m_p}(4p\text{-}1))$ [*for* $m_p \in \mathbf{N^+}$] yield lower integer values than $12r - 5$.
- It is quite challenging to reduce $O_2 \cup O_3$ to the empty set $\varnothing$ by a *computable generalization* in order for us to be able to decisively proclaim that there is no "*divergent*" *Collatz 3x+1 sequence* in the positive integers domain.

Now, C_z is "*divergent*" if and only if its subsequence C_n is "*divergent*" — this also applies to any sequence C_e with starting positive even integer e and its subsequence C_o where $o = \frac{e}{2^k}$ is an odd natural number for some $k \in \mathbf{N}^+$. It is stressed that any $x \in \mathbf{N}^+$ could only be a term of either a convergent or a "*divergent*" *Collatz 3x+1 sequence* but not of both *sequence-types* and x could be the starting number or it could be located anywhere in some suitable sequence. Therefore, without loss of generality, it is reasonable to just examine the *bone-of-contention* "*divergent*" *condensed* *Collatz 3x+1 sequences* with the form:

$$C_z = \left\langle z, \ldots, \frac{2(2^{m_1} n) - 1}{3}, n, \frac{3n+1}{2^{k_1}}, \frac{3^2 n + (3 + 2^{k_1})}{2^{k_1 + k_2}}, \ldots, \frac{3^u n + \sum_{i=0}^{u-1} \left(3^i \cdot 2^{\sum_{j=1}^{u-i-1} k_j} \right)}{2^{\sum_{j=1}^{u} k_j}}, \ldots \right\rangle \quad \text{[D]}$$

where all of their respective terns are positive odd integers with the least value $n > 1$ — here, $k_1 = 1$ and $k_2, k_3, k_4, \ldots, k_u, \ldots$, as well as $m_1, m_2, m_3, \ldots$, are the natural numbers that make their own relevant iterates to be positive odd integers.

- ► What we have contended thus far is that we could transform the *Collatz 3x+1 problem* with its *very-difficult-to-computably-generalize* branching iteration function into some simplified *easy-to-computably-generalize* problem with a restructured *non-branching stop-gap-iteration* function that pertinently involves only all the odd natural numbers. In foresight, the reasonable transformation of a deemed "*computationally complex*" or even "*undecidable*" mathematical problem (*due to ineffective proposed solution-methods*) into some manageable challenge readily discredits the unjustified propaganda that goes with so-called "*computational complexity classes*" cataloging of mathematical problems — in plain truth, the latter categorization is merely a "*measure*" of the unsuitability of the futile proposed *solution-approaches* and not a truthful reflection of the *difficulty-level* nor the *actual solvability* of a given mathematical conundrum.

Let $$b = f^{u}(n) = \frac{3^{u} n + \sum_{i=0}^{u-1}\left(3^{i} \cdot 2^{\sum_{j=1}^{u-i-1} k_j}\right)}{2^{\sum_{j=1}^{u} k_j}} \qquad [3].$$

We could solve for the minimum number n in terms of b in equation [3]:

$$n = \frac{\left(2^{\sum_{j=1}^{u} k_j}\right) b - \sum_{i=0}^{u-1}\left(3^{i} \cdot 2^{\sum_{j=1}^{u-i-1} k_j}\right)}{3^{u}} \qquad [4].$$

We could also apply u-times to b the inverse *Collatz iteration function* $\beta^{-1}(\alpha^{-k_j}(x))$ — but it is computably possible (*the variability of the* k_j*s prohibits any substantial analysis* **[3]**) to arrive at the minimum term n with the value given by equation [4] only if $k_j = 1$ [$\forall j \in \{1, 2, 3, \ldots, u\}$] (*this is also independently noted by Jon Perry in* **[71]** *— for an arbitrarily large* $u \in \mathbf{N}^{+}$, *the negative term in the numerator of equation* [4] *would be dominant and both the minimum term* n *as well as* b *would have to be negative odd integers also* [*in particular, this equation yields the trivial length-1 cycle* <-1, -1, -1, …> *— that is,* $z = y = x = \ldots = b = -1$] *so that any divergent Collatz 3x+1 sequence actually belongs to the negative integers domain*). In this case, one could substitute $x = 3$, $y = 2$, and $e = u$ in the algebraic identity

$$x^{e} - y^{e} = (x - y)\sum_{i=o}^{e-1} x^{i} \cdot y^{e-i-1} \qquad [5]$$

to simplify equation [4] with $b > n > 1$ into

$$3^{u} n = 2^{u} b - (3^{u} - 2^{u}) \qquad \text{or} \qquad 3^{u}(n+1) = 2^{u}(b+1)$$

which could be easily solved, for any $u > 0$, by letting $b = 3^{u} - 1$ and $n = 2^{u} - 1$. However, the sequences with all $k_j = 1$ no longer conform to the regular *Collatx 3x+1 sequences*.

In order to work with just the odd natural numbers, the *standard* Collatz 3x+1 iteration function is further modified **[72, 73]** to the dynamically *streamlined non-branching iteration function*

$$x_i = \frac{3x_{i-1}+1}{2^{k_{i-1}}} \qquad (\textit{with initial positive-odd-integer term } x_0 \in \mathbf{N}^+) \qquad [6]$$

— where k_{i-1} equals the count of *factors-of-*2 in $3x_{i-1}+1$ or the number that makes an odd integer the iterate x_i ($\forall i \in \mathbf{N}^+$). There are numerous approaches presented in the mathematical literature to solve the simplified iteration function [6] — such as attempting to pinpoint the exact trajectory of any starting positive odd integer by using some *look-up tables* **[74]** or *tree-graphs* **[59, 60]** (*in* **[73]**, "*strips*" *of* 2 *or more terms of iteration* [6] *and its inverse function are analyzed using mixed binary-ternary numeral system representation*) — each effort failed to rigorously establish that, $\forall p \in \mathbf{N}^+$, every C_{4p+1} sequence converges to the *cyclic-subsequence* C_1.

► We can further restrict the first term z of the *streamlined* Collatz 3x+1 sequence C_z to be an odd natural number which is divisible by 3 — that is, $z \in \{3, 9, 15, \ldots, 6y+3, \ldots\}$ for $y \in \mathbf{N}$. With $z = 6y+3 = 3(2y+1)$, and an arbitrary $m \in \mathbf{N}$,

$$\beta^{-1}(\alpha^{-m}(z)) = \beta^{-1}(\alpha^{-m}(3(2y+1))) = \beta^{-1}(2^m \cdot 3(2y+1)) = \frac{2[2^m \cdot 3(2y+1)]-1}{3}$$

is never an integer because 3 divides the minuend of the numerator but not its subtrahend and so not also their difference. Therefore, for a, b, c, $d \in \mathbf{N}^+$ and *streamlined* Collatz 3x+1 sequence C_z (*with starting odd natural number* $z > 1$):

- Each C_{3c} is not a subsequence of any C_{3a+1} or C_{3b+2} or C_{3d} with $d \neq c$; *and*
- It suffices to consider only sequences C_z with $z = 3c$ since every C_{3a+1} or C_{3b+2} is always a subsequence of some C_{3c} simply because the former 2 *sequence-types* cannot be *predecessor-extended* indefinitely (*that is, not obtaining some ultimate-predecessor-term which is a positive odd integer divisible by* 3) — "*a collection of infinite iterates with no first term*" is not a sequence at all.

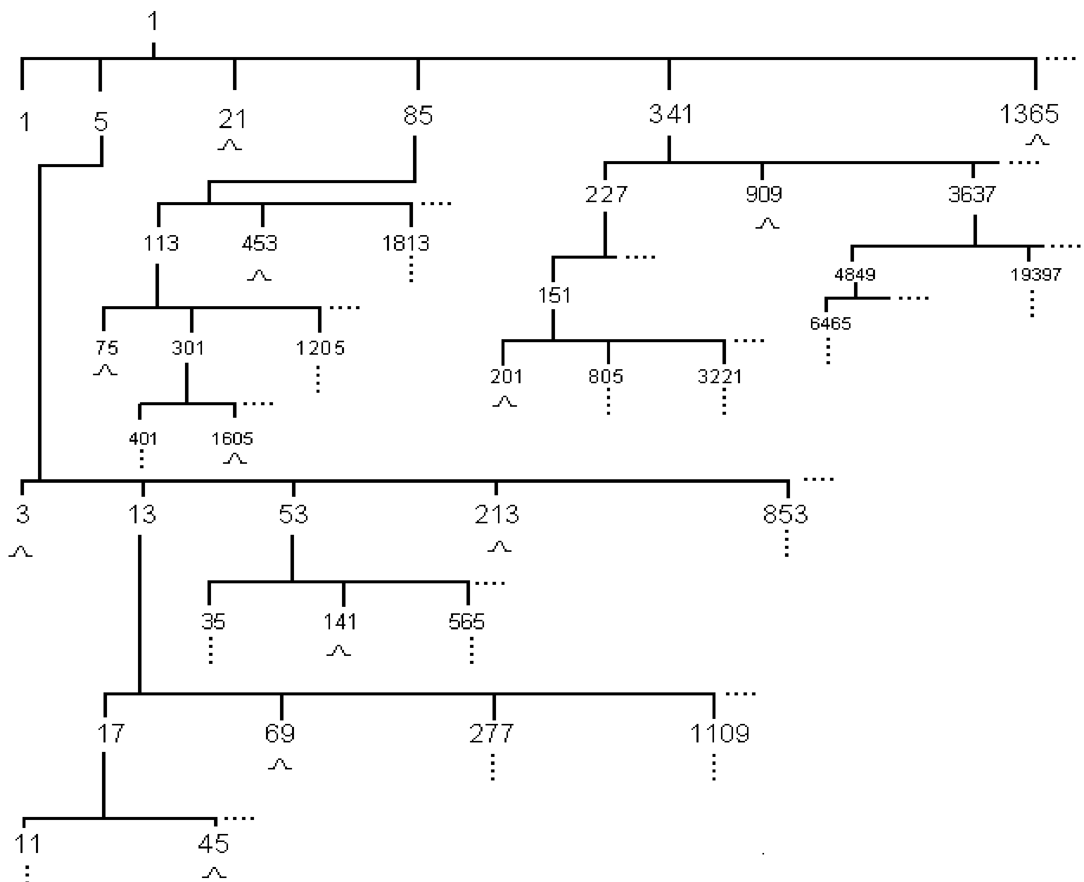
In plain words, all the *streamlined* Collatz 3x+1 sequences C_z (*with* $z > 1$) could be classified into distinct sets of sequences (*and subsequences*) with the same "*ultimate predecessor-extended*" starting odd natural numbers z that are divisible by 3.

As an alternative proof that there are no "*divergent*" *Collatz 3x+1 sequences* in the positive integers domain, we will just simplify and make mathematically rigorous the similar arguments *Kenneth Conrow* presents in his "*Locating an Arbitrary Integer in the Abstract Generation Tree*" Internet webpage **[59]** [*Conrow* 's " $\frac{3n+1}{2^i}$ *Predecessor Tree as a General Tree*" *diagram is shown in* Figure 2 — it displays the *positive-odd-integer* terms of the *streamlined Collatz 3x+1 sequences*] and *Gottfried Helms* portrays in his own "*The Collatz Problem: A view into some* 3x+1*-trees and a new fractal graphic representation*" Internet webpage **[60]**. The essentials of their identical argument is quite simple:

- For arbitrary $p \in \mathbf{N}$, both *Collatz sequences* C_{2p+1} [*or* C_n *for odd* $n \in \mathbf{N}^+$] and C_{8p+5} [*or* C_{4n+1} *for odd* $n \in \mathbf{N}^+$] include the same subsequence $C_{(3p+2)/2^m}$ [*or* $C_{(3n+1)/2^m}$ *for odd* $n \in \mathbf{N}^+$] (*where* m *is the natural number that makes* $\frac{3p+2}{2^m}$ $\left[or\ \frac{3n+1}{2^m}\right]$ *some positive odd integer*) — that is, C_{2p+1} and C_{8p+5} *coalesce* at $C_{(3p+2)/2^m}$ so C_{2p+1} and C_{8p+5} are *co-convergent*|"*divergent*" with $C_{(3p+2)/2^m}$ [*or for odd* $n \in \mathbf{N}^+$, C_n and C_{4n+1} *coalesce* at $C_{(3n+2)/2^m}$ so C_n and C_{4n+1} are *co-convergent*|"*divergent*" with $C_{(3n+1)/2^m}$].
- The convergence of any C_{8p+5} *sequence* ($\forall p \in \mathbf{N}$) is contended *by-construction* of the *Collatz 3x+1 Christmas tree* having the recursive ($b_i = 4b_{i-1}+1$, $\forall i \in \mathbf{N}^+$) *base sequence* ("*trunk*") with first term $b_0 = 1$ and its *same-recursive extension sequences* ("*branches*" — *not to be confused with the prior designation of* "*branch*" *as* "*trajectory*" *or* "*orbit*") with *initial-odd-terms* $u = \beta^{-1}(\alpha^{-m_i}(b_i))$, $\forall i \in \mathbf{N}^+$ [*where* m_i *is the natural number that makes* u *a positive odd integer*]. It is directly verified that any odd natural number u is actually generated by $\beta^{-1}(\alpha^{-m_i}(b_i))$ for some b_i (*not divisible by* 3) and m_i.
- Therefore, the convergence of every "Collatz 3x+1 *Christmas light*" C_{2p+1} ($\forall p \in \mathbf{N}$) sequence is readily established by just basically noting the already known convergence of the sequence C_n with starting number n that is a term of the $8p+5$-*sequence* (*which includes* $2p+1$ *either as the first or as a middle term*) in the recursively (*or inductively*) constructed *Collatz 3x+1 Christmas tree* of $8p+5$-*sequences*.

Figure 2

Kenneth Conrow's $\frac{3n+1}{2^i}$ Predecessor Tree as a General Tree



We summarize our simplification and rigorization of *Kenneth Conrow*'s and *Gottfried Helms*' equivalent arguments as follows:

► Let $T(1)$ be the sequence of all the iterates of the recursion defined by

$$b_i = \begin{cases} 1 & \text{if } i = 0 \\ 4b_{i-1} + 1 & \text{if } i > 0 \end{cases}$$

— that is, $T(1) = \left\langle 1,\ 4+1,\ 4^2+4+1,\ 4^3+4^2+4+1,\ \ldots,\ \sum_{r=0}^{i} 4^r,\ \ldots \right\rangle$

= <1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, 349525, …>.

Then, $3b_i+1$ ($\forall i \in \mathbf{N}^+$) is a *power-of*-2 so their corresponding *Collatz 3x+1 sequence subsequent-terms* successively eventually drop down to 1 — the *algebraic identity* [5] with x = 4, y = 1, and e = i+1 is used to yield $3\left(\sum_{r=0}^{i} 4^r\right) + 1 = 4^{i+1}$ —

$$\beta(b_i) = \frac{3b_i + 1}{2} = \frac{3\left(\sum_{r=0}^{i} 4^r\right) + 1}{2} = \frac{4^{i+1}}{2} = 2^{2i+1};$$

$$\alpha(\beta(b_i)) = 2^{2i}; \quad \ldots; \quad \alpha^{2i}(\beta(b_i)) = 2; \quad \alpha^{2i+1}(\beta(b_i)) = 1$$

— or, equivalently, all the $b_i \in T(1)$ have the same *first-eventually-positive-odd-integer successor-term* $\alpha^{m_i}(\beta(b_i)) = 1$ where m_i (*depending on* b_i) is the corresponding natural number that makes the latter expression to be true. This assertion is tantamount to the declaration that every *Collatz 3x+1 sequence* which includes some $b_i \in T(1)$ as an iterate has the *cyclic-subsequence* <(2, 1)>.

► For $b_i = 3(2q+1) \in T(1)$ with $i \in \mathbf{N}^+$ and $q \in \mathbf{N}$ — that is, $b_i \in \{3, 9, 15, \ldots, 6q+3, \ldots\}$ — only positive even integers $\alpha^{-e}(b_i) = 2^e b_i$ with $e \in \mathbf{N}^+$ are all of its *regular Collatz 3x+1 sequence predecessor-terms* so it has no *streamlined predecessor-terms*.

► For every $i \in \mathbf{N}^+$ such that $b_i \neq 3(2q+1)$ for any $q \in \mathbf{N}$, let $c = \beta^{-1}(\alpha^{-m_i}(b_i))$, where m_i is the least natural number that makes c a positive odd integer. By the same reasoning used for the *base sequence* $T(1)$, we can form $T(1)$'s *first-level* "*extension sequences*"

$$T(1, b_i) = \left\langle c,\ 4c+1,\ 4^2c+4+1,\ 4^3c+4^2+4+1,\ \ldots,\ 4^i c + \sum_{r=0}^{i-1} 4^r,\ \ldots \right\rangle$$

— all terms of which have the same *first-eventually-positive-odd-integer successor-term* $\alpha^{m_i}(\beta(c)) = b_i$ as aptly demonstrated by the following general reasoning (*where the algebraic identity* [5] *with* x = 4, y = 1, *and* e = i *is used to yield* $3\left(\sum_{r=0}^{i-1} 4^r\right) + 1 = 4^i$):

$$\beta\left(4^i c+\sum_{r=0}^{i-1}4^r\right)=\frac{3\left(4^i c+\sum_{r=0}^{i-1}4^r\right)+1}{2}=\frac{3\cdot 4^i c+3\left(\sum_{r=0}^{i-1}4^r\right)+1}{2}=\frac{3\cdot 4^i c+4^i}{2}$$

$$=\frac{4^i(3c+1)}{2}=4^i\cdot\frac{3c+1}{2}=4^i\cdot\beta(c);$$

$$\alpha^{2i}\left(\beta\left(4^i c+\sum_{r=0}^{i-1}4^r\right)\right)=\frac{4^i\cdot\beta(c)}{2^{2i}}=\beta(c)$$

► We could similarly setup the *second-level extension sequences* $T(1,b_i,c_j)$ for each term $c_j \neq 3(2r+1)$ [*for any* $r \in \mathbf{N}$] of $T(1,b_i)$ as well as the *higher-level extension sequences* $T(1,b_i,c_j,d_k)$, $T(1,b_i,c_j,d_k,e_{l,\ldots})$, …, $T(1,b_i,c_j,d_k,e_l,\ldots,z_o,\ldots)$. For instance (*see* Figure 2):

- T(1,5) = <3, 13, 53, 213, …>; T(1,5,3) = ∅; T(1,5,13) = <17, 69, 277, …>; T(1,5,13,17) = <11, 45, 181, …>; T(1,5,13,17,11) = <7, 29, 117, …>; T(1,5,13,17,11,7) = <9, 37, 149, 597, …>; T(1,5,13,17,11,7,9) = ∅; T(1,5,53) = <35, 141, 565, …>; T(1,5,53,35) = <23, 93, 373, …>; …
- T(1,21) = ∅;
- T(1,85) = <113, 453, 1813, …>; T(1,85,113) = <75, 301, 1205, …>; T(1,85,113,75) = ∅; T(1,85,113,301) = <401, 1605, 6421, …>; …
- T(1,341) = <227, 909, 3637, …>; T(1,341,909) = ∅; …
- T(1,1365) = ∅;
- T(1,5461) = <7281, 29125, 116501, 466005, 1864021, …>.

We shall refer to the *base sequence* $T(1)$ and all of its infinite collection of hierarchical *extension sequences* as the "*Collatz 3x+1 Christmas tree*" — indeed, the hierarchy of *extension sequences* actually represents (*in reverse order*) the *positive-odd-integer* iterates (*shown as arguments in the designation of an extension sequence*) of some *Collatz 3x+1 sequence* that converges to 1. The convergence to 1 — the first term of $T(1)$ — is assured by the fact that $\alpha(\beta(1))=\dfrac{\frac{3(1)+1}{2}}{2}=1$.

- If there is another *Collatz 3x+1 Christmas tree* for some odd natural number $n > 1$, then the first term n of its *base sequence* or "*trunk*" $T(n)$ must be engaged in a *cycle*; that is, n must satisfy (*for some* $k \in \mathbf{N}^+$)

$$\alpha^k(\beta(n)) = \frac{\frac{3n+1}{2}}{2^k} = \frac{3n+1}{2^{k+1}} = n \qquad \text{or} \qquad (2^{k+1} - 3)n = 1$$

— this trivial *Diophantine equation* has the sole solution $k = n = 1$.

It might be argued that $\alpha^k(\beta(n))$ does not necessarily have to be equal to n but could be equal to some term of another *base* or *extension sequence* so that the loop presumably involves more than just one *odd-natural-number* term (*in this case, we have a* "*tree*" *with multiple* "*trunks*"). However, our main proof presented in this paper of the *Collatz* 3*x*+1 *conjecture* in the positive integers domain — that there is only one *cyclic-subsequence* <(2, 1)> — readily refutes this claim. Therefore, there is only one *Collatz 3x+1 Christmas tree* with the sole *trunk* $T(1)$ and every sequence $T(n)$ [*for all odd* $n \in \mathbf{N}^+$] is its *branch*.

- The same existential concern about "*divergent*" *Collatz 3x+1 sequences* is also manifest in the existential issue of a *Collatz Christmas tree* having "*branches*" only with no "*trunk*". Because the latter is a nonsensical possibility, then so must be the former. In plain words, any *streamlined Collatz 3x+1 sequence* C_{2p+1} with starting number $2p+1$ [*for* $p \in \mathbf{N}$] is truly connected as a "*Christmas light*" to the *Collatz 3x+1 Christmas tree* through its own $8p+5$-*sequence* "*branch*".

By the *well-ordering principle* for the natural numbers (*that is, any nonempty subset of* **N** *has some first or least element*), if the set of all C_z in [D] is not empty (*that is, if indeed there are* "*divergent*" *streamlined Collatz 3x+1 sequences*) then there is a least *starting-odd-natural-number-divisible-by-*3 N among the zs such that C_N is "*divergent*". We will assert by some *computable generalization* that no such N actually exist.

Every $N \in O_{\div 3}$ = {3, 9, 15, …, 6*n*+3, …} (*for* $n \in \mathbf{N}$) has no *streamlined Collatz 3x+1 sequence predecessor-terms* so each is regarded as a "*leaf*" (*that is, a* "*terminal point*" *or* "*Christmas light*") in our *Collatz 3x+1 Christmas tree*. We could rule out individually every N as the *least-starting-odd-natural-number-divisible-by*-3 of a "*divergent*" *streamlined Collatz* 3*x*+1 *sequence* if we could computably generalize the precise hierarchy of "*branches*" to which the "*leaf*" N is connected to the "*trunk*" $T(1)$ of our *Collatz 3x+1 Christmas tree* — this is exactly conveyed by the iterates of the *streamlined Collatz 3x+1 sequence* C_N which we already know is very difficult to computably generalize ($\forall N \in O_{\div 3}$).

However, by doing a "*trace-back*" of the *predecessor-terms* of a *not-divisible-by*-3 term — say, 5 — of the "*trunk*" $T(1)$, we can rule out all of the *odd-divisible-by*-3 natural numbers $N \in T(1,5)$ = <3, 13, 53, 213, 853, 3413, 13653, …> — that is, {3, 213, 13653, …} — at once. Similarly, by further doing a "*trace-back*" of the *predecessor-terms* of a *not-divisible-by*-3 term — say, 13 — of the "*branch*" $T(1,5)$, we could also rule out every one of the *odd-divisible-by*-3 natural numbers $N \in T(1,5,13)$ = <17, 69, 277, 1109, 4437, 17749, 70997, 283989, …> — that is, {69, 4437, 283989, …} — at once. And so on. The big difference in this "*backward predecessor-term iteration*" with the "*forward successor-term iteration*" is that the latter is "*not known*" to terminate for all starting positive natural numbers while the former is guaranteed to terminate on every *odd-divisible-by*-3 term.

It is clear that we can only rule out all the $N \in O_{\div 3}$ that are less than or equal to some arbitrary *odd-divisible-by*-3 natural number N_0 by proposing some sort of a comprehensive coincident "*parallel or nested trace-back*" processing, starting from every *not-divisible-by*-3 terms of $T(1)$ - {1} and all of its infinite branches, of their respective *predecessor-terms* until N_0 is reached then all processing halts at once. We admit this "*parallel or nested computing*" to be *non-deterministic*, exponential algorithm — but it does decidedly inductively "*maps the terrain of convergences to* C_1" of all the *streamlined Collatz 3x+1 sequences* C_Z.

► Each term b_i (*for* $i \in \mathbf{N}^+$) of the *base sequence* $T(1)$ or of any of its *extension sequences* is congruent to 5 *modulo* 8 — that is, $b_i = 8p+5$ for some $p \in \mathbf{N}$ because, $\forall q \in \mathbf{N}^+$ with $q = 2p+1$, $4q+1 = 4(2p+1)+1 = 8p+5$. For all $p \in \mathbf{N}$, both $2p+1$ and $8p+5$ are odd natural numbers with

$$\beta(2p+1) = \frac{3(2p+1)+1}{2} = \frac{6p+4}{2} = 3p+2$$

and

$$\alpha^2(\beta(8p+5)) = \frac{\frac{3(8p+5)+1}{2}}{2^2} = \frac{24p+16}{8} = 3p+2 .$$

This simply states that the *Collatz 3x+1 sequences* C_{2p+1} and C_{8p+5} ($\forall p \in \mathbf{N}^+$) are convergent|"*divergent*" with the respective convergence|"*divergence*" of their common *subsequence* C_{3p+2} — or, consequently reduced to $C_{(3p+2)/2^m}$ having some odd starting number (*with* m *being the natural number that makes* $\frac{3p+2}{2^m}$ *a positive odd integer*).

It is stressed that this statement is equivalent to the assertion, for every positive odd integer n, about C_n and C_{4n+1} being *co-convergent*|"*divergent*" with $C_{(3n+1)/2^k}$ (*with* k *being the natural number that makes* $\frac{3n+1}{2^k}$ *a positive odd integer*). Plainly put, since p is arbitrary, all the odd natural numbers are involved in this linked list:

- with $p = 1$ — C_3 and C_{13} are *co-convergent* with C_5;
- with $p = 2$ — C_5 and C_{21} are *co-convergent* with C_1;
- with $p = 3$ — C_7 and C_{29} are *co-convergent* with C_{11};
- with $p = 4$ — C_9 and C_{37} are *co-convergent* with C_7;
- with $p = 5$ — C_{11} and C_{45} are *co-convergent* with C_{17};
- *and so on ...*

The second column of Table 1 presents a *computable generalization* (*there is some definite rule for the construction of each row*) of the *co-convergences* of all the *Collatz 3x+1 sequences* — that is, it is an "*exhaustively surveyable*" (*clarified in the appendix*) presentation of the desired relationship for all the positive odd integers.

Furthermore, $\forall p \in \mathbf{N}^+$ with k_p the natural number that makes $\dfrac{3p+1}{2^{k_p}}$ an odd integer,

$$\alpha^{k_p+1}(\beta(4p+1)) = \alpha^{k_p+1}\left(\frac{3(4p+1)+1}{2}\right) = \alpha^{k_p+1}(6p+2) = \frac{3p+1}{2^{k_p}}$$

and $\beta(4p+3) = \dfrac{3(4p+3)+1}{2} = \dfrac{12p+10}{2} = 6p+5$.

The third and fourth columns of Table 1 showcase a *computable generalization* (*there are definite rules for the construction of each row for both columns*) of the convergence of any *Collatz 3x+1 sequence* — say, to confirm the convergence of C_{19} to C_1:

- since 19 = 4*p*+3 with *p* = 4, look up 19 in the 5*th* row and 4*th* column to get its *streamlined Collatz 3x+1 iteration function successor-term* 29;
- since 29 = 4*p*+1 with *p* = 7, look up 29 in the 8*th* row and 3*rd* column to get its *streamlined* Collatz *3x+1 iteration function successor-term* 11;
- since 11 = 4*p*+3 with *p* = 2, look up 11 in the 3*rd* row and 4*th* column to get its *streamlined Collatz 3x+1 iteration function successor-term* 17;
- since 17 = 4*p*+1 with *p* = 4, look up 17 in the 5*th* row and 3*rd* column to get its *streamlined Collatz 3x+1 iteration function successor-term* 13;
- since 13 = 4*p*+1 with *p* = 3, look up 13 in the 4*th* row and 3*rd* column to get its *streamlined Collatz 3x+1 iteration function successor-term* 5; *and finally*,
- since 5 = 4*p*+1 with *p* = 1, look up 5 in the 2*nd* row and 3*rd* column to get its *streamlined Collatz 3x+1 iteration function successor-term* 1.

It is emphasized that every $n \in \mathbf{N}^+$ always appear with the same *successor-term* in any *Collatz* 3*x*+1 *sequence* that includes *n* as an iterate (*whether as the starting number or as a middle-term*). The *computable generalization* encompassed by Table 1 enables us to have an "*all-at-once-view*" (*explained in the appendix*) of the *successor-term relationships* among all the positive odd integers. It is only the unwarranted defective *unsolvability-hype* of the *Collatz* 3*x*+1 *conjecture* from a different perspective that blurs this very simple general perception of convergence of a *Collatz 3x+1 sequence*.

- For the meantime, let us forget the definition of the *Collatz 3x+1 sequences* and focus on columns 3 and 4 of Table 1 with the *rule of construction* specified by [D1] and [D2]. Now, the *Collatz 3x+1 conjecture* is easily established since:
  1) every odd natural number is certainly covered in the *row-listings* in the two columns because of the regularity in the differences of succeeding *terms-pairs* in [D1] and [D2];
  2) the *fixed-preservation-of-distinct-successor-terms* guarantees that one of the *not-divisible-by-*3 "*base sequence*" terms 1, 5, 85, 341, 5461, … [*in the* $(3p+1)/2^{k_p}$ *field of column* 3 *for rows* 1, 3, 5, 13, 21, 28, 53, …] will always be reached from any starting positive odd integer greater than 1 in the $4p+1$ *field of column* 3 or $4p+3$ *field of column* 4.

Table 1
Look-up Table for the Collatz 3x+1 Sequences Terms

p	$2p+1$	$8p+5$	$3p+2$	$\frac{3p+2}{2^{m_p}}$	m_p	$4p+1$	$3p+1$	$\frac{3p+1}{2^{k_p}}$	k_p	[D1]	$4p+3$	$6p+5$	[D2]
0	1	5	2	1	1	1	1	1	0	*	3	5	0
1	3	13	5	5	0	5	4	1	2	0	7	11	1
2	5	21	8	1	3	9	7	7	0	1	11	17	2
3	7	29	11	11	0	13	10	5	2	2	15	23	3
4	9	37	14	7	1	17	13	13	0	3	19	29	4
5	11	45	17	17	0	21	16	1	4	4	23	35	5
6	13	53	20	5	2	25	19	19	0	5	27	41	6
7	15	61	23	23	0	29	22	11	1	6	31	47	7
8	17	69	26	13	1	33	25	25	0	7	35	53	8
9	19	77	29	29	0	37	28	7	2	8	39	59	9
10	21	85	32	1	5	41	31	31	0	9	43	65	10
11	23	93	35	35	0	45	34	17	1	10	47	71	11
12	25	101	38	19	1	49	37	37	0	11	51	77	12
13	27	109	41	41	0	53	40	5	3	12	55	83	13
14	29	117	44	11	2	57	43	43	0	13	59	89	14
15	31	125	47	47	0	61	46	23	1	14	63	95	15
16	33	133	50	25	1	65	49	49	0	15	67	101	16
17	35	141	53	53	0	69	52	13	2	16	71	107	17
18	37	149	56	7	3	73	55	55	0	17	75	113	18
19	39	157	59	59	0	77	58	29	1	18	79	119	19
20	41	165	62	31	1	81	61	61	0	19	83	125	20
21	43	173	65	65	0	85	64	1	6	20	87	131	21
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

[D1] = the count of natural numbers between $3p+1$ and $4p+1$
[D2] = the count of odd natural numbers between $4p+3$ and $6p+5$

Let $\alpha^{m_p}(3p+2) = \dfrac{3p+2}{2^{m_p}} = 2q+1$, for some $q \in \mathbf{N}$, where m_p is the natural number that makes $\alpha^{m_p}(3p+2)$ some positive odd integer. Then, we have the recursive situation that the *Collatz 3x+1 sequences* C_{2q+1} and C_{8q+5} are convergent|"*divergent*" also in the same respect with the convergence|"*divergence*" of $C_{(3q+2)/2^{m_q}}$.

- Viewed from the reverse perspective, each odd natural number $y = 2q+1 > 1$ ($q \in \mathbf{N}^+$) [*which is not divisible by* 3] that is an iterate of a *streamlined Collatz 3x+1 sequence* [6] always has as *immediate-predecessor-term* a $b_0 = 2p+1$ ($p \in \mathbf{N}^+$) [*which is not of the form* $8x+5$ *for any* $x \in \mathbf{N}$] or any of b_0's infinite count of recursive terms $b_{i+1} = 4b_i+1$ [*each of which is of the form* $8c+5$ *for some* $c \in \mathbf{N}$]. In other words, the iterates of any *streamlined Collatz 3x+1 sequence* form some hierarchical structure of $8x+5$-*recursive-sequences*.

While it is not the case that $q < p$ each time, we have a *reverse recursion mechanism* — $c_i = \dfrac{c_{i+1} - 1}{4}$ — that actually takes any term of the form $8c+5$ (*for some* $c \in \mathbf{N}$) to the *least-valued* iterate of its parent $8c+5$-*sequence* which, in point of fact, vividly exposes (*better than the bland perception imparted by the mere successive division-by-*2 *of even iterates*) the true convergence of a *regular Collatz 3x+1 sequence*.

- Every $8c+5$-*sequence* involves the recursive iteration function $c_{i+1} = 4c_i+1$ ($i \in \mathbf{N}^+$) on some *positive-odd-integer* starting number c_0 (*which is not of the form* $8c+5$). Because every term p_i of an $8p+5$-*sequence* has the same *odd-successor-term* $\alpha^m(\beta(p_0)) = q_i$ [*if* $q_i = 1$ *then the* $8p+5$-*sequence is really the base sequence* $T(1)$; *otherwise,* q_i *is a term of an* $8q+5$-*extension-sequence*], we could just consider the *least-valued* initial *positive-odd-integer-term* x_0 of each $8x+5$-*sequence* — this duly dismisses notions of a "*divergent*" sequence since the consideration of arbitrarily large terms are significantly reduced to the computation of only *low-valued* iterates.

- For $p \in \mathbf{N}$, any $2p+1$ is a term in the $8p+5$-*sequence* so we could simply take into consideration the latter sequence's first term $p_0 \leq 2p+1$ which is quickly found by repeated application of the reverse iteration function $p_i = \frac{p_{i+1} - 1}{4}$ starting with $p_n = 2p+1$ until p_{i-1} is no longer a positive odd integer — the last p_i which is still an odd natural number is p_0. If $p_0 = 1$ then we are done — that is, we have demonstrated the desired convergence to 1.
- Now, if $p_0 > 1$ then $\alpha^{m_p}(\beta(p_0)) = 2q+1$ (*for some* $q \in \mathbf{N}$) which is a term in the $8q+5$-*sequence* so we could again just consider the latter sequence's first term $q_0 \leq 2q+1$. If $q_0 = 1$ then we are done; otherwise, we repeat the process until we reach the *base sequence* $T(1)$ with its first term 1.
- The fact that there is only one *base sequence* ("*trunk*") $T(1)$ for the *Collatz 3x+1 Christmas tree* ensures that the *base sequence* $T(1)$ would be reached eventually by the aforementioned recursive process of coalescences.

Basically, we want to sustain the assertion that every positive odd integer is connected to our *Collatz 3x+1 Christmas tree* — this is equivalent to the contention that the latter links all the $8p+5$- or $4(2p+1)+1$-*recursive-sequences* with every starting positive odd integer $2p+1$ ($\forall p \in \mathbf{N}^+$) that is not of the form $8b+5$ (*for any* $b \in \mathbf{N}$) while any positive odd integer of the form $8b+5$ (*for some* $b \in \mathbf{N}$) is a term of some $8p+5$-*sequence* with first term b_0 which is a positive odd integer that is not of the form $8b+5$ (*for any* $b \in \mathbf{N}$). Table 2 delineates some *computable generalization* of the *inter-connections* among all the $4x+1$-*recursive-sequences for each starting odd* $x \in \mathbf{N}^+$. All the odd natural numbers are listed in column 0. Any element in column 0 of row $i \in \{2, 6, 10, 14, \ldots, 4v+2, \ldots\}$ where $v \in \mathbf{N}$, and also all those to the right of column 0 of any row, has the form $8c+5$ (*for* $c \in \mathbf{N}$) — they are all also listed in column 1 [*in fact, all elements to the right of column j in any row are also listed in column j*].

For $i \geq 0$ and $j \geq 0$, suppose $x_{i,j}$ is the *ith-row* and *jth-column* array element in Table 2.

Then $x_{0,j} = \sum_{r=0}^{j} 4^r$ and $x_{i,j} = 2i \cdot 4^j + x_{0,j}$. The following are some defined or easily derived identities that can be "*viewed-all-at-once*" or "*extensively surveyed*" in Table 2:

- $x_{i,j} = 4x_{i,j-1} + 1$ for $i \geq 0,\ j \geq 1$;
- $x_{i,j} = x_{i-1,j} + 2 \cdot 4^j$ for $i \geq 1,\ j \geq 0$;
- $x_{i,0} = 2i + 1$ for $i \geq 0$;
- $x_{i,1} = 8i + 5$ for $i \geq 0$;

  $x_{i,1} \equiv_3 2$ if $i \equiv_3 0$; $x_{i,1} \equiv_3 1$ if $i \equiv_3 1$; $x_{i,1} \equiv_3 0$ if $i \equiv_3 2$;
- $x_{4i+2,j} = x_{i,j+1}$ for $i \geq 0,\ j \geq 0$.

Table 2
Look-up Table for the Collatz 8*p*+5-Sequences Terms

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	…
R_0	1	5	21	85	341	1365	5461	21845	87381	349525	…
R_1	3	13	53	213	853	3413	13653	54613	218453	873813	---
R_2	5	21	85	341	1365	5461	21845	87381	349525	1398101	---
R_3	7	29	117	469	1877	7509	30037	120149	480597	1922389	---
R_4	9	37	149	597	2389	9557	38229	152917	611669	2446677	---
R_5	11	45	181	725	2901	11605	46421	185685	742741	2970965	---
R_6	13	53	213	853	3413	13653	54613	218453	873813	3495253	---
R_7	15	61	245	981	3925	15701	62805	251221	1004885	4019541	---
R_8	17	69	277	1109	4437	17749	70997	283989	1135957	4543829	---
R_9	19	77	309	1237	4949	19797	79189	316757	1267029	5068117	---
R_{10}	21	85	341	1365	5461	21845	87381	349525	1398101	5592405	---
R_{11}	23	93	373	1493	5973	23893	95573	382293	1529173	6116693	---
R_{12}	25	101	405	1621	5973	25941	103765	415061	1660245	6640981	---
R_{13}	27	109	437	1749	6997	27989	111957	447829	1791317	7165269	---
R_{14}	29	117	469	1877	7509	30037	120149	480597	1922389	7689557	---
R_{15}	31	125	501	2005	8021	32085	128341	513365	2053461	8213845	---
R_{16}	33	133	533	2133	8533	34133	136533	546133	2184533	8738133	---
R_{17}	35	141	565	2261	9045	36181	144725	578901	2315605	9262421	---
R_{18}	37	149	597	2389	9557	38229	152917	611669	2446677	9786709	---
R_{19}	39	157	629	2517	10069	40277	161109	644437	2577749	10310997	---
R_{20}	41	165	661	2645	10581	42325	169301	677205	2708821	10835285	---
R_{21}	43	173	693	2773	11093	44373	177493	709973	2839893	11359573	---
R_{22}	45	181	725	2901	11605	46421	185685	742741	2970965	11883861	---
R_{23}	47	189	757	3029	12117	48469	193877	775509	3102037	12408149	---
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

We simply need to establish that any arbitrary positive odd integer which is not of the form $8p+5$ (*for any* $p \in \mathbf{N}$) — that is, a first term b_0 of some $8b+5$-*recursive-sequence* — could be obtained from $\beta^{-1}(\alpha^{-m}(x_{i,0})) = \dfrac{2[2^m(2i+x_{0,0})]-1}{3}$ for some natural numbers *i* and *m* with $x_{i,0}$ not divisible by 3. Thus, with $x_{0,0} = 1$ and $p \in \mathbf{N}^+$ —

$$b_0 = 2p+1 = \beta^{-1}(\alpha^{-m}(x_{i,0})) = \frac{2^{m+1}(2i+1)-1}{3}$$

or $$\frac{3p+2}{2^m} = 2i+1 \qquad [7]$$

or $$p = \frac{2^m(2i+1)-2}{3} \qquad [8].$$

The following are trivial observations:

- $2^m \equiv_3 1$ if m is even and $2^m \equiv_3 2$ if m is odd.
- If $2i+1 \equiv_3 0$, then p is not an integer for any $m \in \mathbf{N}^+$; if $2i+1 \equiv_3 1$ then m must be even, or if $2i+1 \equiv_3 2$ then m must be odd, for p to become a positive integer.

Equation [7] simply states our recursive relation on the coalescence term $\dfrac{3p+2}{2^m}$ of C_{2p+1} and C_{8p+5}. Column 2 of Table 1 lists the first few positive odd integers $2p+1$ with their respective *streamlined Collatz 3x+1 iteration function immediate-successor-terms*. We want to firmly establish the converse claim using column 0 of Table 2 and equation [8] — that every odd natural number n which is not of the form $8b+5$ (*for* $b \in \mathbf{N}$) is an *immediate-predecessor-term* of another element [*which may or may not be of the form* $8c+5$ (*for* $c \in \mathbf{N}$)] in column 0 of Table 2. This would prove that our *Collatz 3x+1 Christmas tree* exhaustively connects all the $8x+5$-*recursive-sequences* and the truth of the *Collatz 3x+1 conjecture* in the positive integers domain immediately follows — that is, each *Collatz 3x+1 sequence* C_n converges to C_1 and there is no divergent C_n.

For $m = 0$, we have $3p+2 = 2i+1$ or $p = \frac{2i-1}{3}$ for which $i \in \{2, 5, 8, \ldots, 3a+2, \ldots\}$ respectively yields $p \in \{1, 3, 5, \ldots, 2b+1, \ldots\}$ or $n \in \{3, 7, 11, \ldots, 4c+3, \ldots\}$ where $a, b, c \in \mathbf{N}$. For examples:

- $i = 2$ yields $p = 1$ or $n = 3$ — that is, $\beta^{-1}(x_{2,0}) = \beta^{-1}(5) = 3$;
- $i = 5$ yields $p = 3$ or $n = 7$ — that is, $\beta^{-1}(x_{5,0}) = \beta^{-1}(11) = 7$; …

For $m > 0$ and $p = 2q$ ($\forall q \in \mathbf{N}^{+}$), equation [7] becomes $\frac{6q+2}{2^m} = \frac{3q+1}{2^{m-1}} = 2i+1$.

For $m = 1$, we have $q = \frac{2i}{3}$ for which $i \in \{3, 6, 9, \ldots, 3d, \ldots\}$ respectively yields $q \in \{2, 4, 6, 8, \ldots, 2e, \ldots\}$ or $n = 2p+1 = 4q+1 \in \{9, 17, 25, \ldots, 8f+1, \ldots\}$ where $d, e, f \in \mathbf{N}^{+}$). For examples:

- $i = 3$ yields $q = 2$ or $n = 9$ — that is, $\beta^{-1}(\alpha^{-1}(x_{3,0})) = \beta^{-1}(\alpha^{-1}(7)) = 9$;
- $i = 6$ yields $q = 4$ or $n = 17$ — that is, $\beta^{-1}(\alpha^{-1}(x_{6,0})) = \beta^{-1}(\alpha^{-1}(13)) = 17$; …

We are now left to consider Table 2's 1st column elements in $\{5, 13, 21, \ldots, 8b+5, \ldots\}$ — or $p \in \{2, 6, 10, \ldots, 4c+2, \ldots\}$ where $b, c \in \mathbf{N}$ — or all the elements $x_{i,j}$ with $j > 0$. Now, any positive odd integer of the form $8b+5$ (*for some* $b \in \mathbf{N}$) is a term of the $T(1)$ *base sequence* or of some $8x+5$-*extension-sequence* for which we can easily find its first term $b_0 = x_{i,0}$ (*which is not of the form* $8b+5$) by the repeated j-applications of the reverse iteration function $b_{j-1} = \frac{b_j - 1}{4}$ starting with $b_j = 8b+5$ until b_{j-1} is no longer a positive odd integer — the last b_j that is still some odd natural number is the first term $b_0 = x_{i,0}$ which (*being not of the form* $8b+5$)) we have already established to be an *immediate-predecessor-term* of some other $x_{k,0}$ (*for some* $k \in \mathbf{N}$, $k \neq i$) in Table 2. For example, for $n = 53 = 8(6)+5$, we just find the first term $b_0 = 3$ (*which we know is an immediate-predecessor-term of* 5) of this $8p+5$-*recursive-sequence* and simply conclude that the positive odd integer 53 is actually connected to our *Collatz 3x+1 Christmas tree*.

► The contention of *recursiveness* — $b_{i+1} = 4b_i+1$ — for the iterates of each *extension sequence* is supported by this simple argument:

Suppose, for an odd $n \in \mathbf{N}^+$, $\beta^{-1}(n) = \frac{2n-1}{3}$ is a positive odd integer. Then

$$\beta^{-1}(\alpha^{-2}(n)) = \beta^{-1}(2^2 n) = \frac{2(2^2 n)-1}{3} = \frac{2^3 n - 1}{3} = 4\left(\frac{2n-1}{3}\right) + 1 = 4\beta^{-1}(n) + 1;$$

$$\beta^{-1}(\alpha^{-4}(n)) = \beta^{-1}(2^4 n) = \frac{2(2^4 n)-1}{3} = \frac{2^5 n - 1}{3} = 4\left(\frac{2^3 n-1}{3}\right) + 1 = 4\beta^{-1}(\alpha^{-2} n)) + 1;$$

$\vdots$

In general, for any $k \in \mathbf{N}^+$:

$$\beta^{-1}(\alpha^{-2k-2}(n)) = \beta^{-2k-2}(2^{2k+2} n) = \frac{2(2^{2k+2} n) - 1}{3} = 4\left(\frac{2^{2k+1} n - 1}{3}\right) + 1$$

$$= \beta^{-1}(\alpha^{-2k} n)) + 1.$$

► A *recursion* is equivalent to a *mathematical induction* **[75]**. Hence, we could justify the "*all-at-once view*" or the "*extensive surveyability*" (*discussed in the appendix*) of the *Collatz 3x+1 Christmas tree*'s recursive *base sequence* $T(1)$ as well as its infinite count of recursive *extension sequences* — hence, also the extensive coverage of all the odd natural numbers in the *Collatz 3x+1 Christmas tree* — by simply invoking the *first principle of* "*parallel mathematical induction*" (*just think of some collection of standing dominoes whose initiated fall of each individual domino also trigger the fall of many other sets of dominoes that are arranged in parallel*). In our case at bar, the infinite terms of the *base sequence* $T(1)$ triggers the *mathematical induction* of $T(1)$'s plethora of infinite parallel or nested recursive *extension sequences*.

- The call on the novel *first principle of* "*parallel mathematical induction*" should not be a source of consternation or controversy. In the same vein, the idea of "*parallel|nested counting|computing|processing*" as some meaningful interpretation of *transfinite sequences* is cited in the appendix.

► Therefore, our *simple deterministic polynomial-time algorithm* for establishing that every *Collatz 3x+1 sequence* with some given starting odd natural number $n = 2p+1$ (*for* $p \in \mathbf{N}^+$) converges to 1 is as follows:

```
Input n
LOOP WHILE n > 1
  LOOP WHILE n is not of the form 8p + 5
    n := (3n + 1) / 2
    LOOP WHILE n is an even natural number
      n := n / 2
    END LOOP
    IF n = 1 THEN STOP
  END LOOP
  LOOP WHILE n is an odd natural number
    n := (n - 1) / 4
    IF n = 1 THEN STOP
  END LOOP
  n := 4n + 1
END LOOP
```

An actually executable computer program is listed in *Program Listing 1* and its output for the starting number 27 is shown in *Sample Output 1*. The indented numbers signify some *Collatz* 3*x*+1-*sequence* iterate with the 8*p*+5 form and a descent to the *first-term* of their respective 8*p*+5-*sequence*. The count of consecutive *positive-odd-integer* iterates that are not divisible by 3 significantly defines the length of a *Collatz* 3*x*+1 *sequence*.

It should not be surprising to find out that this algorithm yields exactly the same terms as by simply computing the consecutive *positive-odd-integer* iterates of the *streamlined* Collatz 3*x*+1 *sequence* C_n. The overriding important difference of the former procedure over the latter is the emphasis placed on the convergence of the *Collatz* 3*x*+1 *sequences* that is conveyed by the very construction of the *Collatz* 3*x*+1 *Christmas tree* of 8*x*+5-*sequences*.

Program Listing 1

Microsoft Visual Foxpro Application Program that Affirms the Collatz Conjecture

```
CLOSE ALL
CLEAR ALL
DELETE FILE CollatzSequence.DBF
CREATE TABLE CollatzSequence (Iterates M)    && To be used for listing the iterates
USE CollatzSequence
APPEND BLANK

CLEAR
INPUT "  Enter starting number > 4 --- " TO z
REPLACE Iterates WITH Iterates + ALLTRIM(STR(z)) + CHR(13)

IF MOD(z, 2) = 0                          && MOD() is the modulo or remainder function
  DO WHILE MOD(z, 2) = 0
    z = z / 2
    IF z > 1
      REPLACE Iterates WITH Iterates + ALLTRIM(STR(z)) + CHR(13)
    ENDIF
  ENDDO
ENDIF

DO WHILE z >= 5
  DO WHILE MOD((z - 5), 8) > 0
    z = (3 * z + 1) / 2
    DO WHILE MOD(z, 2) = 0
      z = z / 2
    ENDDO
    REPLACE Iterates WITH Iterates + ALLTRIM(STR(z)) + CHR(13)
    IF z < 5
      EXIT
    ENDIF
  ENDDO
  IF z >= 5
    u = z
    DO WHILE (u > 3) AND (u - INT(u) = 0)          && INT(x) returns the integer part of x
      u = (u - 1) / 4
      IF MOD(u, 2) = 1
        REPLACE Iterates WITH Iterates + "    " + ALLTRIM(STR(u)) + CHR(13)
      ELSE
        EXIT
      ENDIF
    ENDDO
    DO CASE
      CASE u = 1
        z = 1
      CASE u = 3
        z = 3
        EXIT
      OTHERWISE
        z = 4 * u + 1
    ENDCASE
  ENDIF
ENDDO

DO CASE
  CASE z = 1
    REPLACE Iterates WITH Iterates + "1" + CHR(13)
  CASE z = 3
    REPLACE Iterates WITH Iterates + "5" + CHR(13)
    REPLACE Iterates WITH Iterates + "1" + CHR(13)
  OTHERWISE
    @3,2 SAY "Collatz conjecture not proved!"
ENDCASE

MODIFY MEMO CollatzSequence.Iterates                 && Display all the iterates
CANCEL
```

Sample Output 1

For Starting Number n = 27

```
  27
  41
  31
  47
  71
 107
 161
 121
  91
 137
 103
 155
 233
 175
 263
 395
 593
 445
         111
 167
 251
 377
 283
 425
 319
 479
 719
1079
1619
2429
         607
 911
1367
2051
3077
         769
 577
 433
 325
          81
  61
          15
  23
  35
  53
          13
           3
   5
   1
```

Thus, ***we have surely established the truth of the preferred Collatz 3x+1 conjecture in the positive integers domain***. However, there is still the paramount motivation to defuse the *blown-up* *unsolvability-hype* imprudently hurled to *non-trivial* *Collatz 3x+1-type problems*. In **[64]**, the 3x+1 *problem* is described as a "*mystery*" and the "*remarkable ramifications of unsolvability*" are discussed — reflecting on unsolvable *Diophantine equations*, *Hilbert's 10th problem*, *Turing's computer-program-halting-problem*, *Gödel's incompleteness theorems*, *Goodstein's sequences and theorem*, *fast-growing functions and termination proofs,* as well as the *basic paradigm of mathematical proof* with speculation on some possible *future paradigm shift in axiomatic systems.*

The primary thesis in this paper is *computable generalization*. We fully affirmed in this section what we had already established in the preceding section — the *computable generalizability* of the *all-positive-integer-terms* *Collatz 3x+1 sequences*. We discuss the *unsolvability* of finding all the cycles for the same *Collatz 3x+1 iteration function* in the negative integers domain in the next section VI and provide additional examples of *Collatz 3x+1-type problems* in section VII.

Our simple validation of the *Collatz* 3x+1 *conjecture* in the positive integers domain brings to the forefront the characteristic *Collatz 3x+1 syndrome* — that is, the hasty unwitting labeling of "*unsolvability*" (*due to their ineffective proposed solution-methods*) to actually computable mathematical problems when suitable *solution-approaches* are employed — that precipitated the *modern crisis in mathematics*. In truth, the *streamlined* *Collatz* 3x+1 *iteration function* [6] is indeed very far removed from the "*computational complexity*" or "*undecidability*" concerns (*the falsely alleged lack of computable generalization*) that are openly hyped with the deemed *not-computabty-generalizable branching* *Collatz* 3x+1 *sequences* in the positive integers domain. One might then rightfully contend (*inasmuch as the iteration rule has, in actual fact, been changed*) that the iteration [6] no longer truly manifest the *Collatz* 3x+1 *problem* that stressed so many mathematicians for some time. In an appendix, the *Collatz 3x+1 syndrome* is correlated to the prevalent *flimflams in the foundations of mathematics* that are primarily brought on by flawed applications of *reductio ad absurdum* ("*reduction to self-contradiction*") *proofs* or of *Cantor's anti-diagonal argument* as well as the utterly complacent invocation of the *axiom of choice* to elude the mathematically rigorous requirement of some *computable generalization* such as *mathematical induction*.

VI. *Undecidability of Collatz Conjecture in Negative Integers Domain*

We continue analyzing the *preferred* Collatz 3x+1 problem in the domain of negative integers.

1. For the 4 *branch-nodes* of term $d = f^2(b)$, we evaluate if there is a valid solution for b when either $\alpha(b) = \frac{b}{2} = d$ or $\beta(b) = \frac{3b+1}{2} = d$.

For the *branch-node* $d = 2^2 b = 4b$:

► $\alpha(b) = \frac{b}{2} = 4b$ yields $b = 0$ — the trivial solution C_0 = <(0)>;

► $\beta(b) = \frac{3b+1}{2} = 4b$ yields $b = \frac{1}{5}$ — which is not a valid solution.

For the *branch-node* $d = \frac{2^2 b - 1}{3} = \frac{4b-1}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-1}{3}$ yields $b = \frac{2}{5}$ — which is not a valid solution;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-1}{3}$ yields $b = \frac{2+3}{2^3 - 3^2} = -5$ — which is a valid solution in the negative integers domain corresponding to C_{-7} = <(-7, -10, -5)>.

For the *branch-node* $d = \frac{2^2 b - 2}{3} = \frac{4b-2}{3}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-2}{3}$ yields $b = \frac{4}{5}$ — which is not a valid solution;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-2}{3}$ yields $b = \frac{2^2+3}{2^3 - 3^2} = -7$ — which is a valid solution in the negative integers domain corresponding to C_{-10} = <(-10, -5, -7)>.

For the *branch-node* $d = \frac{2^2 b - (2+3)}{3^2} = \frac{4b-5}{9}$:

► $\alpha(b) = \frac{b}{2} = \frac{4b-5}{9}$ yields $b = \frac{2 \cdot 5}{2^3 - 3^2} = -10$ — which is a valid solution in the negative integers domain corresponding to C_{-5} = <(-5, -7, -10)>;

► $\beta(b) = \frac{3b+1}{2} = \frac{4b-5}{9}$ yields $b = -1$ — the trivial solution C_{-1} = <(-1)>.

Both -7 and -10 are *portal-cycle-terms* with $\alpha^{-1}(-7) = -14$ and $\alpha^{-1}(-10) = -20$. Thus, the *preferred* *Collatz 3x+1 sequences* with *length*-3 *cyclic-subsequences* are the following:

C_{-5} = <(-5, -7, -10)>;

C_{-7} = <(-7, -10, -5)>;

C_{-10} = <(-10, -5, -7)>;

C_{-n_1} = $<-n_1, -m_1, -l_1, \ldots, -14, (-7, -10, -5)>$ for many $n_1 \in \mathbf{N}^{+} - \{5, 7, 10\}$;

C_{-n_2} = $<-n_2, -m_2, -l_2, \ldots, -20, (-10, -5, -7)>$ for many other $n_2 \in \mathbf{N}^{+} - \{5, 7, 10\}$.

2. It is left to the reader to verify that there is no cycle or cyclic subsequence with length 4, 5, 6, 7, 8, 9, or 10 and that there are *length*-11 cycles or cyclic subsequences —

$$f(b) = f^{10}(b) \Leftrightarrow b \in \{-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34\}$$

— whose *branch-nodes* expressions are exhibited in Table 3. The *portal-cycle-terms* are -25, -34, -37, -55, -61, -82, -91, and -136. So, the *preferred* *Collatz 3x+1 sequences* in the negative integers domain with *length*-11 cyclic subsequences are the following:

C_{-17} = <(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)>

C_{-25} = <(-25, -37, -55, -82, -41, -61, -91, -136, -68, -34, -17)>

C_{-34} = <(-34, -17, -25, -37, -55, -82, -41, -61, -91, -136, -68)>

C_{-37} = <(-37, -55, -82, -41, -61, -91, -136, -68, -34, -17, -25)>

C_{-41} = <(-41, -61, -91, -136, -68, -34, -17, -25, -37, -55, -82)>

C_{-55} = <(-55, -82, -41, -61, -91, -136, -68, -34, -17, -25, -37)>

C_{-61} = <(-34, -17, -25, -37, -55, -82, -41, -61, -91, -136, -68)>

C_{-68} = <(-68, -34, -17, -25, -37, -55, -82, -41, -61, -91, -136)>

C_{-82} = <(-82, -41, -61, -91, -136, -68, -34, -17, -25, -37, -55)>

C_{-91} = <(-91, -136, -68, -34, -17, -25, -37, -55, -82, -41, -61)>

C_{-136} = <(-136, -68, -34, -17, -25, -37, -55, -82, -41, -61, -91)>

Table 3

Branch-Nodes **Expressions for** ***Length*****-11 Cycles or Cyclic Subsequences**

— <(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)> —

of many Preferred Collatz 3x+1 Sequences in the Negative Integers Domain

l $f^{10}(b)$	***k*** $f^{9}(b)$	***j*** $f^{8}(b)$	***i*** $f^{7}(b)$	***h*** $f^{6}(b)$	***g*** $f^{5}(b)$	***f*** $f^{4}(b)$	***e*** $f^{3}(b)$	***d*** $f^{2}(b)$	***c*** $f^{1}(b)$	***b***
$\frac{1024b-817}{729}$	$\frac{512b-287}{243}$	$\frac{256b-103}{81}$	$\frac{128b-38}{27}$	$\frac{64b-19}{27}$	$\frac{32b-5}{9}$	$\frac{16b-1}{3}$	$8b$	$4b$	$2b$	-17
$\frac{1024b-1373}{729}$	$\frac{512b-565}{243}$	$\frac{256b-242}{81}$	$\frac{128b-121}{81}$	$\frac{64b-47}{27}$	$\frac{32b-19}{9}$	$\frac{16b-8}{3}$	$\frac{8b-4}{3}$	$\frac{4b-2}{3}$	$\frac{2b-1}{3}$	-25
$\frac{1024b-2207}{729}$	$\frac{512b-982}{243}$	$\frac{256b-491}{243}$	$\frac{128b-205}{81}$	$\frac{64b-89}{27}$	$\frac{32b-40}{9}$	$\frac{16b-20}{9}$	$\frac{8b-10}{9}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-37
$\frac{1024b-3458}{729}$	$\frac{512b-1729}{729}$	$\frac{256b-743}{243}$	$\frac{128b-331}{81}$	$\frac{64b-152}{27}$	$\frac{32b-76}{27}$	$\frac{16b-38}{27}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-55
$\frac{1024b-5699}{2187}$	$\frac{512b-2485}{729}$	$\frac{256b-1121}{243}$	$\frac{128b-520}{81}$	$\frac{64b-260}{81}$	$\frac{32b-130}{81}$	$\frac{16b-65}{81}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-82
$\frac{1024b-2485}{729}$	$\frac{512b-1121}{243}$	$\frac{256b-520}{81}$	$\frac{128b-260}{81}$	$\frac{64b-130}{81}$	$\frac{32b-65}{81}$	$\frac{16b-19}{27}$	$\frac{8b-5}{9}$	$\frac{4b-1}{3}$	$2b$	-41
$\frac{1024b-3875}{729}$	$\frac{512b-1816}{243}$	$\frac{256b-908}{243}$	$\frac{128b-454}{243}$	$\frac{64b-227}{243}$	$\frac{32b-73}{81}$	$\frac{16b-23}{27}$	$\frac{8b-7}{9}$	$\frac{4b-2}{3}$	$\frac{2b-1}{3}$	-61
$\frac{1024b-5960}{729}$	$\frac{512b-2980}{729}$	$\frac{256b-1490}{729}$	$\frac{128b-745}{729}$	$\frac{64b-251}{243}$	$\frac{32b-85}{81}$	$\frac{16b-29}{27}$	$\frac{8b-10}{9}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-91
$\frac{1024b-9452}{2187}$	$\frac{512b-4726}{2187}$	$\frac{256b-2363}{2187}$	$\frac{128b-817}{729}$	$\frac{64b-287}{243}$	$\frac{32b-103}{81}$	$\frac{16b-38}{27}$	$\frac{8b-19}{27}$	$\frac{4b-5}{9}$	$\frac{2b-1}{3}$	-136
$\frac{1024b-4726}{2187}$	$\frac{512b-2363}{2187}$	$\frac{256b-817}{729}$	$\frac{128b-287}{243}$	$\frac{64b-103}{81}$	$\frac{32b-38}{27}$	$\frac{16b-19}{27}$	$\frac{8b-5}{9}$	$\frac{4b-1}{3}$	$2b$	-68
$\frac{1024b-2363}{2187}$	$\frac{512b-817}{729}$	$\frac{256b-287}{243}$	$\frac{128b-103}{81}$	$\frac{64b-38}{27}$	$\frac{32b-19}{27}$	$\frac{16b-5}{9}$	$\frac{8b-1}{3}$	$4b$	$2b$	-34

C_{-n_1} = <$-n_1$, $-m_1$, $-l_1$, …, -50, (-25,-37,-55,-82,-41,-61,-91,-136,-68,-34,-17)>

for many $n_1 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_2} = <$-n_2$, $-m_2$, $-l_2$, …, -23, (-34,-17,-25,-37,-55,-82,-41,-61,-91,-136,-68)>

for many other $n_2 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_3} = <$-n_3$, $-m_3$, $-l_3$, …, -74, (-37,-55,-82,-41,-61,-91,-136,-68,-34,-17,-25)>

for many other $n_3 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_4} = <$-n_4$, $-m_4$, $-l_4$, …, -110, (-55,-82,-41,-61,-91,-136,-68,-34,-17,-25,-37)>

for many other $n_4 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_5} = <$-n_5$, $-m_5$, $-l_5$, …, -122, (-61,-91,-136,-68,-34,-17,-25,-37,-55,-82,-41)>

for many other $n_5 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_6} = <$-n_6$, $-m_6$, $-l_6$, …, -164, (-82,-41,-61,-91,-136,-68,-34,-17,-25,-37,-55)>

for many other $n_6 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_7} = <$-n_7$, $-m_7$, $-l_7$, …, -182, (-91,-136,-68,-34,-17,-25,-37,-55,-82,-41,-61)>

for many other $n_7 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136};

C_{-n_8} = <$-n_8$, $-m_8$, $-l_8$, …, -272, (-136,-68,-34,-17,-25,-37,-55,-82,-41,-61,-91)>

for many other $n_8 \in \mathbf{N}^+$ **-** {17, 25, 34, 37, 41, 55, 61, 68, 82, 91, 136}.

3. Without some *computable generalization* that identifies all the valid cycles, or which establishes that there are no more valid loops with lengths greater than 11 (*nor divergent sequences*), for the *preferred* Collatz 3x+1 sequences in the negative integers domain, this problem would remain undecidable inasmuch as it is exponentially computationally complex — that is, for arbitrary *branch-point* $f^{-u}(b)$, there are 2^{u+1} *cycle-equations* to be evaluated for valid *solution-values* for b and there will not be adequate computing space and time resources to solve all of these *cycle-equations* at some stage, say for $u > 100$ [*in fact, theoretical physicists have computed* **[76]** *that there are less than* 10^{78} ($\sim 2^{260}$) *particles in the entire observable universe while the age of the universe after the deemed* *Big Bang is less than* 10^{40} ($\sim 2^{133}$) *atomic units or* 10^{18} ($\sim 2^{60}$) *seconds*].

VII. *More Examples of Collatz 3x+1-type Problems*

The capable readers could just write their own simple computer programs to easily automate the evaluation of *solution-values* for b in the *general* Collatz 3x+1 sequences' cycles-equations (*which are merely linear equations in* 1 *variable*). The interested readers could solve (*and verify for themselves the reliability of the simple and general method demonstrated in this manuscript*) the following additional examples of *Collatz 3x+1-type problems* taken from the references listed at the ending pages of this paper.

Example 1. The *standard* Collatz 3x+1 sequences **[1, 2, 5]** are defined for positive integers:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if} \quad n \equiv_2 0 \\ 3n+1 & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n (*in the domain of all integers*) has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates. These are known:

- ► 1 trivial *length*-1 cycle — (0);
- ► 1 *length*-2 cycle — (-2, -1); 1 *length*-5 cycle — (-5, -14, -7, -20, -10); 1 *length*-18 cycle — (-17, -50, -25, -74, -37, -110, -55, -164, -82, -41, -122, -61, -182, -91, -272, -136, -68, -34) — in the negative integers domain; *and*
- ► 1 *length*-3 cycle — (4, 2, 1) — with *all-positive-integers* terms.

The same simple argument for the *preferred* Collatz 3x+1 sequences in section IV above readily justifies the conclusion that every *standard* Collatz sequence in the domain of positive integers has one of the following forms:

$C_4 = <(4, 2, 1)>$;

$C_1 = <1, (4, 2, 1)> = <1, C_4>$;

$C_2 = <2, 1, (4, 2, 1)> = <2, 1, C_4>$;

$C_n = <n, m, \ldots, 16, 8, (4, 2, 1)> = <n, m, \ldots, 8, C_4)>; \forall n \in \mathbf{N}^+ - \{1, 2, 4\}$.

Example 2. The *original* Collatz sequences **[5, 64]** are defined for positive integers:

$$f(n)=\begin{cases}\dfrac{2n}{3} & \text{if} \quad n\equiv_3 0\\[2ex] \dfrac{4n-1}{3} & \text{if} \quad n\equiv_3 1\\[2ex] \dfrac{4n+1}{3} & \text{if} \quad n\equiv_3 2\end{cases} \quad ; \quad f(-n)=\begin{cases}-\dfrac{2n}{3} & \text{if} \quad n\equiv_3 0\\[2ex] -\dfrac{4n-1}{3} & \text{if} \quad n\equiv_3 1\\[2ex] -\dfrac{4n+1}{3} & \text{if} \quad n\equiv_3 2\end{cases}$$

It is easily verified that any *original* Collatz sequence C_n, $\forall n \in \mathbf{Z}$, has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates and that they are symmetric with respect to the positive integers and negative integers domain.

In the domain of all integers, it is known that there are:

- ► 3 trivial *length*-1 cycles — (0), (1), and (-1);
- ► 2 *length*-2 cycles — (2, 3) and (-2, -3);
- ► 2 *length*-5 cycles — (4, 5, 7, 9, 6) and (-4, -5, -7, -9, -6); *and*
- ► 2 *length*-12 cycles — (44, 59, 79, 105, 70, 93, 62, 83, 111, 74, 99, 66) and (-44, -59, -79, -105, -70, -93, -62, -83, -111, -74, -99, -66).

Without some computable generalization that would rule out the prospect of valid *length*->-12 cycles, the complete determination of all the valid *cycle-lengths* of the *original* Collatz sequences is truly a *computationally exponentially complex* process and is an *undecidable* problem. Specifically, whether or not

C_8 = <8, 11, 15, 10, 13, 17, 23, 31, 41, 55, 73, 97, ...>

eventually converges to some periodic subsequence would remain to be actually an *undecidable* problem until somebody establishes it otherwise — which is very unlikely since the probability of its being a divergent sequence is higher because there are 2 *sub-functions* that increases, and only 1 *sub-function* that decreases, *successor-term* values.

Example 3. The following *Collatz 3x+1-type sequences* **[5]** are defined for all integers:

$$f(n) = \begin{cases} \dfrac{3n}{2} & \text{if} \quad n \equiv_2 0 \\ \\ \dfrac{3n+1}{2} & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* terms. There are 2 trivial *length*-1 cycles — (0) and (-1). Because each *sub-function* is either monotonic increasing (*for positive integer starting numbers*) or monotonic decreasing (*for negative integer less than* -1 *starting numbers*), then there are no cycles or cyclic subsequences with lengths greater than 1 — that is, each *nontrivial-length*-1 sequence is indeed divergent.

Example 4. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 5 cycles with starting numbers 0, 1, 13, 17, and -1:

$$f(n) = \begin{cases} \alpha(n) = \dfrac{n}{2} & \text{if} \quad n \equiv_2 0 \\ \\ \beta(n) = \dfrac{5n+1}{2} & \text{if} \quad n \equiv_2 1 \end{cases} .$$

It is easily verified that any sequence C_n has either *all*-0 or *all-positive-integers* or *all-negative-integers* iterates. In the *all-integers* domain, these are known:

- ► 1 trivial *length*-1 cycle — (0);
- ► 1 *length*-5 cycle — (1, 3, 8, 4, 2); as well as 2 *length*-7 cycles — (13, 33, 83, 208, 104, 52, 26) and (17, 43, 108, 54, 27, 68, 34) — in the positive integers domain; *and*
- ► 1 *length*-2 cycle — (-2, -1) — in the negative integers domain.

Following the same basic analysis as with the *Collatz 3x+1 sequences*:

- ► The trivial *length*-1 cycle (0) [*that is, for* $u = 0$] as well as the *length*-2 cycle (-2, -1) [*that is, for* $u = 1$] are easily computed.
- ► For $u > 1$, each of the $2^u - 2$ *nontrivial-length*-1 *nodes-expressions* of the *branch-point* $\boldsymbol{f}^{-u}(b)$ has the form

  $$\frac{2^u b - S}{5^v} \quad \text{where } 0 < v \le u, \quad S = \sum (2^p \cdot 5^q) > 0 \text{ with } p, q \in \mathbf{N}.$$

  The *solution-values* for $b_{\min}$ are obtained from either

  $$b = \frac{2S}{2^{u+1} - 5^v} \quad [1] \qquad or \qquad b = \frac{2S + 5^v}{2^{u+1} - 5^{v+1}} \quad [2].$$

  If the domain is restricted only to the negative integers, then there will be a contradiction that equation [2] should yield $b_{\min} < 0$ (*that is, with greater absolute value*) and the fact that it should be applied to odd integers only. Thus, there are no valid cycles other than (-2, -1) in the negative integers domain — hence, every *all-negative-integers-terms* sequence must include (-2, -1) as a *cyclic-subsequence*.

  The possibilities exist in the positive integers domain for sequences having loops with lengths greater than 7 (*including divergent ones*).

Example 5. The following *Collatz 3x+1-type sequences* **[1, 24]** (*defined for all integers*) are claimed to have 3 cycles with starting numbers 0, -1, and -2:

$$f(n) = \begin{cases} \alpha(n) = 2n & \text{if} \quad n \equiv_3 0 \\[2ex] \beta(n) = \dfrac{7n+2}{3} & \text{if} \quad n \equiv_3 1 \\[2ex] \gamma(n) = \dfrac{n-2}{3} & \text{if} \quad n \equiv_3 2 \end{cases} .$$

As Example 2.7 in **[24]**, *Keith Matthews* offers a prize of $100.00 (*Australian*) [*also advertised in* **[1]** *but with a different* $\alpha(n) = 7n + 3$] for a valid proof of his (*not altogether correct*) conjecture that *all divergent trajectories starting in the congruence classes* $n \equiv_3 \pm 1$ *appear to eventually enter the zero residue class mod* 3 and *if* $f^k(n) \equiv_3 \pm 1$, $\forall k \geq 0$*, then the trajectory must eventually enter one of the cycles* (-1) *or* (-2, -4).

It is easily verified that any sequence C_n has either *all-0* or *all-negative-integers* or *all-positive-integers* terms except for $n = \gamma^{-u}(2)$, $\forall u \in \mathbf{N}^+$. There are 2 trivial *length*-1 cycles — (0) and (-1) — but there are no sequences having either <(0)> or <(-1)> as cyclic subsequence.

If $n \in \mathbf{Z} - \{0\}$ and $n \equiv_3 0$, or $n = 3p$ for some $p \in \mathbf{Z} - \{0\}$, then

$$f(n) = \alpha(n) = 2n = 2 \bullet 3p \equiv_3 0$$

and, consequently,

$$C_{3p} = <3p, 2 \bullet 3p, 2^2 \bullet 3p, 2^3 \bullet 3p, \ldots> \text{ (which are divergent sequences).}$$

All the *nontrivial-length-*1*-branch-node-expressions* for *length*-2 *cycle-equations* $f^{-1}(b) = f(b)$ yield only valid *solution-values* for the sole *length*-2 cycle (-4, -2) with -4 as the *portal-cycle-term*. The possibilities for cycles with lengths greater than 2 in the negative integers domain could not be ruled out — so, only for many starting negative integers $-n \notin \{-1, -2, -4\}$ could we claim C_{-n} to converge to C_{-4}:

$$C_{-n} = <-n, -m, -l, \ldots, -f, -e, -10, (-4, -2)>$$

$$= <-n, -m, -l, \ldots, -f, -e, -10, C_{-4}> \text{ where all of } n, m, l, \ldots, f, e \equiv_3 \pm 1.$$

Each *all-negative-integers* divergent sequence has one of the form:

$$C_{-3p} = <-3p, -2 \bullet 3p, -2^2 \bullet 3p, -2^3 \bullet 3p, \ldots>, \forall p \in \mathbf{N}^+ \text{ or}$$

$$C_{-n} = <-n, -m, -l, \ldots, -d, -c, -b, -2b, 2^2 b, -2^3 b, \ldots>,$$

where $n \in \mathbf{N}^+ - \{1, 2, 4\}$ and all of $n, m, l, \ldots, d, c \equiv_3 \pm 1$ while $b \equiv_3 0$.

In the positive integers domain, we first note the following:

$C_2 = <2, C_0> = <2, 0, 0, 0, \ldots>$ —

so, $C_{\gamma^{-u}(2)} = <\gamma^{-u}(2), \gamma^{-u+1}(2), \gamma^{-u+2}(2), \ldots, \gamma^{-2}(2), \gamma^{-1}(2), C_2>, \forall u \in \mathbf{N^+}$;

$C_{3p} = <3p, 2 \bullet 3p, 2^2 \bullet 3p, 2^3 \bullet 3p, \ldots>, \forall p \in \mathbf{N^+}$;

$C_{3p+2} = <3p + 2, C_p>, \forall p \in \mathbf{N^+}$.

For arbitrarily large $n \in \mathbf{N^+}$, it could be established (*by directly going through each starting natural number from* 1 *to* n) that C_n converges to C_0 or diverges with C_{3z} for some $z \in \mathbf{N^+}$. Since $p < 3p+2$, the convergence of C_{3p+2} to C_0 or divergence of C_{3p+2} with C_{3z} immediately follows from the already known convergence of C_p to C_0 or divergence of C_p with C_{3z}.

For the case $n \in \mathbf{N^+}$, $n \equiv_3 1$ or $n = 3p+1$ for some $p \in \mathbf{N}$, it could be rigorously shown that it is impossible for C_{3p+1} not to eventually have a subsequence C_{3z} or C_{3z+2} for some $z \in \mathbf{N^+}$ — that is, there is no starting positive integer $n \equiv_3 1$ such that $f^u(n) \equiv_3 1, \forall u \in \mathbf{N^+}$ simply because:

$C_{3p+1} = \left\langle 3p+1,\ 7p+3,\ \dfrac{7^2 p + 7\cdot 3 + 2}{3}, \ldots \right\rangle$ implies that $p = 3q+1$ with $q < p$,

which upon substituting $3q+1$ for p,

$$C_{3^2 q+2^2} = \left\langle 3^2 q + 2^2,\ 7\cdot 3q + 2\cdot 5,\ 7^2 q + 2^3\cdot 3,\ \frac{7^3 q + 2^3\cdot 3\cdot 7 + 2}{3}, \ldots \right\rangle$$

implies that $q = 3r+1$ with $r < q$; . . . which upon substituting $3r+1$ for q . . . implies that $r = 3s+1$ with $s < r$; . . . which upon substituting $3s+1$ for r . . . implies that $s = 3t+1$ with $t < s$; . . . and so on *ad infinitum*, which is impossible by the *principle of no infinite descent* or by the *well-ordering principle of the natural numbers* (*that is, there is a smallest positive natural number* 1).

Example 6. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 5 cycles with starting numbers 0, 5, 17, -1, and -4:

$$f(n) = \begin{cases} \dfrac{n-3}{3} & \text{if} \quad n \equiv_3 0 \\ \dfrac{n+5}{3} & \text{if} \quad n \equiv_3 1 \\ 10n-5 & \text{if} \quad n \equiv_3 2 \end{cases}.$$

Example 7. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 7 cycles or cyclic subsequences with starting numbers 0, 2 (*length* 7161), 47, -2, -10, -22, and -265138:

$$f(n) = \begin{cases} \dfrac{n}{3} & \text{if} \quad n \equiv_3 0 \\ \dfrac{2n-2}{3} & \text{if} \quad n \equiv_3 1 \\ \dfrac{13n-2}{3} & \text{if} \quad n \equiv_3 2 \end{cases}.$$

Example 8. The following *Collatz 3x+1-type sequences* **[24, 26]** (*defined for all integers*) are claimed to have 3 cycles with starting numbers -1, -8, and 421:

$$f(n) = \begin{cases} 3n-1 & \text{if} \quad n \equiv_3 0 \\ \dfrac{n-16}{3} & \text{if} \quad n \equiv_3 1 \\ \dfrac{-4n-7}{3} & \text{if} \quad n \equiv_3 2 \end{cases}.$$

Example 9. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 7 cycles with starting numbers 0, 1, 7, 10, 514, -2, -749.

$$f(n) = \begin{cases} \dfrac{n}{4} & \text{if} \quad n \equiv_4 0 \\ \dfrac{3n+1}{4} & \text{if} \quad n \equiv_4 1 \\ \dfrac{5n+2}{4} & \text{if} \quad n \equiv_4 2 \\ \dfrac{15n+7}{4} & \text{if} \quad n \equiv_4 3 \end{cases}.$$

Example 10. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have 17 cycles or cyclic subsequences with starting numbers 0, 2, 3, 6 (*length* 1747), 5127, -3, -18, -46, -117, -122, -137, -186, -261, -330, -333, -513 (*length* 1426), and -5205:

$$f(n) = \begin{cases} \dfrac{n}{4} & \text{if} \quad n \equiv_4 0 \\ \dfrac{3n-3}{4} & \text{if} \quad n \equiv_4 1 \\ \dfrac{5n-2}{4} & \text{if} \quad n \equiv_4 2 \\ \dfrac{17n-3}{4} & \text{if} \quad n \equiv_4 3 \end{cases}.$$

Example 11. For this *Collatz 3x+1-type sequences* **[24]** (*defined for positive integers*), it is claimed that every trajectory starting with some even integer eventually cycles while most trajectories starting with odd integers are divergent:

$$f(n) = \begin{cases} \dfrac{3n}{2} & \text{if} \quad n \equiv_4 0 \\ \dfrac{n+1}{2} & \text{if} \quad n \equiv_4 1 \\ \dfrac{n+2}{2} & \text{if} \quad n \equiv_4 2 \\ \dfrac{7n+1}{2} & \text{if} \quad n \equiv_4 3 \end{cases}$$

Example 12. The following *Collatz 3x+1-type sequences* **[16, 24]** (*defined for all integers*) are claimed to have 2 cycles with starting numbers 2 and 6 (*length* 40):

$$f(n) = \begin{cases} \dfrac{2500n+6}{6} & \text{if} \quad n \equiv_6 0 \\ \dfrac{21n-9}{6} & \text{if} \quad n \equiv_6 1 \\ \dfrac{n+16}{6} & \text{if} \quad n \equiv_6 2 \\ \dfrac{21n-51}{6} & \text{if} \quad n \equiv_6 3 \\ \dfrac{21n-72}{6} & \text{if} \quad n \equiv_6 4 \\ \dfrac{n+13}{6} & \text{if} \quad n \equiv_6 5 \end{cases}.$$

Example 13. The following *Collatz 3x+1-type sequences* **[16, 24]** (*defined for all integers*) are claimed to have 9 cycles or *cyclic-subsequences* with starting numbers 0, 2, 4, 2454 (*length* 41), -6, -16, -22, -32, and -78:

$$f(n) = \begin{cases} \dfrac{n}{3} & \text{if} \quad n \equiv_6 0 \\ \dfrac{2n-2}{3} & \text{if} \quad n \equiv_6 1 \\ \dfrac{5n-4}{3} & \text{if} \quad n \equiv_6 2 \\ \dfrac{4n}{3} & \text{if} \quad n \equiv_6 3 \\ \dfrac{5n-8}{3} & \text{if} \quad n \equiv_6 4 \\ \dfrac{4n-2}{3} & \text{if} \quad n \equiv_6 5 \end{cases}.$$

Example 14. The following *Collatz 3x+1-type sequences* **[24, 26]** (*defined for all integers*) are claimed to have 13 cycles or *cyclic-subsequences* with starting numbers 0, 1, 10, 13, 61, 158, 205, 3292, 4244, -2, -11, -12, and -18:

$$f(n) = \begin{cases} \dfrac{n}{4} & \text{if} \quad n \equiv_8 0 \\ \dfrac{n+1}{2} & \text{if} \quad n \equiv_8 1 \\ 20n - 40 & \text{if} \quad n \equiv_8 2 \\ \dfrac{n-3}{8} & \text{if} \quad n \equiv_8 3 \\ 20n + 48 & \text{if} \quad n \equiv_8 4 \\ \dfrac{3n-13}{2} & \text{if} \quad n \equiv_8 5 \\ \dfrac{11n-2}{4} & \text{if} \quad n \equiv_8 6 \\ \dfrac{n+1}{8} & \text{if} \quad n \equiv_8 7 \end{cases}.$$

Example 15. The following *Collatz 3x+1-type sequences* **[24]** (*defined for all integers*) are claimed to have no cycles or *cyclic-subsequences*:

$$f(n) = \begin{cases} \dfrac{3n+2}{2} & \text{if} \quad n \equiv_4 0 \\ \dfrac{n-1}{2} & \text{if} \quad n \equiv_4 1 \\ \dfrac{n}{2} & \text{if} \quad n \equiv_4 2 \\ \dfrac{7n-1}{2} & \text{if} \quad n \equiv_4 3 \end{cases}.$$

VIII. *Conclusion*

The truly successful applicability of our very simple and general approach to decide many *Collatz 3x+1-type problems* guarantees its own effective tenability as a *solution-method*. When viewed from its beginning *non-periodic* iterates, all the *Collatz 3x+1 sequences* in the positive integers domain defied every attempt of *computable generalization* — hence, the unjustified clamors of "*unsolvability*" of the very simply stated *Collatz 3x+1 conjecture*. However, a straightforward look at the ending periodic terms provides a *clear-cut* *computable generalization* that readily rules out the *existential-possibility* of an *all-positive-integer-terms* *Collatz 3x+1 sequence* with *cyclic-subsequence* other than C_2 = <(2, 1)> or which does not converge to C_2 — hence, quickly proving the *Collatz 3x+1 conjecture*. Moreover, by transforming the *branching* *Collatz 3x+1 iteration function* into its equivalent streamlined *non-branching* iteration function in the domain of all odd natural numbers, an equally tenable *constructive-inductive solution-approach* is found.

As explained in details in the *Appendix* — "*The Collatz 3x+1 Syndrome and Flimflams in the Foundations of Mathematics*" **[77]** — it is purely the lack of a known *appropriate computable generalization* that exposes a *non-self-contradictory* mathematical problem to misbranding as "*computationally complex*" or "*undecidable*" or "*unsolvable*". In reality, the *computability* concern and the "*computational complexity classes*" categorization do not properly apply to the mathematical problem but distinctly to each of its diverse proposed *solution-methods* in suitable domains — that is, the *Collatz 3x+1 problem* is easily solvable (*by the very simple and general solution-technique that we have demonstrated in this manuscript*) despite its having numerous unsuccessful proposed *solution-approaches* or that the *brute-force solution-method* of evaluating individually its 2^{u+1} ($u\to\infty$) *cycle-equations* to find all of its valid *cyclic-subsequences* and rule out "*divergent*" ones is *exponentially computationally complex* (*therefore, we have also affirmed that computational complexity classifications of mathematical problems are unsound ab initio*).

The pervasive ramifications of the presence of a simple proof of the *Collatz 3x+1 conjecture* — in relation to, say, the *incomplete totality of the non-computably generalizable collection of all the prime natural numbers* (*the domain of the most popular prevailing unsolvable problems in number theory*), the solvability of *Hilbert*'s *entscheidungsproblem* ("*decision problem*"), the equality of the **P** and **NP** "*computational complexity classes*", etc. — to the resolution of the prevalent *modern crisis in the foundations of mathematics* should not be overlooked.

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