Irreversible Investment with Tax Policy Uncertainty

Chuck Coleman
NPA Data Services, Inc.
1424 16th St., N.W., Suite 700
Washington, DC 20036

and

Department of Mathematical Sciences
George Mason University, MSN 3F2
Fairfax, VA 22030

October 13, 1997

Abstract

The effects of tax policy uncertainty on irreversible investment in a continuous-time scenario are studied. A firm invests in a single constant-cost project whose revenues are generated by a geometric Brownian process and whose profits are taxed. The tax rate is subject to a fixed change at an unknown future time. This uncertainty is modelled by considering the project to be the sum of two projects: one taxed at the post-tax change rate and a project taxed at one less the tax rate change, subject to bankruptcy. The effects on investment are ambiguous, depending on a variety of parameters.

1 Introduction

Irreversible investment models rarely consider taxation and then that taxation is deterministic.[1] Aizenman and Marion [1] do a two-period model of stochastic taxation. Uncertain tax policy requires persistence, defined as a greater than 1/2 probability that the current policy will persist into the second period, to negatively affect investment value. Coleman [7] creates a model of endogenous tax reform, in which firms purchase stochastically expiring tax breaks from the government. This paper considers the problem of investment subject to a known tax change occurring at an unknown time. A firm consists of a single taxable project in continous time. The project's cost is constant and its returns (or, equivalently, output price) are generated by a geometric Brownian diffusion. Project value and optimal investment rules are determined

[1] For example, see Mackie-Mason [9] for a deterministic model of taxation.

when the project is taxed at a constant, deterministic rate. The bankruptcy process is then introduced. This process consists of a single jump to zero, remaining there forever. This process is assumed to be generated by a Poisson process of constant hazard. A firm subject to stochastic taxation is assumed to consist of two projects: a project taxed forever at the post-tax change rate and another project taxed at the difference of the two tax rates, subject to bankruptcy.

Section 2 sets up and solves the baseline model assuming no taxation and no bankruptcy. Section 3 solves for deterministic taxation. Section 4 introduces the bankruptcy process and solves the corresponding models. Section 5 does the same for stochastic taxes. Section 6 concludes this paper.

2 Base Model

The firm is assumed to possess an infinitely-lived investment opportunity which it may exercise at any time at a fixed cost. During each (infinitesimal) period, the firm, when operating incurs a constant cost c. Its revenues, P, follow a geometric Brownian process.[2] One way to think of this model is that the firm produces one unit of output at constant cost per period and receives a stochastic price for it. Here, price and revenue are equivalent. In general, P is assumed to be generated by the continuous-time process:[3]

$$dP = \alpha P\,dt + \sigma P\,dz + P \sum_i \gamma_i\,dq_i - P \sum_i \lambda_i\,\mathbf{E}(\gamma_i - 1)dt \tag{1}$$

where t is time, α is the expected growth rate of P, $\sigma\,dz$ is an increment of a Wiener process with variance σ^2, dq_i is a marked Poisson process with (possibly stochastic) increments γ_i, and $\mathbf{E}$ is the expectation operator. The three terms on the right form a martingale. That is, for $t > s$, $\mathbf{E}(P_t \mid \mathcal{I}_s) = P_s$, where $\mathcal{I}_s$ is the information available at time s and $P = P_s$ at time s.[4] When $\gamma_i \equiv 0$, then P is said to be generated by a geometric diffusion process:

$$dP = \alpha P\,dt + \sigma P\,dz \tag{2}$$

This assumption will be maintained from here through the next few Sections and dropped in Section 4.

By assuming that the capital asset pricing model (CAPM) holds,[5] we can interpret the trend component

[2] Geometric Brownian diffusion is not the only reasonable continuous process. For example, the process may be mean-reverting of the Ornstein-Uhlenbeck type: $dP = \eta(\bar{P} - P)P\,dt + \sigma P\,dz$, where dz is a standard Wiener process. See Dixit and Pindyck [8] pp. 161–167. Pindyck and Rubinfeld [12] perform unit-root tests on the prices of crude oil, copper, and lumber. They find that their tests fail to reject random walks when 30 years' data are used, but do reject when 117 years are used, indicating possible long-term mean-reversion. The mean reversion may reflect long-run average costs. Most applications, however, have horizons much shorter than 100 years, so random walks such as the geometric diffusion should suffice.

[3] This exposition generally follows Pindyck [11].

[4] Several other technical assumptions need to be made. See Brémaud [3] Appendix A for a more complete explanation.

[5] This assumption is mainly for analytical convenience. It permits the use of market interest rates and stochastic optimization.

of this model. Let μ be the expected return of a portfolio which replicates P, r be the risk-free interest rate, ϕ be the market price of risk, and ρ be the correlation between P and the market portfolio. Then, $\mu = r + \phi\rho\sigma$. Let $\alpha = \mu - \delta$. δ can be considered to be the opportunity cost of investment. It is analogous to the dividend rate on a continuously paying stock. It has several interpretations: the decay rate of cash flows, it can reflect the entry and exit of competitors, or it can be the *net marginal convenience yield* from storage of the commodity produced and is assumed constant. Note, that if $\delta \leq 0$, then the firm would never invest, as it never loses by retaining the option to invest. One advantage of this specification is that changes in P over a finite interval are lognormally distributed.[6]

Whenever the firm chooses to invest, it expends I to receive project value $V(P)$. It can never abandon the project to receive its original investment, so investment is irreversible. Since it can defer investment, it receives value from the option to invest $F(P)$ when it chooses not to invest. When the project is active, it has a constant fixed cost c. Thus, its net revenue is $P - c$. It may also have shutdown and subsequent startup costs k and ℓ, respectively.

2.1 Valuing the Project

The differential equation governing $V(P)$ is obtained by forming a hedging portfolio with the same expected return.[7] The CAPM assumption permits this portfolio to be formed by appealing to spanning.[8] This portfolio is formed by being long in the project and short V_P units of output.[9,10] This portfolio has value $V(P) - V_P P$ with instantaneous cash flow $j(P-c)dt - \delta V_P P\,dt$, where $j = 1$ when the project is operating, and zero when it is not. $\delta V_P P$ is the cost required to maintain the short position in the portfolio.[11] The total return on this portfolio is the sum of its differential and its cash flow: $dV - V_P dP + j(P-c)dt - \delta V_P\,dt$. Since the portfolio is risk-free, its return is set equal to $r(V - V_P P)dt$:

$$dV - V_P dP + j(P-c)dt - \delta P V_P\,dt = r(V - PV_P)dt \tag{3}$$

The first term dV is expanded by Ito's Lemma:

$$dV = V_P dP + (1/2)V_{PP}(dP)^2$$

When it fails, more complicated dynamic programming becomes necessary, which does not substantially alter the results.

[6] See Merton [10] pp. 79–80.

[7] This is only possible when the process generating P has no jump component.

[8] Spanning exists when any asset can be replicated by a combination of other assets.

[9] This is the standard way of forming hedging portfolios. See Black and Scholes [4] for a general explanation.

[10] Derivatives are denoted by subscripts.

[11] This is analogous to a dividend payment. Since the portfolio is short in output, the portfolio holder must pay out dividends.

Substituting (2) for dP and rearranging obtains:

$$\frac{1}{2}\sigma^2 P^2 V_{PP} + (r-\delta)PV_P + j(P-c) - rV = 0 \tag{4}$$

This equation is used throughout this paper with only minor changes.

Alternatively, dynamic programming may be used to evaluate V. This is necessary when spanning does not obtain. That is, the project contains systematic, nondiversifiable risk. In this case μ becomes the required risk-adjusted return. Let $\pi(P,t)$ be the profit function, then the Bellman equation for optimality is:[12]

$$V(P,t) = \pi(P,t) + e^{-\mu\, dt}\mathbf{E}_t\left[V(P+dP,\, t+dt)\right] \tag{5}$$

Under the above assumption that P is a geometric diffusion, application of Itô's Lemma obtains the following partial differential equation for V:

$$\frac{1}{2}\sigma^2 P^2 V_{PP}(P,t) + \alpha P V_P(P,t) + V_t(P,t) - \mu V(P,t) + \pi(P,t) = 0 \tag{6}$$

If V does not depend on t, then (6) simplifies to

$$\frac{1}{2}\sigma^2 P^2 V(P)_{PP} + \alpha P V(P)_P - \mu V(P) + \pi(P) = 0 \tag{7}$$

This is identical in form to (4) when μ is substituted for r and $\delta = \mu - \alpha$. This means that instead of forming a riskless portfolio to value the project, one can form a hedging portfolio with a required return of μ. The upshot is that μ may be substituted for r in solutions of (4) and similar equations.

This paper assumes that shutdown and subsequent startup are costless.[13] The stochastic differential equation for this problem, given no jump process, is just (4).

$$\frac{1}{2}\sigma^2 V_{PP} + (r-\delta)PV_P + j(P-c) - rV = 0 \tag{8}$$

The boundary conditions are obtained as follows. When $P = 0$, the value of the project is also zero:

$$V(0) = 0 \tag{9}$$

[12] I use $\pi(P,t)$ to emphasize the generality of this result. The details may be found in Dixit and Pindyck [8] pp. 121–122. t, of course, is time and is included for use in later chapters.

[13] The notation in this and subsequent Sections follows Pindyck [11] as does the analysis in this Section.

V must be continuous and smooth at $P = c$:

$$
\begin{aligned}
V(c^-) &= V(c^+) && (10)\\
V_P(c^-) &= V_P(c^+) && (11)
\end{aligned}
$$

where c^- and c^+ are the left and right limits at c, respectively. Finally, as $P \to \infty$, the probability of shutdown goes to zero, since $P - c \to \infty$. Thus, the value of the project converges to the value of perpetuity obtained by operating the firm forever. P is discounted by the required rate of return μ, but since it is expected to grow at rate α, the revenues are discounted by their difference, δ. c is discounted by the risk-free interest rate r. This condition rules out speculative bubbles, as the system has a homogeneous solution above $P = c$ which satisfies the boundary conditions.[14] Thus,

$$\lim_{P \to \infty} V = P/\delta - c/r \tag{12}$$

The solution to this system is

$$V(P) = \begin{cases} A_1 P^{\beta_1}, & P < c \\ A_2 P^{\beta_2} + P/\delta - c/r, & P \geq c \end{cases} \tag{13}$$

where A_1, A_2, β_1, β_2 are constants. β_1 and β_2 are roots of the quadratic equation $\mathcal{Q}(\beta) = (1/2)\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0$. These evaluate to

$$\beta_1 = 1/2 - (r - \delta)/\sigma^2 + \left\{\left[(r - \delta)/\sigma^2 - 1/2\right]^2 + 2r/\sigma^2\right\}^{\frac{1}{2}}$$

and

$$\beta_2 = 1/2 - (r - \delta)/\sigma^2 - \left\{\left[(r - \delta)/\sigma^2 - 1/2\right]^2 + 2r/\sigma^2\right\}^{\frac{1}{2}}$$

It can be shown that $\beta_1 > 1$, $\beta_2 < 0$. Both β_1 and β_2 remain the same throughout this thesis. Equation (12) yields the constants A_1 and A_2 as

$$A_1 = \frac{r - \beta_2(r - \delta)}{r\delta(\beta_1 - \beta_2)} c^{1 - \beta_1}$$

[14] Dixit and Pindyck [8] pp. 181–182 use the CAPM to rule out speculative bubbles.

and

$$A_2 = \frac{r - \beta_1(r-\delta)}{r\delta(\beta_1 - \beta_2)} c^{1-\beta_2}$$

Equation (13) has the following interpretation. When $P < c$, the firm is not operating, $A_1 P^{\beta_1}$ is the value of its option to produce at a later date, should P rise above c. When $P \geq c$, the firm produces. The term $P/\delta - c/r$ comes from the asymptotic condition. The value of the option to shutdown should the price fall below c is $A_2 P^{\beta_2}$. This value falls as P increases since the probability that P will fall below c goes to zero.

2.2 Valuing the Option to Invest

The option to invest is valued similarly to the project itself. Let $F(P)$ be the value of the option to invest. The hedging portfolio is $F - F_P P$. This portfolio, unlike the project itself, generates no cash flows. It still, however, requires payment of $\delta F_P P$. The rest of the argument is the same as that determining $V(P)$, with F substituted for V and the term $j(P-c)$ deleted. This results in the equation for F:

$$\frac{1}{2}\sigma^2 P^2 F_{PP} + (r-\delta) P F_P - rF = 0 \tag{14}$$

The boundary conditions may be determined from the investment decision. Let P^* be the value of P at which it just becomes optimal to invest amount I in the project.[15] That is:[16]

$$F(P^*) = V(P^*) - I \tag{15}$$

It also must be true that when $P = 0$, the option to invest is worthless:

$$F(0) = 0 \tag{16}$$

Finally, the "smooth pasting" or "high-order contact" condition must be satisfied:

$$F_P(P^*) = V_P(P^*) \tag{17}$$

The smooth pasting condition is a necessary condition for stochastic optimal control.[17]

[15] This makes this into an optimal stopping time problem. Brock et al. [5] emphasize this view with various one-dimensional geometric price evolution processes and boundary conditions.

[16] The net present value rule is equivalent to setting $V - I = 0$.

[17] Brekke and Øksendal [2] prove that any solution to a stopping problem which satisfies "smooth-pasting" is optimal.

This system has the solution

$$F(P) = \begin{cases} aP^{\beta_1}, & P \leq P^* \\ V(P) - I, & P > P^* \end{cases} \tag{18}$$

From equation (17) the constant $a = \frac{\beta_2 A_2}{\beta_1}(P^*)^{\beta_2-\beta_1} + \frac{1}{\delta\beta_1}(P^*)^{1-\beta_1}$. The critical price P^* is obtained from (15) as the solution to the equation

$$(\beta_1 - \beta_2)A_2(P^*)^{\beta_2} + (\beta_1 - 1)\frac{P^*}{\delta} - \beta_1\left(\frac{c}{r} + I\right) = 0 \tag{19}$$

This equation must be solved numerically.

3 Deterministic Taxation

Let us now introduce a uniform tax on profits at the rate of τ. Thus, whenever profit of $P - c$ is received, the firm pays $\tau(P - c)$ to the government and receives aftertax net income of $(1 - \tau)(P - c)$. Equation (8) becomes

$$\frac{1}{2}\sigma^2 P^2 \acute{V}_{PP} + (r - \delta)P\acute{V}_P + j(1 - \tau)(P - c) - r\acute{V} = 0 \tag{20}$$

where the acute accents on V refer to the value of a taxed project. Since the homogeneous part of (20) is unchanged from (8), that part of the solution is unchanged. The inhomogeneous part is solved by using the asymptotic boundary condition

$$\lim_{P\to\infty} \acute{V}(P;\tau) = (1 - \tau)(P/\delta - c/r) \tag{21}$$

This is the aftertax value of the difference between the annuities used to determine the original asymptotic boundary condition (12). The other boundary conditions are unchanged. Combining (20) with its boundary conditions obtains

$$\acute{V}(P;\tau) = \begin{cases} \acute{A}_1(\tau)P^{\beta_1}, & P < c \\ \acute{A}_2(\tau)P^{\beta_2} + (1 - \tau)(P/\delta - c/r), & P \geq c \end{cases} \tag{22}$$

where β_1 and β_2 are defined as before.[18] $\acute{A}_1(\tau)$ and $\acute{A}_1(\tau)$ are given by

$$\begin{aligned} \acute{A}_1(\tau) &= (1 - \tau)\frac{r - \beta_2(r - \delta)}{r\delta(\beta_1 - \beta_2)}c^{1-\beta_1} = (1 - \tau)A_1 \\ \acute{A}_2(\tau) &= (1 - \tau)\frac{r - \beta_1(r - \delta)}{r\delta(\beta_1 - \beta_2)}c^{1-\beta_2} = (1 - \tau)A_2 \end{aligned}$$

[18] Of course, when $\tau = 0$, (22) is identical to (13), as are (20) and (8).

Therefore

$$\acute{V}(P;\tau) = \begin{cases} (1-\tau)A_1 P^{\beta_1} = (1-\tau)V(P), & P < c \\ (1-\tau)A_2(\tau)P^{\beta_2} + (1-\tau)(P/\delta - c/r) = (1-\tau)V(P), & P \geq c \end{cases} \tag{23}$$

Thus, we find that taxation reduces the value of the firm in proportion to the tax rate.

The differential equation governing the value of the option to invest is unchanged from (14), since a nonexistent project cannot be taxed. Thus, $\acute{F}$ has an identical form to $\acute{V}$, with different coefficients resulting from the use of $\acute{V}$ instead of V:

$$\acute{F}(P;\tau) = \begin{cases} \acute{a}(\tau)P^{\beta_1}, & P \leq \acute{P}^*(\tau) \\ (1-\tau)V(P) - eI, & P > \acute{P}^*(\tau) \end{cases} \tag{24}$$

where e, $0 < e \leq 1$ is a factor applied to I to reflect its tax treatment. Important values of e are $e = 1$ and $e = 1 - \tau$. The former indicates that investment is not expensed at all. The latter occurs when investment is fully expensed and either a tax refund is received or $\max(P_0, P^*) - c > I$, in which case I is less than the operating profit. In this case, equation (24) is identical to (18) up to multiplication by $1-\tau$. P^* is unchanged and $\acute{a}(\tau) = (1-\tau)a$, so that the option to invest is reduced in proportion to the tax rate,[19]

When $e \neq 1 - \tau$, we evaluate anew $\acute{a}(\tau)$ and $\acute{P}^*$:

$$\acute{a}(\tau) = (1-\tau)\left[\frac{\beta_2 A_2}{\beta_1}(\acute{P}^*)^{\beta_2-\beta_1} + \frac{1}{\delta\beta_1}(\acute{P}^*)^{1-\beta_1}\right] \tag{25}$$

and $\acute{P}^*$ is the solution to the equation

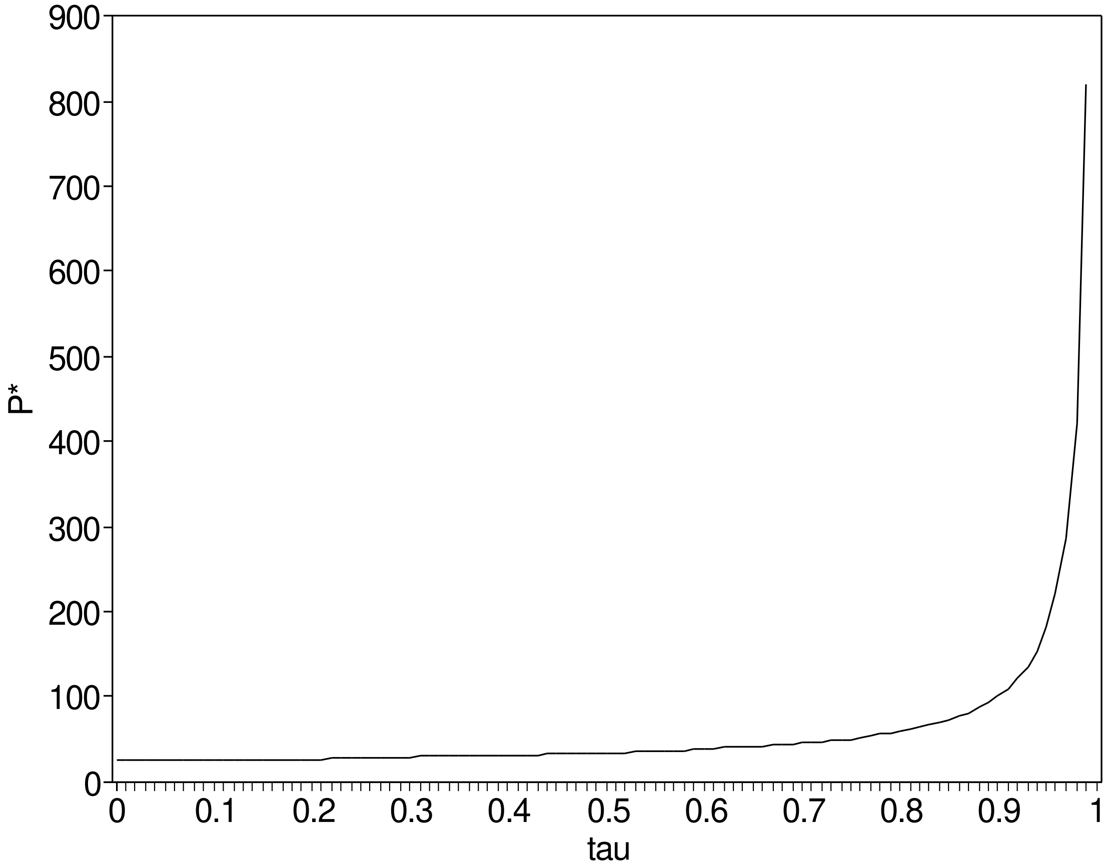
$$(1-\tau)\left[(\beta_1-\beta_2)A_2(\acute{P}^*)^{\beta_2} + (\beta_1-1)\frac{\acute{P}^*}{\delta} - \beta_1\frac{c}{r}\right] - \beta_1 eI = 0 \tag{26}$$

We take derivatives to determine the effects of changes in τ on $\acute{a}$, $\acute{P}^*$. First, we apply the implicit function theorem to equation (26) to find $d\acute{P}^*/d\tau$:

$$\frac{d\acute{P}^*}{d\tau} = -\frac{\partial(26)/\partial\tau}{\partial(26)/\partial\acute{P}^*} = \frac{\left[(\beta_1-\beta_2)A_2(\acute{P}^*)^{\beta_2} + (\beta_1-1)\frac{\acute{P}^*}{\delta} - \beta_1\left(\frac{c}{r}\right)\right] + \beta_1 I\frac{de}{d\tau}}{(1-\tau)\left[(\beta_1-\beta_2)\beta_2 A_2(\acute{P}^*)^{\beta_2-1} + \frac{\beta_1-1}{\delta}\right]} \geq 0 \tag{27}$$

The proofs of this inequality and inequality (28), assuming that $de/d\tau \geq -1$, are in Coleman [6], Appendix A.[20] Figure 1 shows a plot of $\acute{P}^*$ as a function of τ for a set of values for r, δ, c, σ, I with $e = 1$. Next, we

[19] This result is also reported in Mackie-Mason [9] p. 322.
[20] Note that the rules $e = 1$ and $e = 1 - \tau$ satisfy this assumption.



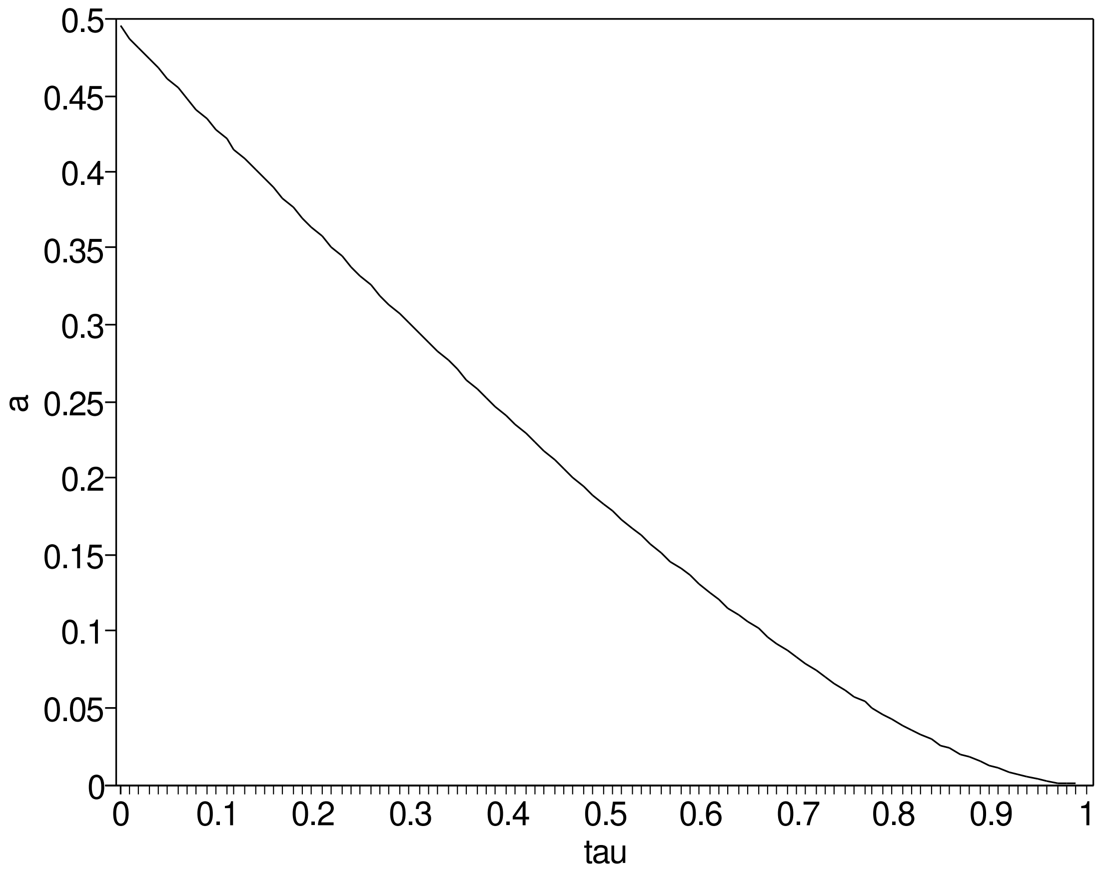
Note: $r = .04$, $\delta = .04$, $c = 10$, $\sigma = .2$, $I = 100$

Figure 1: P^* as a Function of τ

find $d\acute{a}/d\tau$:

$$
\begin{aligned}
\frac{d\acute{a}}{d\tau} &= \frac{\partial \acute{a}}{\partial \tau} + \frac{\partial \acute{a}}{\partial P^*}\frac{dP^*}{d\tau} \\
&= -\frac{\beta_2 A_2}{\beta_1}(\acute{P}^*)^{\beta_2-\beta_1} - \frac{1}{\delta\beta_1}(\acute{P}^*)^{1-\beta_1} + (1-\tau)\left[\frac{(1-\beta_1)}{\beta_1\delta}(\acute{P}^*)^{-\beta_1} + \frac{A_2\beta_2(\beta_2-\beta_1)}{\beta_1}(\acute{P}^*)^{\beta_2-\beta_1-1}\right]\frac{dP^*}{d\tau} \\
&< 0
\end{aligned}
$$

Figure 2 plots $\acute{a}(\tau)$ for the same parameter values as in Figure 1.



Note: $r = .04$, $\delta = .04$, $c = 10$, $\sigma = .2$, $I = 100$

Figure 2: $\acute{a}(\tau)$

4 The Bankruptcy Process

When dP contains a jump component, $\gamma_i \neq 0$ for some i. The bankruptcy process is a special case of a marked Poisson process with $\gamma_i \equiv -1$. We may refer to a single process, because the combination of any two Poisson bankruptcy processes is itself a Poisson bankruptcy process.[21] Let $S(t)$ be the cumulative distribution function of seizure times. The *hazard rate* is defined as $S'(t)/S(t) \equiv \lambda(t)$, the intensity of the Poisson process in the jump-diffusion case. In this case, the intensity may be interpreted as the conditional probability of seizure during the infinitesimal interval $(t,\, t+dt)$, given that seizure has not occurred by time t. Let X be the event of seizure:

$$\Pr(X \in (t,\, t+dt)|X > t) = \frac{\Pr\{X \in (t, t+dt),\, X > t\}}{\Pr\{X > t\}} = \frac{\Pr\{X \in (t, t+dt)\}}{\Pr\{X > t\}} = \frac{S'(t)}{S(t)} dt \tag{28}$$

Conversely, the hazard rate also completely determines distribution of seizure times. If $\lambda_t \equiv \lambda$, then the seizure times are distributed exponentially with mean $1/\lambda$. In other cases, $S(t)$ is given by $S(t) = 1 - \exp\left\{-\int_0^t \lambda(s)\, ds\right\}$.[22] Note that if $\lambda(t)$ does not decay to zero fast enough, if at all, the integral is divergent. Bankruptcy occurs at some t almost surely (i.e. with probability one.) The hazard rate for bankruptcy is most likely constant. The hazard rate for confiscation may be a function P or V. In the tax reform model, the firm's return after receiving the tax break is the sum of the pre-tax break return and the incremental return from the tax break, the latter subject to a bankruptcy process. In the stochastic tax model, the firm's return is the sum of the return at the base tax rate and a countable number of returns from bankruptcy processes with constant hazard.

The bankruptcy process results in a simplification of equation (1). The sole Poisson term dq takes values $\{1, 0\}$, according to whether confiscation at that instant occurs or not. Equation (2) thus becomes:

$$dP = \alpha P\, dt + \sigma P\, dt - P\, dq \tag{29}$$

Note that λ does not occur in the above expression. This is because $\mathbf{E}(\gamma_i - 1) = \mathbf{E}(1-1) = 0$.[23]

[21] To see this, consider two Poisson processes with cumulative hazard functions $\Lambda(t)$, $\Lambda'(t)$ (the cumulative hazard function is simply the integral from $t = 0$ of the hazard rate.) Let $N(s,t)$, $N'(s,t)$ be the numbers of arrivals in each respective process between times s and t, $0 \leq s < t$. The probabilities of zero arrivals in the two processes (i.e. $N(s,t), N'(s,t) = 0$) are $\exp\left[-(\Lambda(t) - \Lambda(s))\right], \exp\left[-(\Lambda'(t) - \Lambda'(s))\right]$. The probability that $N(s,t) = N'(s,t) = N(s,t) + N'(s,t) = 0$ is $\exp\left[-((\Lambda(t) + \Lambda'(t)) - (\Lambda(s) + \Lambda'(s)))\right]$. Thus, the combined process is a Poisson bankruptcy process whose cumulative hazard function is the sum of the cumulative hazard functions of the original processes.

[22] This exposition follows Ross [13] pp. 168–170.

[23] This should really be $\mathbf{E}(-\gamma_i - 1) = 0$, since γ_i is defined to have range $[-1, \infty)$ above.

4.1 Valuing Projects Subject to Bankruptcy

Again, we form the hedging portfolio $V - V_P P$. Since the jump risk is uncorrelated with P and V, and, in fact, is independent of those variables, the required return equals $r + \lambda$. The addition of λ is required to make the expected return to the portfolio be r. To see this, consider receiving the constant infinite stream $\bar{V}$, subject to bankruptcy hazard rate λ. The present value of this stream after discounting is $\int_0^\infty e^{-\lambda t} e^{-rt} \bar{V}\, dt = \int_0^\infty e^{-(r+\lambda)} \bar{V}\, dt = \bar{V}/(r+\lambda)$. This stream is valued identically to the certain stream discounted at $r + \lambda$. If the required return incorporates a risk premium, λ in the equations below may be increased. Again, the total return from the hedging portfolio is $dV - V_P dP + j(P - c)dt - \delta P V_P\, dt$, with the same argument. However, this is now equated to $(r + \lambda)(V - V_P P)dt - (V - PV_P)dq$. The last term, $-(V - PV_P)dq$, reflects the unneutralizable jump risk. The two expressions are equated:

$$dV - PV_P dP + j(P - c)dt - \delta P V_P\, dt = (r + \lambda)(V - V_P P)dt - (V - PV_P)dq \tag{30}$$

dV is expanded by the jump-diffusion version of Ito's Lemma:[24]

$$dV = (\frac{1}{2}\sigma^2 P^2 V_{PP} + (\alpha - \lambda) P V_P)dt + \sigma P V_P dz - V dq \tag{31}$$

$V_P dP$ is also expanded, using equation (2):

$$V_P dP = \alpha P V_P dt + \sigma P V_P dz - P V_P dq \tag{32}$$

Substituting equations (31) and (32) into (30) obtains

$$\begin{aligned} &\left[\tfrac{1}{2}\sigma^2 P^2 V_{PP} + j(P - c) - (\delta - \lambda) P V_P\right] dt - (V - PV_P)dq \\ &\quad = (r + \lambda)(V - V_P P)dt - (V - PV_P)dq \end{aligned} \tag{33}$$

Cancelling the terms in dq_i, dividing through by dt, and rearranging obtains

$$\frac{1}{2}\sigma^2 P^2 V_{PP} + (r - \delta) P V_P + j(P - c) - (r + \lambda) V = 0 \tag{34}$$

The option to invest is valued similarly, assuming that the bankruptcy event affects the option to invest. Otherwise, equation (14) still obtains. Again, the hedging portfolio is $F - F_P P$. Again, we eliminate the

[24] Merton [10] p. 92.

term $j(P-c)$ and substitute F for V to obtain the differential equation for F:

$$\frac{1}{2}\sigma^2 P^2 F_{PP} + (r-\delta)PF_P - (r+\lambda)F = 0 \tag{35}$$

To solve this system, we set $r^* = r + \lambda$ and $\delta^* = \delta + \lambda$:

$$\tfrac{1}{2}\sigma^2 P^2 V_{PP} + (r^* - \delta^*)PV_P + j(P-c) - r^* V = 0 \tag{36}$$

$$\tfrac{1}{2}\sigma^2 P^2 F_{PP} + (r^* - \delta^*)PF_P - r^* F = 0 \tag{37}$$

This system is identical to equations (4) and (14) in Section 2, with r^* and δ^* substituted for r and δ, respectively.[25] These equations point out the main effect of bankruptcy: it raises the effective discount rate of an ongoing project by the hazard rate. In addition, it raises the effective δ by the same amount.

Another effect shows up when the option to defer investment is not affected by the bankruptcy-generating process. Equations (34) and (2.2) combine to show that the effective discount and depreciation rates applied to the project and the option to defer investment in the project differ, with higher rates being applied to the project itself.

The boundary conditions are unaffected by the presence of bankruptcy. The nonasymptotic conditions may easily be seen to be unchanged. The asymptotic condition is given by the discounted value of the difference of the annuities of P and c, given the bankruptcy process. This value is given by:

$$\begin{aligned}\textstyle\int_0^\infty e^{-\lambda t}(e^{-\delta t}P - e^{-rt}c)dt &= \textstyle\int_0^\infty e^{-(\delta+\lambda)t}P\,dt - \int_0^\infty e^{-(r+\lambda)t}c\,dt \\ &= \textstyle\int_0^\infty e^{-\delta^* t}P\,dt - \int_0^\infty e^{-r^*}c\,dt = \frac{P}{\delta^*} - \frac{c}{r^*}\end{aligned} \tag{38}$$

Therefore,

$$\lim_{P\to\infty} V(P) = \frac{P}{\delta^*} - \frac{c}{r^*} \tag{39}$$

which is exactly equation (12) with δ^*, and r^* substituted for δ and r, respectively.

The solution for $V(P)$ has the same form as (12):

$$V(P) = \begin{cases} A_1^* P^{\beta_1^*}, & P < c \\ A_2^* P^{\beta_2^*} + P/\delta^* - c/r^*, & P \geq c \end{cases} \tag{40}$$

[25] Of course, the discount rates for F and r and δ if the option to defer investment is unaffected.

where

$$\begin{aligned}\beta_1^* &= 1/2 - (r^* - \delta^*)/\sigma^2 + \left\{\left[(r^* - \delta^*)/\sigma^2 - 1/2\right]^2 + 2r^*/\sigma^2\right\}^{\frac{1}{2}} \\ &= 1/2 - (r - \delta)/\sigma^2 + \left\{\left[(r - \delta)/\sigma^2 - 1/2\right]^2 + 2(r + \lambda)/\sigma^2\right\}^{\frac{1}{2}}\end{aligned}$$

and, likewise,

$$\beta_2^* = 1/2 - (r - \delta)/\sigma^2 - \left\{\left[(r - \delta)/\sigma^2 - 1/2\right]^2 + 2(r + \lambda)/\sigma\right\}^{\frac{1}{2}}$$

Note that β_1^*, β_2^* differ from β_1, β_2 by only the term including λ inside the square root. Since $2\lambda/\sigma^2 > 0$, $\beta_1^* > \beta_1$, $\beta_2^* < \beta_2$. Likewise,

$$A_1^* = \frac{r^* - \beta_2^*(r^* - \delta^*)}{r^*\delta^*(\beta_1^* - \beta_2^*)} c^{1-\beta_1^*}$$

and

$$A_2^* = \frac{r^* - \beta_1^*(r^* - \delta^*)}{r^*\delta^*(\beta_1^* - \beta_2^*)} c^{1-\beta_2^*}$$

The analysis of taxation is identical to Section 3, with starred variables replacing their unstarred counterparts.

We value $F(P)$, assuming that the bankruptcy process affects F:

$$F(P) = \begin{cases} a^* P^{\beta_1^*}, & P \leq P^* \\ V(P) - I, & P > P^* \end{cases} \tag{41}$$

where $a^* = \frac{\beta_2^* A_2^*}{\beta_1^*}(P^*)^{\beta_2^* - \beta_1^*} + \frac{1}{\delta^*\beta_1^*}(P^*)^{1-\beta_1^*}$. The critical price P^* is the solution to the equation

$$\frac{A_2^*(\beta_1^* - \beta_2^*)}{\delta^*\beta_1^*}(P^*)^{\beta_2^*} - + \frac{\beta_1^* - 1}{\delta^*\beta_1^*}P^* - \frac{c}{r^*} - eI = 0 \tag{42}$$

where e again reflects the expensing of I.

5 Stochastic Tax Rates

The tax rate changes exactly once, at a stochastic time. This is modeled as the sum of two projects: one project is taxed at the final rate, τ_f, and the other project follows a bankruptcy process and is taxed at the rate $\Delta\tau \equiv \tau_i - \tau_f$, the difference between the initial and final tax rates. Let λ be the (constant) hazard rate of bankruptcy. This results in the pre-tax change system

$$\grave{V}(P) = \begin{cases} (1 - \tau_f)A_1 P^{\beta_1} + (\tau_i - \tau_f)A_1^* P^{\beta_1^*} & P < c \\ (1 - \tau_f)(A_2 P^{\beta_2} + P/\delta - c/r) + (\tau_i - \tau_f)(A_2^* P^{\beta_2^*} + P/\delta^* - c/r^*) & P \geq c \end{cases} \tag{43}$$

where $\delta^* = \delta + \lambda$, $r^* = r + \lambda$, and A_1^*, A_2^*, β_1^*, β_2^* are defined as in Section 4. After the tax change, of course, the starred terms fall out.

The value of the pre-tax change option to invest is likewise formed by summing the values of the parts of the project:

$$\grave{F}(P) = \begin{cases} \grave{a}(\tau_f)P^{\beta_1} + \grave{a}^*(1 - \tau_i + \tau_f)P^{\beta_1^*}, & P \leq \grave{P}^* \\ \grave{V}(P) - e(\tau_i)I, & P > \grave{P}^* \end{cases} \tag{44}$$

where e again reflects the expensing of I and $\grave{a}(\tau)$ and $\grave{a}^*(\tau)$ are defined by

$$\begin{aligned} \grave{a}(\tau) &= (1-\tau)\left[\frac{\beta_2 A_2}{\beta_1}(\grave{P}^*)^{\beta_2-\beta_1} + \frac{1}{\delta\beta_1}(\grave{P}^*)^{1-\beta_1}\right] && (45) \\ \grave{a}^*(\tau) &= (1-\tau)\left[\frac{\beta_2^* A_2^*}{\beta_1^*}(\grave{P}^*)^{\beta_2^*-\beta_1^*} + \frac{1}{\delta^*\beta_1^*}(\grave{P}^*)^{1-\beta_1^*}\right] && (46) \end{aligned}$$

where $\grave{P}^*$ solves

$$\begin{aligned} & (1-\tau_f)\left[\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2} + \frac{\beta_1 - 1}{\delta\beta_1}\grave{P}^* - \frac{c}{r}\right] \\ + \; & (\tau_i - \tau_f)\left[\frac{A_2^*(\beta_1^* - \beta_2^*)}{\beta_1^*}(\grave{P}^*)^{\beta_2^*} + \frac{\beta_1^* - 1}{\delta^*\beta_1^*}\grave{P}^* - \frac{c}{r^*}\right] - e(\tau_i)I \\ = \; & 0 \end{aligned} \tag{47}$$

The analysis of the effect of the stochastic tax component can be analyzed similarly to Section 3. We form $d\grave{P}^*/d\Delta\tau$, where $\Delta\tau = \tau_i - \tau_f$:

$$\begin{aligned} \frac{d\grave{P}^*}{d\Delta\tau} &= -\frac{\partial(47)/\partial\Delta\tau}{\partial(47)/\partial\grave{P}^*} \\ &= -\frac{\frac{A_2^*(\beta_1^*-\beta_2^*)}{\beta_1^*}(\grave{P}^*)^{\beta_2^*} + \frac{\beta_1^*-1}{\delta^*\beta_1^*}\grave{P}^* - \frac{c}{r^*} - I\frac{de}{d\tau_i}}{(1-\tau_f)\left[\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2-1} + \frac{\beta_1-1}{\delta\beta_1}\right] + \Delta\tau\left[\frac{A_2^*\beta_2^*(\beta_1^*-\beta_2^*)}{\beta_1^*}(\grave{P}^*)^{\beta_2^*-1} + \frac{\beta_1^*-1}{\delta^*\beta_1^*}\right]} \end{aligned} \tag{48}$$

This is similar to (27). Assuming again that $de/d\tau_i \geq -1$, when $\Delta\tau \geq 0$, $d\grave{P}^*/d\Delta\tau \geq 0$. This means that raising the initial tax rate increases the critical price for investment, a hardly surprising result. However, I cannot prove this result for all $\Delta\tau < 0$ as a result of the different discounting of the post-tax cut process. Still, by continuity, for some range $z < \Delta\tau < 0$, the inequality holds, thus lowering the initial tax rate reduces the critical price for this range.

Instead of holding τ_f constant and varying τ_i, we can do the opposite. We find $d\grave{P}^*/d\tau_f$:

$$\frac{d\grave{P}^*}{d\tau_f} = -\frac{\partial(47)/\partial\tau_f}{\partial(47)/\partial\grave{P}^*} = \frac{\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2} + \frac{\beta_1-1}{\delta\beta_1}\grave{P}^* - \frac{c}{r} + \frac{A_2^*(\beta_1^*-\beta_2^*)}{\beta_1^*}(\grave{P}^*)^{\beta_2^*} + \frac{\beta_1^*-1}{\delta^*\beta_1^*}\grave{P}^* - \frac{c}{r^*}}{(1-\tau_f)\left[\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2-1} + \frac{\beta_1-1}{\delta\beta_1}\right] + \Delta\tau\left[\frac{A_2^*\beta_2^*(\beta_1^*-\beta_2^*)}{\beta_1^*}(\grave{P}^*)^{\beta_2^*-1} + \frac{\beta_1^*-1}{\delta^*\beta_1^*}\right]} \tag{49}$$

The sign of this equation is ambiguous since the sign of neither cluster of starred or unstarred terms in the numerator is known a priori. They may very well have different signs. Instead, we can find the value of λ, given r, δ, σ, c, $\grave{P}^*$, if one exists, which sets the numerator equal to zero. This defines a surface in $\Re^5_+$. On one side of this surface a future tax cut (raise) reduces (increases) $\grave{P}^*$ on the margin. On the other side, the opposite is true. The sign of $d\grave{P}^*/\tau_f$ also depends on the sign of the denominator. Again, the denominator is positve when $\Delta\tau \geq z$ for some $z < 0$.

The difficulty with the preceeding comparative statics is the statement of the derivatives in terms of $\grave{P}^*$. Instead of examining the derivatives, $\grave{P}^*$ can be computed as a function of $r, \delta, \sigma, \lambda, \tau_i, \tau_f$. This also has the advantage of being easier to compute: we only solve equation (47) for each set of parameters. We need not then locate the previously defined surface. The asymptotic cases of $\lambda = 0$ and $\lambda = \infty$ are the easiest to examine. When $\lambda = 0$, equation (47) converges to equation (26) with τ_i substituted for τ. This has the obvious interpretation that a tax change which never occurs has no effect on the critical price for investment. As $\lambda \to \infty$, equation (47) converges to

$$\begin{aligned} & (1-\tau_f)\left[\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2} + \frac{\beta_1-1}{\delta\beta_1}\grave{P}^* - \frac{c}{r}\right] \\ - \; & (\tau_i - \tau_f)\frac{c}{r^*} - e(\tau_i)I \\ = \; & 0 \end{aligned}$$

Thus, an initial tax cut (increase) which expires immediately with probability one acts as an increase (decrease) in the cost of the investment, relative to the investment taxed at τ_f. This is true even though the expected value of the effect on post-tax returns is zero. Thus, a probability zero event (the continuation of the initial tax rate) has a real effect. Another implication is that transient (i.e. high λ) tax changes have the expected effect.

Combining the asymptotic results indicates that as λ increases from 0 to infinity, $\grave{P}^*$ goes from $\grave{P}^*(\tau_i)$ to a value close to $\grave{P}^*(\tau_f)$. However, it is not clear that this relationship is monotonic. Consider

$$\frac{d\grave{P}^*}{d\lambda} = -\frac{\partial(47)/\partial\lambda}{\partial(47)/\partial\grave{P}^*} \tag{50}$$

The sign of the denominator is positive; therefore, $d\grave{P}^*/d\lambda$ has the sign of the numerator. Expanding this numerator is a highly nontrivial task, due to its nonlinearity.

5.1 Multiple Tax Changes

This technique can be extended. Consider a countable set $\Delta\tau_j$, with the property that there is no sum $\sum_{k\in j}\Delta\tau_k$ such that $\tau_f+\sum_{k\in j}\Delta\tau_k>1$. That is, there is never a tax rate greater than 1.[26] The initial tax rate is equal to $\tau_f+\sum_j\Delta\tau_j$. Each $\Delta\tau_j$ has an independent hazard rate λ_k. Then, the value of the project becomes

$$\grave{V}(P)=\begin{cases}(1-\tau_f)A_1P^{\beta_1}+\sum_{k\in j}\Delta\tau_kA_1^kP^{\beta_1^k} & P<c\\ (1-\tau_f)(A_2P^{\beta_2}+P/\delta-c/r)+\sum_{k\in j}\Delta\tau_k(A_2^kP^{\beta_2^k}+P/\delta_k-c/r_k) & P\geq c\end{cases}\tag{51}$$

where $\delta_k=\delta+\lambda_k$ and $r_k=r+\lambda_k$. The set $k\in j$ is the set of tax changes excluding those which have occurred.[27] A_1^k, A_2^k, β_1^k, β_2^k are defined as in Section 4, substituting λ_k for λ.

The option to invest takes a similar form:

$$\grave{F}(P)=\begin{cases}\acute{a}(\tau_f)P^{\beta_1}+\sum_{k\in j}\acute{a}^k(1-\Delta\tau_k)P^{\beta_1^k}, & P\leq\grave{P}^*\\ \grave{V}(P)-e(\tau_i)I, & P>\grave{P}^*\end{cases}\tag{52}$$

where $\acute{a}(\tau)$ is defined as above and $\acute{a}^k(\tau)$ is formed by substituting the k-superscripted variables for the starred ones. $\grave{P}^*$ now solves the equation

$$\begin{aligned}&(1-\tau_f)\left[\frac{A_2(\beta_1-\beta_2)}{\beta_1}(\grave{P}^*)^{\beta_2-\beta_1}+\frac{\beta_1-1}{\delta\beta_1}\grave{P}^*-\frac{c}{r}\right]\\ +&\sum_{k\in j}\Delta\tau_k\left[\frac{A_2^k(\beta_1^k-\beta_2^k)}{\beta_1^k}(\grave{P}^*)^{\beta_2^k-\beta_1^k}+\frac{\beta_1^k-1}{\delta_k\beta_1^k}\grave{P}^*-\frac{c}{r_k}\right]-e(\tau_i)I\\ =&\ 0\end{aligned}\tag{53}$$

6 Conclusion

References

[1] Joshua Aizenman and Nancy Marion. Policy uncertainty, persistence and growth. Working Paper 3848, National Bureau of Economic Research, September 1991.

[26] We can optionally disallow stochastic subsidies, i.e. $\tau_f+\sum_{k\in j}\Delta\tau_k<0$, without affecting the analysis.

[27] Remember, $\Delta\tau_j$ is added to the *final* tax rate, so that $-\Delta\tau_j$ is the actual tax rate change incurred.

[2] Kjell Arne Brekke and Bernt Øksendal. The high contact principle as a sufficiency condition for optimal stopping. In Diderik Lund and Bernt Øksendal, editors, *Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems*, pages 187–208. North Holland, New York, 1991.

[3] Pierre Brémaud. *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, New York, 1981.

[4] Michael J. Brennan and Eduardo S. Schwartz. Evaluating natural resource investments. *Journal of Business*, 58(2):135–157, 1985.

[5] William A. Brock, Michael Rothschild, and Joseph E. Stiglitz. Stochastic capital theory. In George R. Ferwel, editor, *Joan Robinson and Modern Economic Theory.* Macmillan, New York, 1988.

[6] Charles D. Coleman. *Irreversible Investment under Uncertainty with Taxation.* PhD thesis, George Mason University, Fairfax, VA, 1995.

[7] Charles D. Coleman. A simple model of endogenous tax reform with irreversible investment and tax breaks. Submitted to *Journal of Public Economics*, 1997.

[8] Avinash K. Dixit and Robert S. Pindyck. *Investment under Uncertainty.* Princeton University Press, Princeton, New Jersey, 1994.

[9] Jeffrey K. MacKie-Mason. Some nonlinear tax effects on asset values and investment decisions under uncertainty. *Journal of Public Economics*, 42:301–328, August 1990.

[10] Robert C. Merton. *Continuous-Time Finance.* Blackwell, Cambridge, MA, 1990.

[11] Robert S. Pindyck. Irreversibility, uncertainty, and investment. *Journal of Economic Literature*, 29:1110–1148, September 1991.

[12] Robert S. Pindyck and Daniel L. Rubinfeld. *Econometric Models and Economic Forecasting.* McGraw-Hill, New York, New York, third edition, 1991.

[13] Sheldon Ross. *A First Course in Probability.* MacMillan Publishing Company, New York, second edition, 1984.