

Ch04: Time value of money, aka compounding-discounting arithmetic

Nominal interest rates, as quoted by banks, etc., are always > 0 .

Thus, \$1 today will become $> \$1$ in the future.

Conversely, \$1 in the future is worth $< \$1$ today.

The process of finding the future value (FV) given the present value (PV), the annual interest rate (i), the # of years (n), and frequency at which the interest is calculated annually (m) is called *compounding*.

The process of finding the PV given the FV_n , the i, the n, and the m is called *discounting*.

Compounding, as suggested semantically, results in $FV > PV$.

Discounting, as suggested semantically, results in $PV < FV$.

The only fundamental equation that underlies the concept on time value of money is:

$$FV_n = PV * \left(1 + \frac{i}{m}\right)^{mn}$$

where $m = 1$ if interest is compounded annually;

$m=2$ if interest is compounded semiannually;

$m=4$ if interest is compounded quarterly;

$m=12$ if interest is compounded monthly;

$m=52$ if interest is compounded weekly, and;

$m=365$ if interest is compounded daily.

Subscript 'n' functions as an index, and has no arithmetic function. Superscript 'mn' functions as exponentiation.

Example 1: Today, you deposit \$125 into a savings account that pays 9% interest that is cpded monthly. What will be your account balance one year from today?

$$FV_n = 125 * (1 + .09/12)^{12*1} = 125 * 1.0938069 = \underline{136.73}$$

N=12; I/Y=.75; PV= -125; cpt FV

Example 2: Using the numbers from the previous example except now that the bank compounds the interest annually. What will be FV_1 ?

$$FV_1 = 125 * (1 + .09/1)^{1*1} = 125 * 1.09 = \underline{136.25}$$

$N=1$; $I/Y = 9$; $PV = -125$; cpt FV .

In the above e.g., the \$125 can be called *principal* instead of PV .

Of the \$136.25, \$125 is the principal, and the remaining \$11.25 is called the *compound interest*.

Compound interest is not to be confused with *simple interest*.

Simple interest is simply estimated as: **simple interest = $PV * i * n$** .

In U.S., we use compound interest by default and almost exclusively unless otherwise stipulated.

From examples 1 and 2 above, we see that increasing the compounding frequency results in _____ future value, *ceteris paribus*.

If m is not specifically specified, $m=1$ by default.

In the above e.g., the 1.09 is called future value interest factor, abbreviated $FVIF$.

$FVIF$ is double-subscripted. E.g., $FVIF_{9\%,1} = 1.09$.

$$FVIF_{0.75\%,12} = 1.0938069.$$

Thus, $FV_n = PV * FVIF_{i\%,n}$

$FVIF_{i\%,n}$ is always > 1.000 .

Example 3: (Discounting example)

A gambler-friend wants to borrow an amt from you today for an unknown cause. He insists in paying you 10% interest compounded semiannually. He proves to you that he'll have \$1k cash inflow in two years' time, and you believe him. How much are you willing to lend to him based on the \$1k future cash inflow?

$$1,000 = PV * (1 + .10/2)^{2*2}$$

$$PV = 1,000 * 1/(1.05)^4 = 1,000 * .8227024 = \underline{822.70}$$

$N=4$; $I/Y = 5$; $FV = -1000$; cpt PV

The .8227024 is called the present value interest factor, abbreviated PVIF, and it is double-subscripted.

Thus, $PV = FV_n * PVIF_{i\%,n}$

$PVIF_{i\%,n}$ is always < 1.000 .

$PVIF_{i\%,n}$ is the reciprocal of $FVIF_{i\%,n}$, i.e.,

$$PVIF_{i\%,n} = \frac{1}{FVIF_{i\%,n}}$$

They are reciprocal if and only if the corresponding subscripts are of the same values, otherwise they are not reciprocal to each other.

At times, you'll have to find i given FV_n , PV and n .

Other times, you'll have to find n given FV_n , PV and i .

Find n given FV_n , PV and i

Example: At 9% interest annually, how long does it take your principal to double in value?

$$FV_n = 2 * PV = PV * (1 + .09)^n$$

$$2 = 1.09^n$$

Solving for n , we have $\ln 2 = n * \ln 1.09$,

$$n = \ln 2 / \ln 1.09 = \underline{8.04 \text{ years}}$$

$I/Y = 9$; $PV = -1$; $FV = 2$; $\text{cpt } N$ (note that either PV or FV must be $-ve$)

(It doesn't matter if we've used \log instead of \ln as long as we are consistent in the choice. This is why I said in my course syllabus that I need you to have a calculator with the logarithmic key.)

As a result of the above example, let's discuss something highly pertinent called Rule 72.

Rule 72 \equiv a well-known rule-of-thumb used frequently to quickly estimate the # of years needed for an investment to double in value.

How does Rule 72 work?

Simply divide 72 by the annual interest rate expressed in percent. In the above e.g., $n = 72/9 = 8$ years.

Rule 72 can *only* be used in conjunction with doubling in principal. It is inapplicable for tripling or quadrupling.

Find i given FV_n , PV and n

Example: You're offered an investment that requires you to put up \$4k today in exchange for \$10k eight yrs from now. What is the annual rate of return on this investment?

$$10k = 4k * (1 + i)^8$$

$$10/4 = 2.5 = (1 + i)^8$$

$$(2.5)^{1/8} = 1 + i$$

$$(2.5)^{1/8} - 1 = i$$

$$1.1213534 - 1 = i$$

$$.1213534 = i$$

$$12.14\% = i$$

$N=8$; $PV = -4$; $FV = 10$; cpt I/Y.

Another example: Congress approved to pay \$25k to President George Washington in 1789. Congress paid President George W. Bush \$400k in 2001. What is the implicit annual inflation rate over the 212-year period?

$$400k = 25k (1 + i)^{212}$$

Solving for i , we get $i = 1.3164\%$ p.a.

$N=212$; $PV = -25$; $FV = 400$; cpt I/Y

Continuous compounding or discounting

When the compounding frequency becomes so rapid that $m \rightarrow \infty$, then the fundamental equation is replaced by:

$$FV = PV * e^{i*n} \text{ where } e \approx 2.718281828459045 \dots$$

Derivation of $FV_n = PV * e^{i*n}$

$dV/dt = kV$, rate of change V is proportional to existing level of V with k being the proportionality constant.

Rearrange, we get $dV/V = k*dt$

Now, integrate on both sides, we get

$\ln V = kt + C$, where C is a constant.

Assume initial condition of $t=0$, $V = V_0$.

$\ln V_0 = 0 + C$, or simply $C = \ln V_0$

$\ln V = kt + \ln V_0$

$\ln V - \ln V_0 = \ln (V/V_0) = kt$

$V/V_0 = e^{kt}$

$$V = V_0 * e^{kt}$$

The above equation is a growth equation when k takes positive value; a decay equation when k takes negative value.

The same equation is used to estimate exponential growth phenomena or nuclear decay.

In finance, $V \equiv FV$, $V_0 \equiv PV$, $k \equiv i$, and $FV_n = PV * e^{in}$ is exactly the same equation as $V = V_0 e^{kt}$ in physics or mathematics.

Example: You take a \$1k loan from a bank that cpds interest continuously at 11%. How much will you owe in one year's time?

$$\begin{aligned} FV_1 &= 1,000 * e^{.11*1} \\ &= 1,000 * 1.1162781 = \underline{\$1,116.28} \end{aligned}$$

Keystrokes: 0.11 2nd e^x * 1000 =

Finding m given FV_n , PV , i and n is rather rare. However, cpding frequency does give rise to another complication as illustrated by the example below,

You want to borrow \$ X for one year. Bank A quotes you 9% interest cpded annually whereas Bank B quotes 9% interest cpded quarterly. In this case, the choice is transparent and easy: you prefer A to B because $FV_1(A) <$

$FV_1(B)$.

However, what if Bank B quotes 8.8% interest cpded quarterly and Bank A quotes 9% interest cpded annually?

$$FV_1(A) = X * (1.09) = 1.09X$$

$$FV_1(B) = X * (1 + .088/4)^{4*1} = X * 1.022^4 = 1.0909468X.$$

Therefore you'd choose A as a borrower and choose B as a lender.

The way to standardize the various cpding frequencies is to estimate the so called **effective annual rate**, abbreviated **EAR**.

$$EAR = (1 + \frac{i}{m})^m - 1$$

$$EAR_A = (1 + .09/1)^1 - 1 = .09 = 9\%$$

$$EAR_B = (1 + .088/4)^4 - 1 = .0909468 = 9.09\%$$

In general, **effective k-year rate** = $(1 + i/m)^{mk} - 1$. Notice how the word 'annual' has camouflaged the k=1 situation.

Example: What is the effective 2-year rate for a loan that charges 9% p.a. quarterly?

Effective 2-year rate = $(1 + .09/4)^{4*2} - 1 = \underline{19.483\%}$.

Annuity \equiv a series of *equal* cash flows for a *fixed # of yrs.*

Perpetuity (perpetual annuity) \equiv a series of *equal* cash flows occurring *forever.*

4-yr \$50 ordinary annuity versus 4-yr \$50 annuity due

4-year \$50 ordinary annuity	4-yr \$50 annuity due

For ordinary annuity (default mode), each CF occurs at the end-of-year.
 For annuity due, each CF occurs at the beginning-of-year.

Given $i = 12\%$, find PV and FV_4 for the ordinary annuity.

$$\begin{aligned}
 PV &= 50/1.12^1 + 50/1.12^2 + 50/1.12^3 + 50/1.12^4 \\
 &= 44.64 + 39.86 + 35.59 + 31.76 \\
 &= \underline{151.87}
 \end{aligned}$$

$N = 4$; $I/Y = 12$; $PMT = 50$; **cpt PV**

Short-cut formula: (show derivation on chalk/marker board)

$$PV = A * \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right] = A * PVIFA_{i\%,n}$$

To obtain $PVIFA_{i\%,n}$ in the financial calculator:

$N=n$, $I/Y=i\%$, $PMT = -1$, **cpt PV.**

Thus, $PV = 50 * [(1 - 1/1.12^4)/.12] = 50 * 3.0373493 = \underline{151.87}$.

The 3.0373493 is called the present value interest factor annuity, abbreviated PVIFA. PVIFA is double-subscripted as $PVIFA_{i\%,n}$.

If interest rate were 0%, then $PVIFA_{0\%,4} = 4$, and $PV = 50 * 4 = 200$.

Since $i > 0\%$, then $PVIFA_{i\%,n} < n$ is a good checking condition.

$$FV_4 = 50(1.12)^3 + 50(1.12)^2 + 50(1.12)^1 + 50(1.12)^0$$

$$= 70.25 + 62.72 + 56 + 50 = \underline{238.97}$$

$N=4; I/Y=12; PMT=50; \text{cpt } FV$

Short-cut formula: (show derivation on chalk/marker board)

$$FV_n = A * \left[\frac{(1+i)^n - 1}{i} \right] = A * FVIFA_{i\%,n}$$

To obtain $FVIFA_{i\%,n}$ in the financial calculator:

$N=n; I/Y=i\%; PMT = -1; \text{cpt } FV$.

$$FV_4 = 50 * [(1.12^4 - 1)/.12] = 50 * 4.779328 = \underline{238.97}$$

The 4.779328 is called the future value interest factor annuity, abbreviated FVIFA, and double-subscripted $FVIFA_{i\%,n}$.

If interest rate were 0%, $FVIFA_{0\%,4} = 4$, and $FV_4 = 50 * 4 = 200.00$

Since interest rate $> 0\%$, $FVIFA_{i\%,4} > 4.000$.

Hence, $FVIFA_{i\%,n} > n$ is a good checking condition.

Note: $FVIFA_{i\%,n}$ is *not* the reciprocal of $PVIFA_{i\%,n}$.

Find the PV and FV_4 for the annuity due for $i=12\%$.

$$PV = 50/1.12^0 + 50/1.12^1 + 50/1.12^2 + 50/1.12^3$$

$$= 50 + 44.64 + 39.86 + 35.59 = \underline{170.09}$$

Alternatively, using the short-cut formula:

$$PV = 50/1.12^0 + 3\text{-year ordinary annuity}$$

$$= 50 + 50[(1 - 1/1.12^3)/.12] = 50 + 50 * 2.4018313 = \underline{170.09}$$

$$FV_4 = 50(1.12)^4 + 50(1.12)^3 + 50(1.12)^2 + 50(1.12)^1$$

$$= 78.68 + 70.25 + 62.72 + 56 = 267.65$$

What do you realize between 170.09 and 151.87, and between 267.65 and 238.97?

Check: $170.09 = 151.87(1.12)$? and $267.65 = 238.97(1.12)$?

If the equality holds, then:

Annuity due = (Ordinary annuity)*(1+i)

Present value of a perpetuity: $PV(\text{perpetuity}) = \frac{A}{i}$

Future value of a perpetuity, $FV(\text{perpetuity}) = \infty$ theoretically.

Home mortgage problem:

You take a \$170k 30-year mortgage, and the bank charges 7¼% per annum.

Estimate your monthly mortgage payment?

PV = \$170k.

$i = 7.25\%$ per year, or $.0725/12 = .006041$ per month

$n = 30$ years, or 360 months.

So, $170,000 = A * [(1 - 1/1.006041^{360})/.006041]$

$170,000 = A * 146.58968$

$A = 170,000/146.58968 = \underline{1159.70}$.

$N=360$; $I/Y=7.25/12$; $PV=170000$; cpt PMT

Re-estimate the above problem assuming a 15-yr mortgage instead.

$A = 170,000/109.54548 = \underline{1,551.87}$.

$N=180$; cpt PMT

IRA Problem:

A 30 year-old individual sets aside \$2,250 per year into an IRA. She plans to retire at the age of 60. Assuming the IRA pays her 10.5% per year, and the investor makes end-of-year payment, estimate her IRA balance upon retirement.

$FV_{30} = 2,250 * [(1.105^{30} - 1)/.105] = 2250(180.88149) = \underline{406,983.36}$.

The 180.88149 is called the $FVIFA_{10.5\%,30}$.

Now, at the age of 60, the same individual expects to live for another 20 yrs. She plans to spread the \$406,983.36 evenly over the next two decades, and receiving the annual lump sum at the end-of-year. How much will she receive each yr if the IRA now returns 12% annually?

$406,983.36 = A [(1 - 1/1.12^{20})/.12]$

$406,983.36 = A [7.4694436]$

$A = \underline{\$54,486.44}$

Calculating EAR with Points (common in mortgages)

You are looking at a one-year loan of \$500. The interest rate is quoted as 12% plus two points. A point on a loan is simply 1% (one percentage point) of the loan amount. Quotes similar to this one are very common with home mortgages. The interest rate quotation in this example requires the borrower to pay two points to the lender up front and repay the loan later with 12% interest. What rate would you actually be paying here?

Solutions:

$$\text{Up front receipt} = (1 - .02) * 500 = 490.00$$

$$\text{Repayment one year later} = 500 * (1 + .12) = 560.00$$

$$560 = 490 (1 + i)$$

$$i = 560/490 - 1 = \underline{14.29\%}$$

Cash Discount

A discount is often given by the supplier to induce prompt payment by the buyer. For example, *2/10, net 30* is a commonly encountered cash discount quotation. What is the effective annual rate (EAR) for the quotation?

Solutions: 2/10, net 30 means take 2% off the invoiced amount if paid within 10 days, else the full amount is due in 30 days.

So, the buyer either pays 98 cents for every dollar on the 10th day or the full \$1 on the 30th day.

$$1 = .98 (1 + i)$$

$$1/.98 = 1.020408163 = (1 + i)^{0.054794521}$$

$$1.020408163^{1/.054794521} = 1 + i$$

$$1.4458529273 = 1 + i$$

$$i = .4458529273 \approx \underline{44.59\%}$$

Amortization schedule

For a 30-year fixed rate mortgage of \$170k at 6% per year, the amortization schedule follows:

$$6\% \text{ per year} = .5\% \text{ per month} = i$$

$$30 \text{ years} = 360 \text{ months} = n$$

$$PV = \$170k$$

$$\text{Hence, monthly payment, } A = \$1,019.24$$

t	Beginning balance	Monthly payment	Interest	Principal reduction	Ending balance
1	\$170k	\$1,019.24	\$850	\$169.24	\$169,830.76
2	169,830.76	1,019.24	849.15	170.09	169,660.67
3	169,660.67	1,019.24	848.30	170.94	169,489.73
:	:	:	:	:	:
:	:	:	:	:	:
360	1,014.17	1,019.24	5.07	1014.17	0

Show how to create the above table in Texas Instrument BAII Plus financial calculator using the 2nd Amort keys.

Gasoline-only versus hybrid car decision

A gas-only Honda civic sells for \$18k while a Honda civic hybrid sells for \$24k. Assume the gas-only car yields 30 mpg, and the hybrid 45 mpg. Assume annual mileage at 12k miles, and gas sells at 3.50 \$/gallon, and interest rate at 6% per annum. Justify the purchase of the hybrid civic.

Gas efficiency of the gas-only car = 11.67 cents/mile

Hybrid efficiency = 7.78 cents/mile

Savings per year = (11.67 – 7.78) cents/mile*12,000miles = 466.67 \$/yr

PV of addition cost of the hybrid = 24k – 18k = \$6k.

So, we get:

I/Y=6; PV = –6000, PMT = 466.67, cpt N = 25 yrs 4 mths (not very viable since the engineering life of the hybrid will not be that long, ☹)

What if the federal government allows the \$6k to be taken against tax deduction for someone who is in the 30% tax bracket?

Then, I/Y=6; PV = –4200; PMT=466.67, cpt N = 13yrs4mths

What if the government allows 55% tax write-off of the \$6k?

Then, I/Y=6; PV = –2700; PMT=466.67, cpt N = 7yrs4mth

1/2m now and 1/2m 3 decades later problem

You take a \$1/2m 30-year mortgage at 6% p.a. with a balloon payment of \$1/2m to be paid upon maturity with your retirement bonus which had been promised in writing to you. Estimate your monthly payment.

$$.5m = A(PVIFA_{360mths,.5\%}) + .5m/(1.005^{360})$$

Solving using BAII Plus, we get \$2,500 per month.

N=360; I/Y=0.5; PV=500000; FV= -500000; cpt PMT = -2500.

This problem resembles a bond issue such that the entire loan amount is repaid at maturity, and the \$2,500 per month amount is merely the interest payment used to service the principal of \$½m, i.e., 0.5%* ½ m = 2,500.

In summary, you have learned:

$$FV_n = PV * \left(1 + \frac{i}{m}\right)^{mn}$$

$$FV_n = PV * e^{i*n} \text{ where } e \approx 2.718281828459045 \dots$$

$$PV = A * \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right]$$

$$FV_n = A * \left[\frac{(1+i)^n - 1}{i} \right]$$

$$PV(\text{perpetuity}) = \frac{A}{i}$$

$$EAR = \left[1 + \frac{i}{m} \right]^m - 1$$

Rule 72: # of years for doubling = 72/i

Microsoft Excel syntaxes:

To find:	Syntax
Future value	=fv(rate,nper,pmt,pv,type)
Present value	=pv(rate,nper,pmt,fv,type)
Rate	=rate(nper,pmt,pv,fv,type)
# of periods	=nper(rate,pmt,pv,fv,type)
Annuity or payment	=pmt(rate,nper,pv,fv,type)

Once in the **Financial Function** mode, you can also **Insert, Function** to edit the input numbers.

Type is a variable that denotes the timing of cash flows in an annuity such that a **0** means the default *ordinary annuity* whereas a **1** means *annuity due*.

Type has no effect on non-annuity estimations.

The **pmt**, **pv**, and **fv** within the parentheses are **sign-sensitive** such that cash *outflow* must be entered as a negative value whereas cash inflow is to be entered as a default positive value.

Rate within parentheses can be entered as decimal format or percent form with the % sign. So, **.11** and **11%** are the same entry. Also, you can perform simple arithmetic operation when entering the rate. For example, 9% per year translates into 0.75% per month. Instead of performing 9%/12 outside in a calculator, one can simply enter **.09/12** as the monthly interest rate.

Example: Find the monthly payment for a 30-year mortgage for a principal value of 169,900 at 7.125% per year interest.

$$=PMT(0.07125/12,360,-169900,0,0)$$

Answer: \$1,144.65

