

Title: **Collatz $3x+1$ Conjecture Proved!**
(*Research Announcement*)

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Abstract

The *Collatz $3x+1$ conjecture*
in the positive integers domain
is proved by establishing that
any preferred Collatz $3x+1$ sequence
in the domain of all integers
with cyclic subsequence other than $\langle(2, 1)\rangle$,
or which has no cyclic subsequence,
must have *all-zero* or *all-negative-integer* iterates.

Collatz 3x+1 Conjecture Proved!

The most commonly cited orientation manuscript for the *Collatz 3x+1 conjecture* is [1] — *Jeffrey Lagarias* declares:

"The $3x+1$ problem, also known as the *Collatz problem*, the *Syracuse problem*, *Kakutani's problem*, *Hasse's algorithm*, and *Ulam's problem* concerns the behavior of the iterates of the function which takes odd integers n to $3n+1$ and even integers n to $\frac{n}{2}$. The $3x+1$ Conjecture asserts that, starting from any positive integer n , repeated iteration of this function eventually produces the value 1. . . . The $3x+1$ Conjecture is simple to state and apparently intractably hard to solve. . . .

The known results on the $3x+1$ problem are most elegantly expressed in terms of iterations of the function

$$T(n) = \begin{cases} \frac{3n+1}{2} & n \equiv 1 \pmod{2} \\ \frac{n}{2} & n \equiv 0 \pmod{2} \end{cases} .$$

. . . We call the sequence of iterates $\langle n, T(n), T^{(2)}(n), T^{(3)}(n), \dots \rangle$ the *trajectory* of n .

There are three possible behaviors for such trajectories when $n > 0$.

- (i) *Convergent trajectory*. Some $T^{(k)}(n) = 1$;
- (ii) *Non-trivial cyclic trajectory*. The sequence $T^{(k)}(n)$ eventually becomes periodic and $T^{(k)}(n) \neq 1$ for any $k \geq 1$;
- (iii) *Divergent trajectory*. $\lim_{k \rightarrow \infty} T^{(k)}(n) = \infty$.

The $3x + 1$ Conjecture asserts that all trajectories of positive n are convergent.

The difficulty of settling the $3x+1$ *problem* seems connected to the fact that it is a deterministic process that simulates '*random*' behavior. We face this dilemma: On the one hand, to the extent that the problem has structure, we can analyze it — yet, it is precisely this structure that seems to prevent us from proving that it behaves '*randomly*'. On the other hand, to the extent that the problem is *structureless* and '*random*', we have nothing to analyze and, consequently, cannot rigorously prove anything. **Of course, there remains the possibility that someone will find some hidden regularity in the $3x+1$ problem that allows some of the conjectures about it to be settled.** The existing general methods in *number theory* and *ergodic theory* do not seem to touch the $3x+1$ *problem*; in this sense, it seems intractable at present."

In [2], *Marc Chamberland* exclaims:

"The $3x+1$ *problem* is perhaps today's most enigmatic unsolved mathematical problem — it can be explained to a child who has learned how to divide by 2 and multiply by 3; yet, there are relatively few strong results toward solving it. *Paul Erdős* was correct when he stated: '*Mathematics is not ready for such problems*' [1, 3]. . . .

The map T is usually favored in the literature. . . . The structure of the positive integers forces any orbit of T to iterate to one of the following:

1. *the trivial cycle* (1; 2);
2. *a non-trivial cycle*;
3. *infinity* (the orbit is divergent).

The $3x + 1$ *Problem* claims that option 1 occurs in all cases. . . ."

Currently, the most authoritative *up-to-date bibliographical references* to publications about the *Collatz $3x+1$ problem* are [4, 5] — *Jeffrey Lagarias* declares that (*as of* 15 April 2007):

"*At present, the $3x+1$ conjecture remains unsolved.*"

A comprehensive reading in the mathematical literature of the many proposed *solution-methods* to prove the **Collatz 3x+1 conjecture** reveals that each of them draws their respective unsuccessful conclusion from some failed attempt to *computably generalize* the claimed identical behavior of the iterates of a **Collatz 3x+1 sequence** — $C_n = \langle n, T(n), T^{(2)}(n), T^{(3)}(n), \dots \rangle$ — to eventually "converge to 1" [an unfortunately misleading contention in the **Lagarias** quote but which is aptly rectified with emphasis on the ending **cyclic subsequence** $\langle (2,1) \rangle$ in the **Chamberland** quote] for all sequences with an infinite count of distinct positive integer starting number n .

A very simple and general *solution-approach* for deciding many **Collatz 3x+1-type problems** — that is, determining all of the valid *ending cyclic subsequences* (for all lengths $u \in \mathbf{N}^+$) of integer sequences recursively defined by some *branching function* — is presented in [6]. This effective *solution-method* — which takes into consideration the *cycle-terms* of an ending cyclic subsequence (if any) rather than the beginning *non-periodic* iterates of a representative **Collatz 3x+1 sequence** — is succinctly applied in this paper to decide the *preferred Collatz 3x+1 problem* in the domain \mathbf{N}^+ of all positive integers that involves the *2-branch* iteration function defined by:

$$T(n) = \begin{cases} \alpha(n) = \frac{n}{2} & \text{if } n \text{ is even} \\ \beta(n) = \frac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases} ; \quad T^{-1}(m) = \begin{cases} \alpha^{-1}(m) = 2m & \left(\begin{array}{l} \text{no pre-conditions} \Rightarrow \\ \alpha^{-1}(m) \text{ always an integer;} \\ \beta^{-1}(m) \text{ valid if an integer} \end{array} \right) \\ \beta^{-1}(m) = \frac{2m-1}{3} & \end{cases}$$

- ▶ Every *preferred Collatz 3x+1 sequence* involves some arbitrary *branch (or path or trajectory or orbit)* that takes the form of some infinite sequence

$$C_n = \langle n, T(n), T^{(2)}(n), T^{(3)}(n), \dots, T^{(k-1)}(n), T^{(k)}(n), T^{(k+1)}(n), \dots \rangle$$

of *branch-points* $T^{(k)}(n)$ [$\forall k \in \mathbf{N}$ with $T^{(0)}(n) = n$] where the starting number $n \in \mathbf{N}^+$.

- ▶ Each *preferred Collatz 3x+1 sequence* C_n [$\forall n \in \mathbf{N}^+$] has *all-positive-integer* iterates:

Let x be any positive integer. Then, $T(x)$ — either $\alpha(x) = \frac{x}{2}$ (for x even) or

$$\beta(x) = \frac{3x+1}{2} \text{ (for } x \text{ odd) — is also a positive integer.}$$

- If all the iterates of C_n are distinct then C_n is **divergent**; otherwise, C_n is **fully cyclic** [with cycle-length $k+1$] if and only if $T^{(k+1)}(n) = n$ for some $k \in \mathbf{N}$ while C_n has some **cyclic subsequence** $C_m = \langle T^{(k-h)}(n), T^{(k-h+1)}(n), T^{(k-h+2)}(n), \dots, T^{(k-1)}(n), T^{(k)}(n) \rangle$ [with cycle-length $h+1$] if and only if the **cycle-equation** $T^{(k+1)}(n) = T^{(k-h)}(n)$ holds for some $h, k \in \mathbf{N}^+$ with $h < k$ — in other words, there exists some **first-duplicated term** $T^{(k+1)}(n) = T^{(k-h)}(n)$ [with $0 \leq h \leq k$] in sequence C_n and the group of distinct iterates $\langle T^{(k-h)}(n), T^{(k-h+1)}(n), T^{(k-h+2)}(n), \dots, T^{(k-1)}(n), T^{(k)}(n) \rangle$ [in that fixed successor-term order] repeats *ad infinitum* [C_n is then said to be **convergent**].

For arbitrary convergent sequence C_n , let $b = T^{(k)}(n)$ — that is, b is the immediate *predecessor-cycle-term* of the *first-duplicated term* $T^{(k+1)}(n)$. Then, we could simply express the *cycle-equations* $T^{(k+1)}(n) = T^{(k-h)}(n)$ as $T(b) = T^{(-h)}(b)$, and C_n 's *cycle-terms* as $(T^{(-h)}(b), T^{(-h+1)}(b), T^{(-h+2)}(b), \dots, T^{(-1)}(b), b)$, in terms of b and the inverse **Collatz $3x+1$ iteration function** T^{-1} instead of n and T .

- To fully decide the *preferred Collatz $3x+1$ problem*, it is necessary to find all its valid *solution-cycles* $(T^{(-h)}(b), T^{(-h+1)}(b), T^{(-h+2)}(b), \dots, T^{(-1)}(b), b)$ — by solving *one-by-one* for b the *cycle-equations* $T^{(-h)}(b) = T(b)$ — for each *cycle-length* $h \in \mathbf{N}^+$. Since there are two possible immediate *predecessor-terms* $\alpha^{-1}(T^{(-h)}(b))$ and $\beta^{-1}(T^{(-h)}(b))$ for every *branch-point* $T^{(-h)}(b)$ [$\forall h \in \mathbf{N}$], each of the latter has 2^h **branch-point-nodes** that are represented by their respective expressions involving combination of "powers" of the *sub-functions* α^{-1} and β^{-1} of T^{-1} . Because b could be an odd or an even integer, then there are 2^{h+1} *cycle-equations* $T^{(-h)}(b) = T(b)$ for each *branch-point* $T^{(-h)}(b)$ [$\forall h \in \mathbf{N}^+$]:

$$T^{(-h)}(b) = T(b) = \begin{cases} \alpha(b) = \frac{b}{2} & \text{if } b \text{ is even} & [1] \\ \beta(b) = \frac{3b+1}{2} & \text{if } b \text{ is odd} & [2] \end{cases}$$

It is stressed that only 1 of the 2 equations [1] and [2] would yield a valid *solution-value* for b — however, both equations still need to be computed to determine which one yields the valid *solution-value* for b . Hence, this proposed *solution-approach* to the *preferred Collatz $3x+1$ problem* is truthfully **exponentially computationally complex**.

► The presupposed *solution-cycle branch-point-nodes* expressions are shown below:

$T(b)$	$T^{(2)}(b)$	\dots	$T^{(3)}(b)$	$T^{(2)}(b)$	$T^{(1)}(b)$	b
$\left\{ \begin{array}{l} \frac{b}{2} \\ \dots \\ \frac{3b+1}{2} \end{array} \right\}$	$\left\{ \begin{array}{l} \frac{b}{2^2} \\ \frac{3b+2}{2^2} \\ \dots \\ \frac{3b+1}{2^2} \\ \frac{3^2b+(3+2)}{2^2} \end{array} \right\}$	\dots	$2^3 b$	$\left\{ \begin{array}{l} 2^2 b \\ \dots \\ \frac{2^2 b - 1}{3} \end{array} \right\}$	$\left\{ \begin{array}{l} 2b \\ \dots \\ \frac{2b-1}{3} \end{array} \right\}$	$\left. \vphantom{\begin{array}{l} 2b \\ \dots \\ \frac{2b-1}{3} \end{array}} \right\}, b$
			$\frac{2^3 b - 1}{3}$			
			$\frac{2^3 b - 2}{3}$			
			$\frac{2^3 b - (2+3)}{3^2}$			
			$\frac{2^3 b - 2^2}{3}$			
			$\frac{2^3 b - (2^2 + 3)}{3^2}$			
			$\frac{2^3 b - (2^2 + 2 \cdot 3)}{3^2}$			
			$\frac{2^3 b - (2^2 + 2 \cdot 3 + 3^2)}{3^3}$			

• We first solve for all valid trivial *length-1 solution-cycles* (b) — that is, $T(b) = b$:

- ◆ $\alpha(b) = \frac{b}{2} = b$ yields $b = 0$ — the *solution-cycle* (0) = $\langle 0, 0, 0, \dots \rangle$;
- ◆ $\beta(b) = \frac{3b+1}{2} = b$ yields $b = -1$ — the *solution-cycle* (-1) = $\langle -1, -1, -1, \dots \rangle$.

For any *length-(h+1) cycle-equation* $T^{(h)}(b) = T(b)$, the *branch-point-node* equation $\alpha^{(h)}(b) = \alpha(b)$ [this corresponds to the top row *branch-point-node* expression in each *right-column* above] always yield only the trivial *length-1 solution-cycle* (0) while the *branch-point-node* equation $\beta^{(h)}(b) = \beta(b)$ [this corresponds to the bottom row *branch-point-node* expression in each *right-column* above] always yield the trivial *length-1 solution-cycle* (-1) — both are not valid in the domain of positive integers.

► For any valid *solution-cycle* $(T^{(-h)}(b), T^{(-h+1)}(b), T^{(-h+2)}(b), \dots, T^{(-1)}(b), b)$ [$\forall h \in \mathbf{N}^+$ or for all *cycle-lengths* > 1], the $h+1$ cyclic permutations of its *cycle-terms* demand that each *cycle-term* is a valid *solution-value* for some b of a corresponding *cycle-equation* $T^{(-h)}(b) = T(b)$. Thus, in addition to the 2 *branch-point-nodes* expressions that produce the 2 trivial *length-1 solution-cycles* (0) and (-1), there are also $h+1$ *branch-point-node* expressions whose own *cycle-equations* correspondingly yield the valid *solution-values* $b_1 = b, b_2 = T^{(-1)}(b), b_3 = T^{(-2)}(b), \dots, b_{h-1} = T^{(-h+1)}(b), b_h = T^{(-h+1)}(b), b_{h+1} = T^{(-h)}(b)$ — forming the *solution-cycle* $(b_{h+1}, b_h, b_{h-1}, \dots, b_3, b_2, b_1)$.

Hence, there are $h+1$ fully cyclic sequences C_i for $i \in \{b_1, b_2, b_3, \dots, b_{h-1}, b_h, b_{h+1}\}$ corresponding to the $h+1$ cyclic permutations of $(b_{h+1}, b_h, b_{h-1}, \dots, b_3, b_2, b_1)$ and there are also many sequences C_z that respectively include C_i where i is a *portal-cycle-term* [that is, a *cycle-term* b_j with at least one of its valid $T^{-1}(b_j) — \alpha^{-1}(b_j)$ or $\beta^{-1}(b_j)$ — values not in $\{b_{h+1}, b_h, b_{h-1}, \dots, b_3, b_2, b_1\}$ so that some other sequences could include the loop as a periodic subsequence through b_j]. This assertion is true for as many distinct $h \in \mathbf{N}$ that yields valid *solution-cycles* with *length*-($h+1$). Thus, all of the **Collatz $3x+1$ -type sequences** are partitioned into the "**equivalence classes**" of:

- 1) **fully cyclic sequences** — those formed by the cyclic permutations of the terms of each *solution-cycle* — their *first-duplicated* term is their own starting number;
- 2) **sequences with cyclic subsequences** — those formed from valid respective *non-cycle-predecessor-terms* of the *portal-cycle-terms* of each *solution-cycle*;
- 3) **divergent sequences**.

Case 2 sequences do not exist if every valid *solution-cycle* has no *portal-cycle-term* — in this scenario, there are only fully cyclic sequences as well as divergent sequences with starting numbers that are not *cycle-terms* of any of the *solution-cycles*.

There is no Case 3 sequence if there are only a finite count of Case 1 sequences and at least one of the latter has a *portal-cycle-term* — that is, each *non-fully-cyclic Collatz $3x+1$ sequence* with starting number not equal to any *cycle-term* of any *solution-cycle* includes exactly one of the cyclic subsequences with *portal-cycle-terms* [this reasoning immediately trivially follows from the very definition (the form) of the preferred Collatz $3x+1$ iteration function and its sequences].

- We next solve for all valid *length-2 solution-cycles* — that is, $T(b) = T^{(-1)}(b)$:

For the *branch-point-node* expression $T^{(-1)}(b) = 2b$ —

- ◆ $\alpha(b) = \frac{b}{2} = 2b$ yields $b = 0$ — this corresponds to (0);
- ◆ $\beta(b) = \frac{3b+1}{2} = 2b$ yields $b_1 = \frac{1}{2^2 - 3} = 1$.

For the *branch-point-node* expression $T^{(-1)}(b) = \frac{2b-1}{3}$ —

- ◆ $\alpha(b) = \frac{b}{2} = \frac{2b-1}{3}$ yields $b_2 = \frac{2}{2^2 - 3} = 2$;
- ◆ $\beta(b) = \frac{3b+1}{2} = \frac{2b-1}{3}$ yields $b = -1$ — this corresponds to (-1).

Thus, we have one valid *length-2 solution-cycle* (2, 1). Other than the *cycle-term 1*, the *cycle-term 2* also has another valid immediate *predecessor-term* $\alpha^{-1}(2) = 2(2) = 4$ which is not a *cycle-term*. Thus, 2 is the *portal-cycle-term* of the loop (2, 1) — that is, many sequences C_n with $n \notin \{1, 2\}$ includes $C_2 = \langle 2, 1 \rangle$ as a cyclic subsequence.

- By the very definition (*the form*) of the *preferred Collatz 3x+1 iteration function* and its sequences — $C_n = \langle n, T(n), T^{(2)}(n), T^{(3)}(n), \dots, T^{(-1)}(b), b, T(b), \dots \rangle$ with $T^{(k)}(n) = b$ for some $k \in \mathbf{N}^+$ — $T^{(-1)}(b) = T(b)$ always hold and yields the valid *solution-cycle* (2, 1). Hence, if we could categorically rule out other *all-positive-integer-terms solution-cycle* with length > 2 — that is, if we could establish that $T^{(-h)}(b) = T(b)$ [$\forall h \in \mathbf{N}^+, h > 2$] could not have valid *solution-value* for b in the positive integers domain then every C_n indeed includes C_2 as a cyclic subsequence.

Because the possible *cycle-length* $h \rightarrow \infty$, it is clear that we could not determine, by just our *brute-force solution-approach*, all of the valid *solution-cycles* of the *preferred Collatz 3x+1 sequences* in the positive integers domain. Fortunately, there is a very simple *computable generalization* of all the *cycle-equations* for *cycle-length* $h > 2$ that immediately absolutely rules out any valid *all-positive-integer-terms solution-cycle*.

- For any valid *solution-cycle* $(b_{h+1}, b_h, b_{h-1}, \dots, b_3, b_2, b_1)$ [$\forall h \in \mathbf{N}^+$ or for all *cycle-lengths* > 1], the count $h+1$ of *solution-cycle-terms* is finite so there is, among the *cycle-terms*, some *minimum-valued cycle-term*

$$b_{\min} = \text{MINIMUM } (b_{h+1}, b_h, b_{h-1}, \dots, b_3, b_2, b_1)$$

— which must be a positive odd integer in the positive integers domain in order for its successor *cycle-term* $T(b_{\min})$ to be greater than the *designated-minimum cycle-term* b_{\min} .

- As can be readily generalized from the presumed *solution-cycle branch-nodes expressions* depicted earlier, for the h -times [$h > 1$] applications of the inverse function T^{-1} (that is, either α^{-1} or β^{-1} as appropriate) on b_1 , each of the $2^h - 2$ *nontrivial-length-1-nodes-expressions* of the *branch-point* $T^h(b_1)$ has the form

$$\frac{2^h b_1 - S}{3^g} \quad \text{where} \quad 0 < g \leq h \quad \text{with} \quad h > 1;$$

$$S = \sum (2^p \cdot 3^q) > 0 \quad \text{with} \quad p, q \in \mathbf{N}.$$

Therefore, for the *solution-cycle-term* b_1 , the 2 *cycle-equations* $T^{(-h)}(b_1) = \alpha(b_1)$ and $T^{(-h)}(b_1) = \beta(b_1)$ [$\forall h > 1$ or for all *cycle-lengths* greater than 2] that have to be evaluated for valid *solution-values* for b_1 are, respectively —

- for *even-integer solution-cycle-term* b_1 :

$$\frac{2^h b_1 - S}{3^g} = \frac{b_1}{2} \quad \text{or} \quad b_1 = \frac{2S}{2^{h+1} - 3^g} \quad [1]$$

- for *odd-integer solution-cycle-term* b_1 :

$$\frac{2^h b_1 - S}{3^g} = \frac{3b_1 + 1}{2} \quad \text{or} \quad b_1 = \frac{2S + 3^g}{2^{h+1} - 3^{g+1}} \quad [2].$$

We emphasize that the numerator in the *right-hand-side* of equation [1] is less than the numerator in the *right-hand-side* of equation [2] while the denominator in the former is greater than the denominator in the latter — that is, equation [1] would indeed yield a smaller *solution-value* for b_1 (*in the specified domain*) than equation [2]. Let us consider each *solution-cycle-terms* list where their respective *minimum-valued cycle-term* b_{\min} ($\forall h > 1$ or for all cycle-lengths greater than 2) is the "last term" b_1 — that is, $(b_{h+1}, b_h, b_{h-1}, \dots, b_2, b_{\min})$.

It is stressed that the *left-hand-side branch-node* expression involving \mathbf{S} for b_{\min} are the same in both equations [1] and [2] (*for examples, please refer back to the evaluation of the lengths 1 and 2 cycle-equations earlier*). It is not known whether every *solution-cycle's* respective b_{\min} is odd or even, but by comparing equations [1] and [2], it could indeed be deduced that each b_{\min} must always be an even integer (*that is, b_{\min} is actually evaluated from equation [1]*); therefore, b_{\min} (*as well as each of the other respective solution-cycle-terms*) could not be a positive integer since, otherwise, $0 < \alpha(b_{\min}) = \frac{b_{\min}}{2} < b_{\min}$ — contradicting b_{\min} 's supposed minimality among its *solution-cycle-terms* in the domain of positive integers. Thus, any *preferred Collatz $3x+1$ sequence* which is fully cyclic, or that includes a cyclic subsequence, with length > 2 actually belongs to the domain of negative integers.

- Because there are no cycles other than (2, 1), and 2 is a valid *portal-cycle-term*, then every *Collatz $3x+1$ sequence* C_n in the domain of positive integers, must include C_2 as a periodic subsequence — hence, there is indeed no "*divergent*" *all-positive-integer-terms* sequence C_n .

The summarized arguments presented in this *research announcement* are discussed in expository details in [6], with additional supporting arguments and alternative proof for the nonexistence of any divergent *preferred Collatz $3x+1$ sequence* including refutations of plainly flawed possible objections and counterexamples to the fundamental logic of the *solution-approach* just presented.

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