A Life-Cycle Model of Fame

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Abstract

A model of investment in human capital is developed, following Levy [5], in which utility depends on both income and the accumulated stock of human capital. Utility may also have a contribution from legacy after death. Investment in human capital is in the form of time, to the exclusion of wages. In particular, I model the accumulation of fame derived by investment in scientific research. I then investigate the comparative statics for the professionalization of an amateur discipline. Generally, an individual must derive great utility from fame for him to devote significant time to science.

1 Introduction

This paper models a process of human capital accumulation, the investment being in the form of time spent in scientific pursuits at the expense of monetary pursuits. This model is similar to capital accumulation models in the literature[1] in that the individual chooses the level of saving, and, hence, also of consumption, out of his assets at each point in time. The key differences here are that the amount of savable assets is fixed at all times, because the length of the day cannot be changed and that the return to savings is not at a fixed rate. It can also be considered an extension of Diamond's [2] model of scientific capital accumulation. The model itself comes from Levy.[2]

The primary motivation for human capital investment is to increase the future earnings stream. A secondary motivation is the good obtained from the investment itself, e.g. fame, education, etc. A final motivation is legacy, in our case, the utility obtained from fame in the afterlife. This investment, for simplicity, is assumed to come at the expense of time spent on income-producing activities. The wage itself is a function of fame. For our purposes, we consider only the case of the individual scientist's effort to maximize the present discounted value of his utility stream over his lifetime. I assume that he has a finite life span and perfect information about his production possibilities. Section 2 formally presents and solves the model and includes the effects of legacy. Section 3 examines the effects of the professionalization of an amateur discipline. Section 4 looks at a purely amateur profession, in which fame does not increase income. Section 5 concludes this paper.

*I would like to thank David M. Levy for giving me this problem to work on; Samuel L. Braunstein for invaluable help with derivations and writing advice; and Daniel Neuhauser for help with derivations. Additionally, I am indebted to Roger Congleton, Hany Eldeib, Ken Koford and David Levy for comments and suggestions. Needless to say, I bear full responsibility for any remaining errors and obscurities.

[1] See Ramsey [6], Samuelson [7], and Samuelson & Solow [8].

[2] Levy [5], pp. 616–617.

2 The Model and Its Solution

The individual in this model has to decide how to allocate his time between the production of fame and income. The individual produces fame in order to increase his income. He then seeks to maximize the present discounted value of his utility stream over his lifetime. His utility function U at any point in time t is given by

$$U(t) = U\big(F(t), I(t)\big) \tag{1}$$

where $F(t)$ and $I(t)$ the stock of fame and flow of income at time t, respectively.[3] Formally, let $F(t)$ be given by the equation

$$F(t) = \int_0^t f\big(s(\tau)\big)\, d\tau \tag{2}$$

where $s(t)$ is the amount of time devoted to scientific work at date t. The function $f\big(s(t)\big)$ is then the flow of fame, or, rather, the rate at which the stock of fame changes. To incorporate the effects of fame on the wage, we assume the following

$$w(t) = w_0 + g\big(F(t)\big) \tag{3}$$

where $w(t)$ is the wage at time t, $w_0 > 0$ is the base wage with no fame, and g, $g(0) = 0$ and $g' > 0$, is the change in the wage caused by a change in fame. g is monotonic so that increases in fame always increase the wage. Income at time t is thus

$$I(t) = m(t)w(t) \tag{4}$$

where $m(t)$ is the amount of time devoted to market activities. We then have to constrain w and m so that they are nonnegative and add up to 24. The individual cannot devote negative hours to an activity and is assumed to only spend his time on either scientific or market pursuits. We write these constraints as

$$m(t) + s(t) = 24 \tag{5}$$

and

$$m(t)\,, s(t) \geq 0 \tag{6}$$

We then assume that the individual maximizes the present discounted value of his utility stream over his lifetime. Assuming that his lifespan is fixed at some value T and that his psychic discount rate is r, we obtain the following objective function.

$$V = \int_0^T e^{-rt}\, U\big(F(t), I(t)\big)\, dt \tag{7}$$

In order to solve (7) we must use the calculus of variations. Setting the first variation equal to zero obtains[4]

$$\int_t^T e^{-r\tau}\left[\frac{\partial U}{\partial F}\bigg|_\tau + \frac{\partial U}{\partial I}\bigg|_\tau \big(24 - s(\tau)\big)\frac{dg}{dF}\bigg|_\tau\right] d\tau = \frac{e^{-rt}\left[\frac{\partial U}{\partial I}\big|_t\, w(t) - \lambda(t) - \mu(t)\right]}{\frac{df}{ds}\big|_t} \tag{8}$$

where $\lambda(t)$, $\mu(t)$ are the Lagrange multipliers associated with the constraints (5) and (6) on $s(t)$.[5] The numerical solutions are derived by differentiating (8) with respect to t. This differentiation results in loss of information because the result is the derivative of an infinite number of equations differing from (8) by a constant.[6] Thus, all solutions satisfy the same marginal condition when the constraints on $s(t)$ are not binding:

$$e^{-rt}\left[\frac{\partial U}{\partial F} + \frac{\partial U}{\partial I}\big(24 - s(t)\big)\frac{dg}{dF}\right] = -\frac{d}{dt}\left[\frac{e^{-rt}\frac{\partial U}{\partial I}w(t)}{\frac{df}{ds}}\right] \tag{9}$$

[3] Strictly speaking, U has to be a cardinal utility function, invariant up to a constant.
[4] See Appendix A for the derivations.
[5] Specifically, $\lambda(t)$ is associated with the constraint $s(t) \geq 0$ and $\mu(t)$ is associated with the constraint $m(t) \geq 0$ or, equivalently, $s(t) \leq 24$.
[6] Hany Eldeib provided this insight.

Equation (9) is known as the "implicit equation of motion." To interpret (8), we difference it over the interval $[t, t+\Delta t]$, again assuming that $\lambda(t) = \mu(t) = 0$.[7] This is equivalent to integrating (9) over the same interval:

$$\int_t^{t+\Delta t} e^{-r\tau} \left[\frac{\partial U}{\partial F} \Big|_\tau + \frac{\partial U}{\partial I} \Big|_\tau (24 - s(\tau)) \frac{dg}{dF} \Big|_\tau \right] d\tau$$
$$= \frac{e^{-rt} \frac{\partial U}{\partial I} \big|_t \, w(t)}{\frac{df}{ds} \big|_t} - \frac{e^{-r(t+\Delta t)} \frac{\partial U}{\partial I} \big|_{t+\Delta t} \, w(t+\Delta t)}{\frac{df}{ds} \big|_{t+\Delta t}} \tag{10}$$

Equation (10) has the usual economic interpretation: the marginal utility of fame earned over the interval $[t, t+\Delta t]$ equals the marginal utility of income foregone over the same interval. The term df/ds in the R.H.S. of the denominator of (8) is the shadow price of income in terms of fame. The larger its value, the smaller is the price of income in terms of fame. This reflects the substitution of fame for income.

Since (8) has no closed form solution, one must resort to numerical approximation. This reduces the dimensionality of the problem from infinity to a large finite number. Appendix B documents the numerical solution. Simply put, a discrete approximation to (9) is obtained and time paths generate until the utility-maximizing path is found. These paths are obtained by an initial grid search on values of $s(0)$. The value of $s(0)$ on this grid which maximizes V is then used to start a search by bisection. The bisection is run for eight steps, providing a precision of $(.5)^8 \approx .004$ for the maximizing $s(0)$. The initial use of grid searching reduces the computations needed to account for legacy as described in Subsection 2.1.

2.1 Fame in the Afterlife

As stated in the introduction, the scientist may be motivated by fame in the afterlife. We model this by altering the objective function in (7) by adding a term reflecting the utility gained by the possession of fame at the end of life $h[F(T)]$,[8] $h' > 0$

$$\hat{V} = \int_0^T e^{-rt} U\big(F(t), I(t)\big) dt + h\left[F(T)\right] \tag{11}$$

Setting the first variation of (11) equal to zero obtains

$$\int_t^T e^{-r\tau} \left[\frac{\partial U}{\partial F} \Big|_\tau + \frac{\partial U}{\partial I} \Big|_\tau (24 - s(\tau)) \frac{dg}{dF} \Big|_\tau \right] d\tau = \frac{e^{-rt} \left[\frac{\partial U}{\partial I} \big|_t \, w(t) - \lambda(t) - \mu(t) \right] - \frac{dh}{dF}\big|_T}{\frac{df}{ds} \big|_t} \tag{12}$$

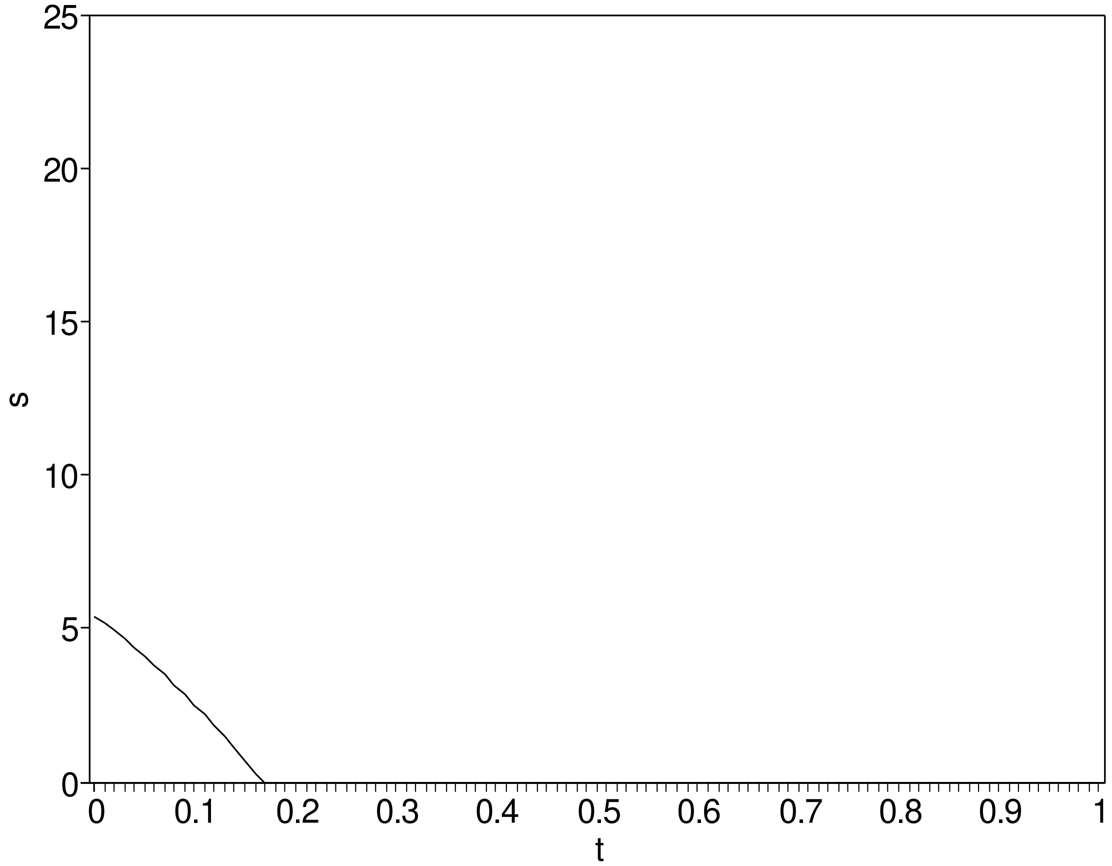
The derivative and interpretation of (12) are identical to those of (8). The addition of a legacy term has no effect on the marginal conditions. Instead, the effect must lie in a change in $s(0)$, which can only be determined by numerical solution.

2.2 Numerical Results

Appendix B describes the numerical approximation technique. In these $U(F(t), I(t)) = \log(\alpha F(t) + \beta I(t))$ and $h[F(t)] = h_0 F(T)$. T is taken to be 1. $f(s(t)) = f_0 s(t)$ and $g(F(t)) = F(t)$. The calculations were done with 100 time steps (i.e., $\Delta t = .01$.)

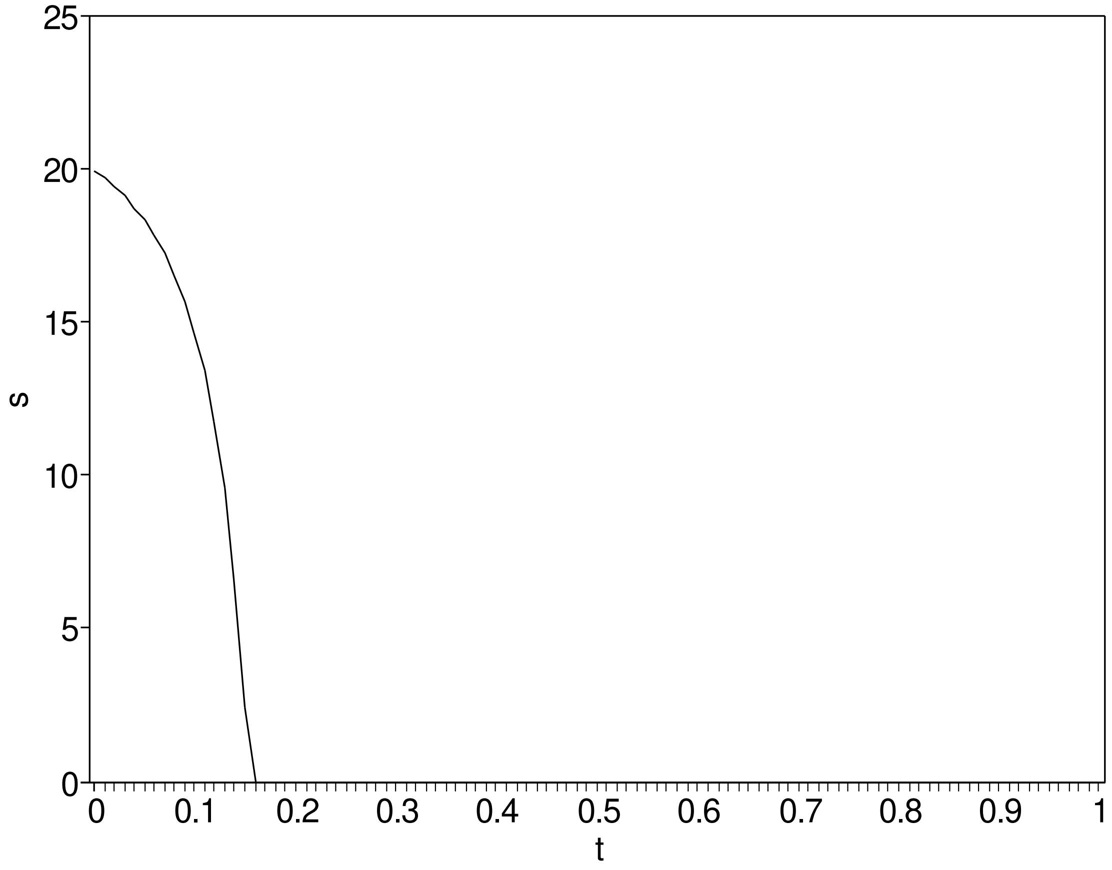
Figures 1 and 2 show typical time paths of solutions to (8). These paths are strictly decreasing. Figures 3 and 4 show a different sort of path. They are characterized by a sharp rise in the level of scientific effort at the beginning of the individual's career, followed by a gradual decline, often to zero. One can think of this as meaning that the individual goofs off as an undergraduate, then goes to graduate school where he devotes a great deal of time to research, and finally ends up living off of his stock of accumulated fame.[9] These time paths seem to be driven by the heavy weight that the scientist puts on fame. In Figure 3, $\alpha = 100$. In figure 4, $h_0 = .009$, which is a very high level in this context. These paths are consistent with the evidence in Cole [1], a general depiction of which is shown as Figure 1 of Diamond [2].

[7] Appendix Subsection A.3 interprets $\lambda(t)$, $\mu(t)$ when the constraints bind.
[8] This forces us into an even stricter cardinal utility environment. There is *no* utility invariance.
[9] Kenneth Koford is responsible in part for this interpretation.



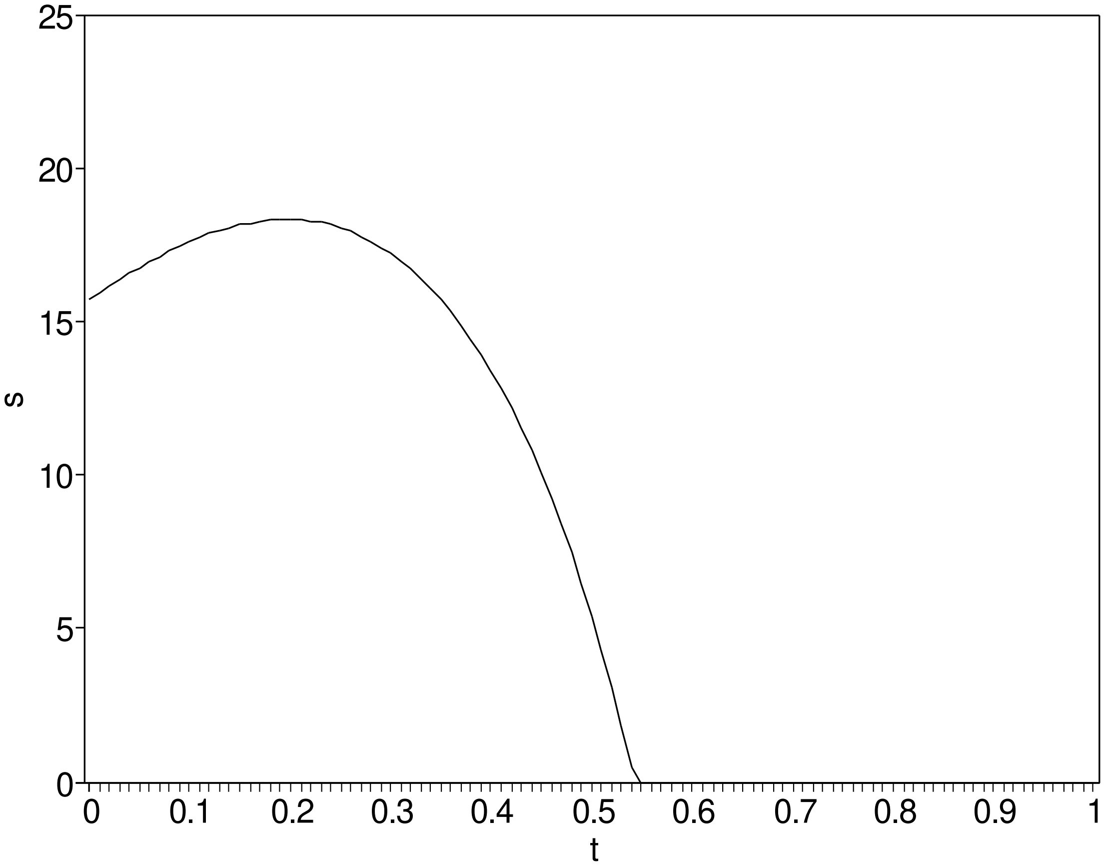
Note: $\alpha = 30$, $\beta = 1$, $r = .8$, $w_0 = 30$, $f_0 = 1$, $T = 1$, $h_0 = 0$

Figure 1: $s(t)$



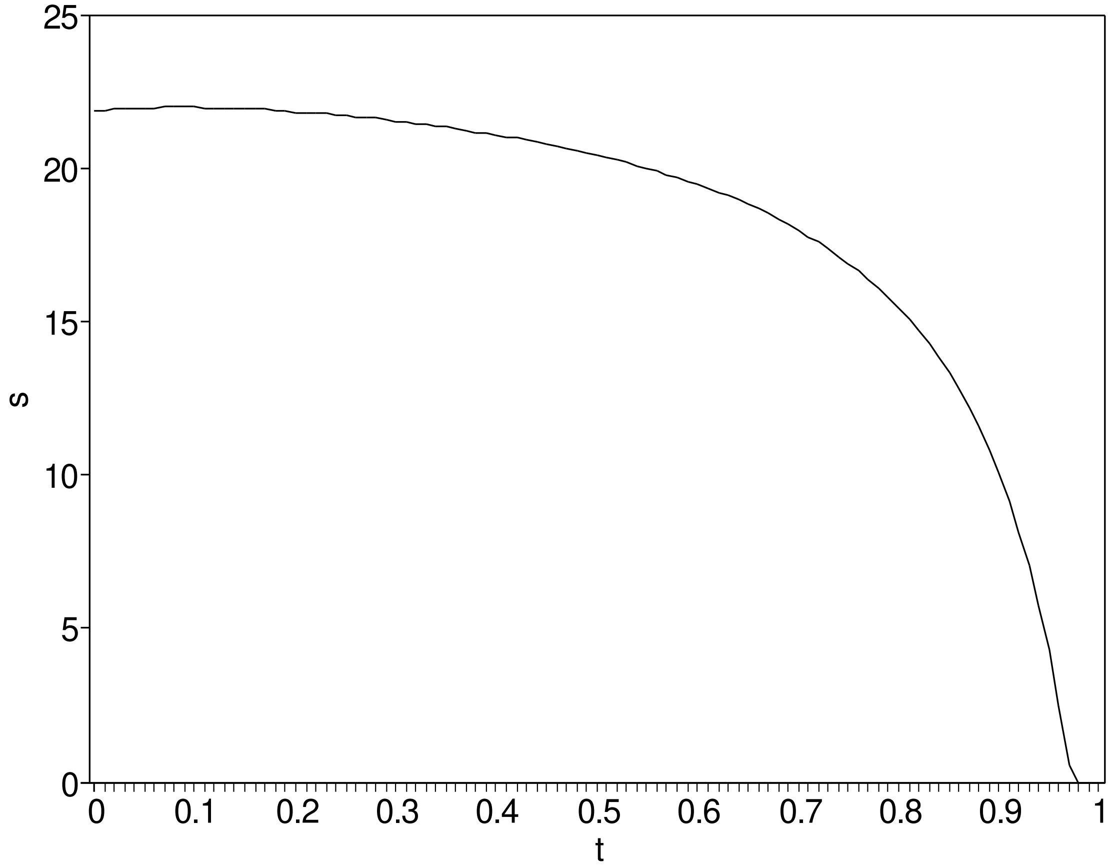
Note: $\alpha = 3$, $\beta = 10$, $r = .8$, $w_0 = 30$, $f_0 = 6$, $T = 1$, $h_0 = 0$

Figure 2: $s(t)$



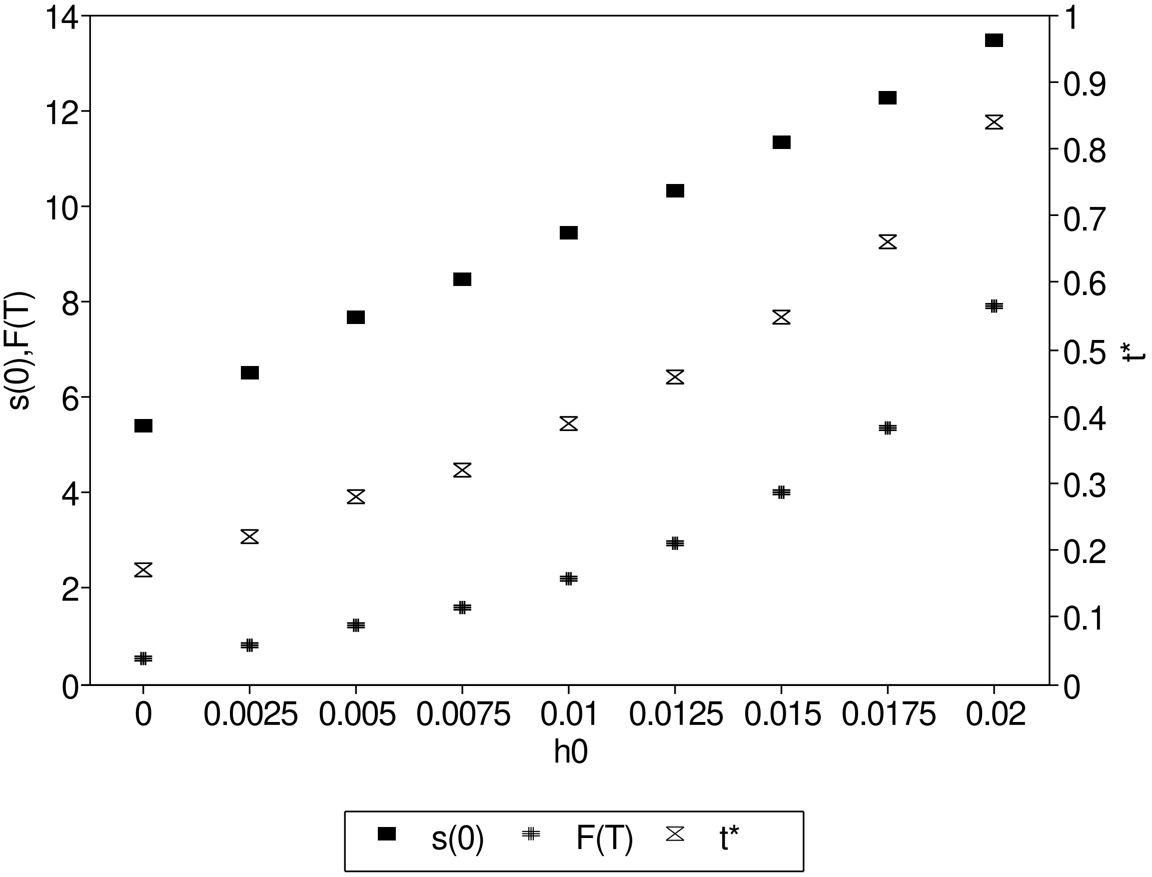
Note: $\alpha = 100$, $\beta = 1$, $r = .8$, $w_0 = 30$, $f_0 = 1$, $T = 1$, $h_0 = 0$

Figure 3: $s(t)$



Note: $\alpha = 3$, $\beta = 10$, $r = .8$, $w_0 = 30$, $f_0 = 6$, $T = 1$, $h_0 = 0.009$

Figure 4: $s(t)$



Note: $\alpha = 30$, $\beta = 1$, $r = .8$, $w_0 = 30$, $f_0 = 1$, $T = 1$

Figure 5: The effect of legacy (h_0) on $s(t)$, $F(T)$, t^*

2.2.1 The Effects of Legacy

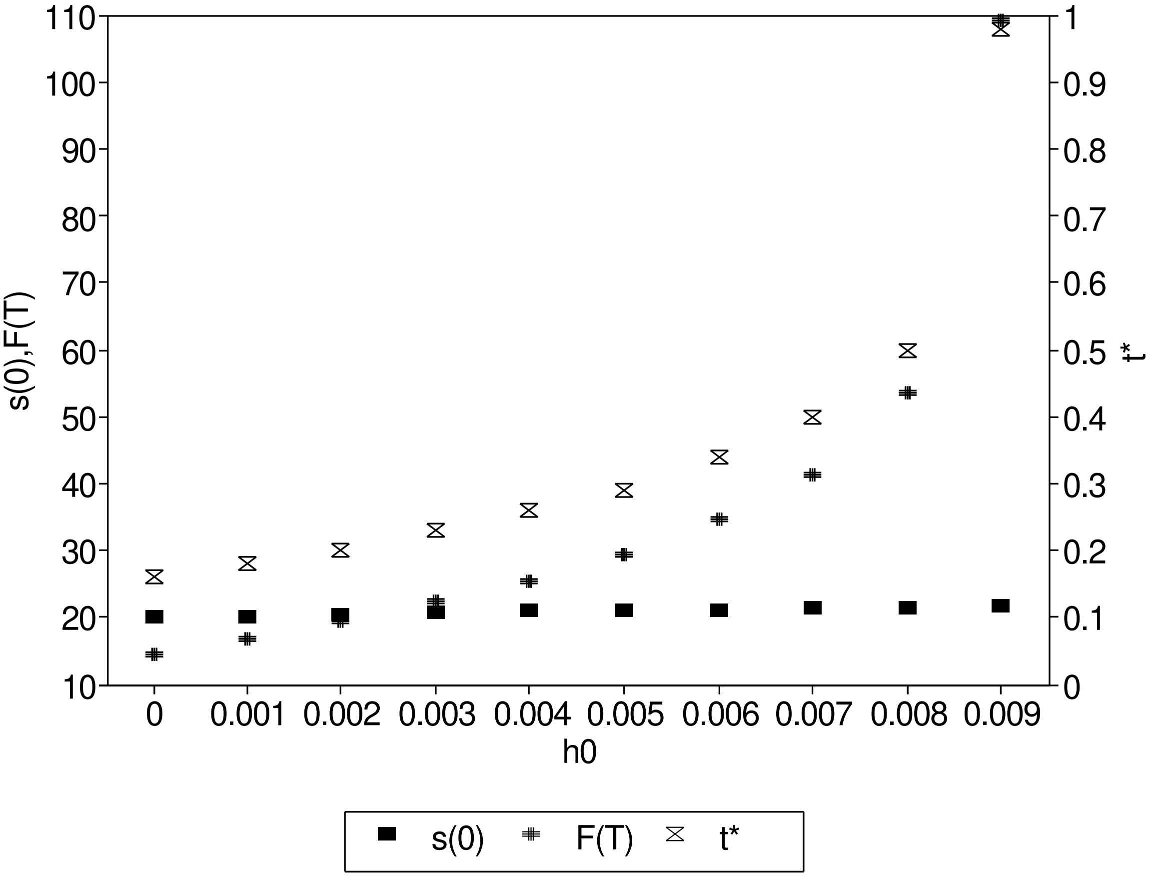
Figures 5 and 6 show $s(0)$, $F(T)$, t^* as functions of h_0. Each of these is increasing, indicating that the substitution effect of legacy dominates.

3 Model for the Amateur-Professional Transition

During the professionalization of an amateur discipline, the demand for scientists increases. We assume that this increase in demand manifests itself in a rise in the base wage w_0. As more scientists enter the field, competition decreases the rewards to scientific work, resulting in less fame for equal amounts of effort (i.e $f^p(\cdot) \leq f^a(\cdot)$ for all nonnegative arguments, where the superscripts p and a refer to the professional and amateur periods, respectively). This is modeled by lowering f_0.

Generally, in cases in which no utility is derived from the possession of human capital per se, decreases in f lead to decreases in human capital accumulation.[10] However, in this model, as Levy hypothesized, the presence of utility from the stock of human capital itself could lead to increased scientific effort. Tables 1 and 2 show $s(0)$ for various values of f_0 and w_0. $s(0)$ falls in both f_0 and w_0. These refute Levy's hypothesis, in keeping with the results from the literature. This also points out the difficulties in working in dynamic

[10] I owe this point to Kenneth Koford. I unfortunately was unable to obtain a reference.



Note: $\alpha = 3$, $\beta = 10$, $r = .8$, $w_0 = 30$, $f_0 = 6$, $T = 1$

Figure 6: The effect of legacy (h_0) on $s(t)$, $F(T)$, t^*

contexts. Levy derived his hypothesis using static indifference curve theory. The intuitions from this theory do not necessarily carry over to dynamic contexts.

Table 1: First Table for Testing Levy's Hypothesis

f_0	w_0	$s(0)$
1	30	5.41
.9	30	2.96
1	31	4.72
.9	31	2.13
.9	50	.66*
.5	50	.66*

Note: $\alpha = 30$, $\beta = 1$, $r = .8$, $h_0 = 0$.
*Indistiguishable from 0 due to numerical inaccuracy.

Table 2: Second Table for Testing Levy's Hypothesis

f_0	w_0	$s(0)$
1	30	.66*
2	30	3.71
3	30	13.84
4	30	17.46
5	30	19.10
6	30	19.93
3	50	.66*
3	31	13.46

Note: $\alpha = 3$, $\beta = 10$, $r = .8$, $h_0 = 0$.
*Indistiguishable from 0 due to numerical inaccuracy.

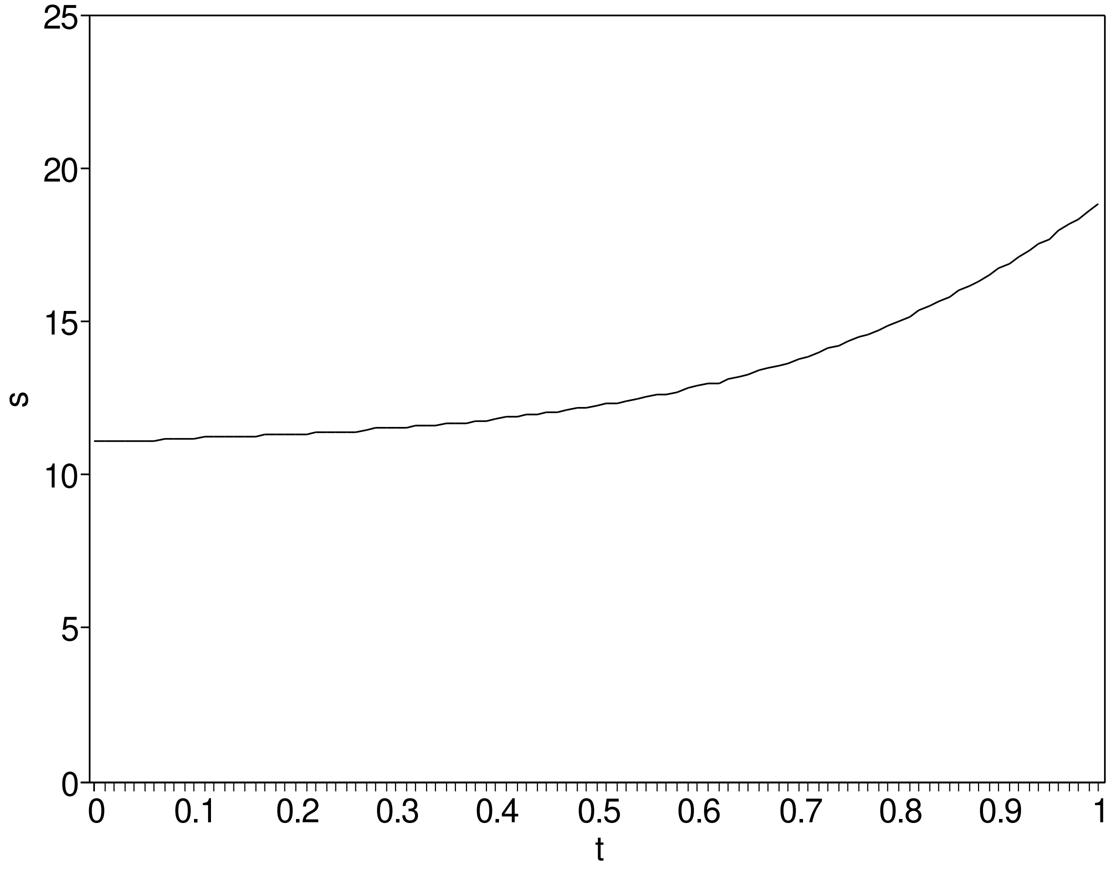
4 A Purely Amateur Profession

Appendix Section B.2 develops the analysis for a purely amateur profession. Equation (32) generates paths with assumption of logarithmic utility. Figures 7 and 8 show two examples of the paths generated when fame does not affect the wage. These correspond to Figures 2 and 3, respectively, with the exception that in Figure 7 h_0 is set to 0.01. Lower values of h_0 do not generate significant scientific effort. The interpretation of these graphs is clear: when scientific effort has no effect on the wage, an individual's utility function must place a heavy weight on fame in order for that individual to undertake efforts towards its acquisition. The shapes of these curves are are interesting. While Figure 8 reproduces the hump of Figure 3, Figure 7 shows a constant increase in $s(t)$. This may reflect the imminence of death and the utility of legacy reflected by the positive h_0. In fact, for values of $h_0 < .00632$ and other parameters as shown, only a tiny amount of scientific effort occurs at the beginning of the individual's career. For greater values of h_0, $t^* > 1 = T$. That is, scientific effort continues throughout life, except for the last instant of the individual's career, at which $s(T) = 0$.[11]

5 Discussion and Conclusions

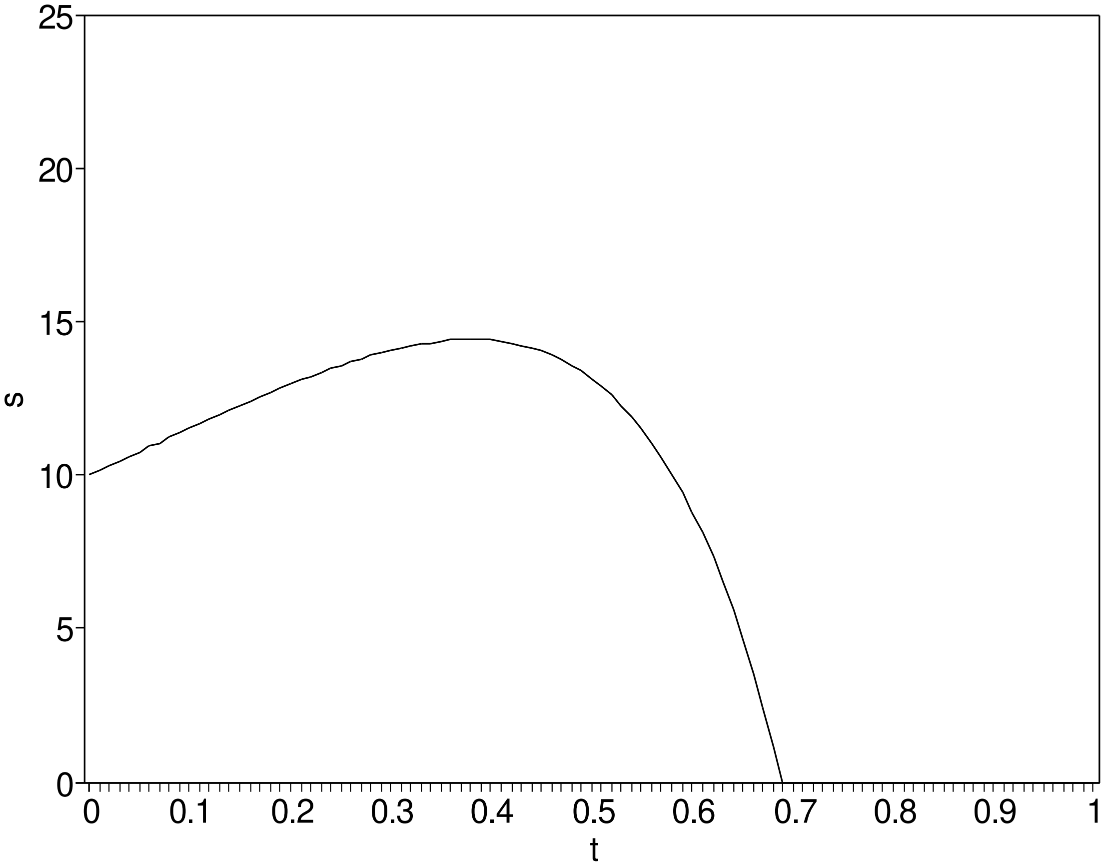
This paper has developed a model of human capital accumulation, when utility is derived, in part, from the stock of capital itself. It has characterized the individual's investment path. These paths can be very different from those of Diamond [2] and Levin and Stephan [4]. In these papers, the amount of time invested in scientific effort is constantly decreasing. In contrast, this paper has shown that, under various scenarios,

[11] This is an example of the discontinous paths discussed in Appendix Section A.4.



Note: $\alpha = 3$, $\beta = 10$, $r = .8$, $w_0 = 30$, $f_0 = 6$, $T = 1$, $h_0 = 0.01$.

Figure 7: $s(t)$ when fame has no effect on income



Note: $\alpha = 100$, $\beta = 1$, $r = .8$, $w_0 = 30$, $f_0 = 1$, $T = 1$, $h_0 = 0$.

Figure 8: $s(t)$ when fame has no effect on income

this time path may be humped, i.e., rising and then falling, or even rising throughout the scientist's career. A general result is that, for the individual to make any significant investment in scientific effort, he must receive a substantial reward, either by a rapid increase of his wage in fame, or simply from the utility generated by fame.

This paper differs from Diamond [2] and Levin and Stephan [4] in other ways. Diamond [2] equates utility with income in a discrete-time framework. Using this utility function in continuous time with a linear translation of scientific effort to fame generates a null Lagrangian. Other choices of utility functions produce complicated expressions with partial derivatives, whose interpretation is unclear. Levin and Stephan [4] use research output instead of fame as a source of utility and increased income. The scientist's utility function is log-additive.[12] Their result that scientific effort decreases over time may be dependent on this choice of utility function.

This framework can be used to explore the results generate by other utility functions. This work can be extended to cover cases involving variable discount rates and the uncertainty of time of death. Another path is to investigate imperfect information of the production possibility set. This work may possibly be used to derive testable hypotheses about human capital accumulation in which utility flows from the stock of capital itself.

A Derivation of Solution Equations

This paper uses a single control variable, $s(t)$, and a single state variable, $F(t)$. The calculus of variations is the solution method. Due to the integral in the state variable (fame), the Euler-Lagrange Equation cannot be used.[13] Therefore, the brute-force approach is used. The control variable is perturbed by a disturbance variable, the perturbed versions of dependent variables are calculated, and the first variation of the objective function is obtained by subtracting the original objective function from its perturbed version. Setting this first variation equal to zero obtains the solution.

A.1 No Utility from Fame in the Afterlife

Equation (7) is repeated for convenience:[14]

$$V = \int_0^T e^{-rt} U\big(F(t), I(t)\big)\, dt$$

The Lagrangean is formed by augmenting (7) with terms representing the contraints (5) and (6)

$$\mathcal{L} = \int_0^T \left[e^{-rt} U\big(F(t), I(t)\big) + \lambda(t)s(t) + \mu(t)\left(s(t) - 24\right)\right] dt$$

where $\lambda(t)$, $\mu(t)$ are the Lagrange multipliers corresponding to the constraints $s(t) \geq 0$ and $s(t) \leq 24$, respectively. Variable $s(t)$ is perturbed by adding variable $\epsilon(t)$:

$$s(t) \to s(t) + \epsilon(t)$$

$F(t)$, $I(t)$, $w(t)$ are then perturbed by substituting $s(t) + \epsilon(t)$ for $s(t)$:

$$\begin{aligned}
F(t) \to \int_0^t f\left(s(\tau) + \epsilon(\tau)\right) d\tau &= \int_0^t \left[f\big(s(\tau)\big) + \epsilon(\tau) \left.\frac{df}{ds}\right|_\tau\right] d\tau + O(\epsilon^2) \\
&= F(t) + \int_0^t \epsilon(\tau) \frac{df}{ds} d\tau + O(\epsilon^2) \\
w(t) \to w_0 + g\left(F(t) + \delta F(t)\right) &= w(t) + \left.\frac{dg}{dF}\right|_t \delta F(t) + O(\delta F^2) \\
I(t) \to I(t) - \epsilon(t) w(t) + \left(24 - s(t)\right) &\left.\frac{dg}{dF}\right|_t \delta F(t) + O(\delta F^2)
\end{aligned}$$

[12] That is, $U(r, I) = \theta_1 \log r + \theta_2 \log I$, where r is research output and I is income.

[13] The Euler-Lagrange Equation gives the solution in terms of derivatives of the state and control variables.

[14] The original, unconstrained derivations in this and Subsection A.2 are by Samuel L. Braunstein and Daniel Neuhauser.

where δ is the first variation operator. We summarize the first variations:[15]

$$\begin{aligned}
\delta s(t) &= \epsilon(t) \\
\delta F(t) &= \int_0^t \epsilon(\tau)\, \left.\frac{df}{ds}\right|_\tau d\tau \\
\delta I(t) &= -\epsilon(t)w(t) + (24 - s(t)) \left.\frac{dg}{dF}\right|_t \delta F(t)
\end{aligned}$$

$\mathcal{L}$ is now perturbed

$$\mathcal{L} \to \mathcal{L} + \int_0^T e^{-rt} \left\{ \left.\frac{\partial U}{\partial F}\right|_t \delta F(t) + \left.\frac{\partial U}{\partial I}\right|_t \delta I(t) + \lambda(t)\,(s(t) + \epsilon(t)) + \mu(t)\,(s(t) + \epsilon(t) - 24) \right\} dt$$

Therefore

$$\begin{aligned}
\delta\mathcal{L} &= \int_0^T e^{-rt} \Big\{ \left.\frac{\partial U}{\partial F}\right|_t \int_0^t \epsilon(\tau)\, \left.\frac{df}{ds}\right|_\tau d\tau \\
&+ \left.\frac{\partial U}{\partial I}\right|_t \left[-\epsilon(t)w(t) + (24 - s(t)) \left.\frac{dg}{dF}\right|_t \int_0^t \epsilon(\tau)\, \left.\frac{df}{ds}\right|_\tau d\tau \right] + \lambda(t)\epsilon(t) + \mu(t)\epsilon(t) \Big\} dt
\end{aligned}$$

We then interchange the integrals, $\int_0^T dt \int_0^t d\tau = \int_0^T d\tau \int_\tau^T dt$:

$$\begin{aligned}
\delta\mathcal{L} = \int_0^T dt \int_t^T d\tau\, e^{-r\tau} &\left\{ \left.\frac{\partial U}{\partial F}\right|_\tau \epsilon(t) \left.\frac{df}{ds}\right|_t + \left.\frac{\partial U}{\partial I}\right|_\tau (24 - s(\tau)) \left.\frac{dg}{dF}\right|_\tau \epsilon(t) \left.\frac{df}{ds}\right|_t \right\} \\
&+ \int_0^T dt\, e^{-rt} \epsilon(t) \left[\lambda(t) + \mu(t) - \left.\frac{\partial U}{\partial I}\right|_t w(t) \right] \qquad (13)
\end{aligned}$$

For V to be a maximum $\delta\mathcal{L}$ must be zero to first order in ϵ change in s, i.e. $\delta\mathcal{L} = 0$. But, $\epsilon(t)$ is an arbitrary change, so the integral of (13) must vanish. Therefore, equation (9) obtains.

A.1.1 Restrictions on Utility Functions

The choice of utility function is important. $U(F, I) = \alpha F + \beta I$ leads to $s(t) \equiv 0$. This an example of a null Lagrangian.[16] Appendix Subsubsection B.1.1 shows the functions which generate null Lagrangians when $f(s(t)) = f_0 s(t)$ and $g[F(t)\} = g_0 F(t)$. Other utility functions lead to singularities when $F(0) = 0$, i.e., $\partial U/\partial F|_{F=0} = \infty$. $U(F, I) = F^\alpha I^\beta$ has partial derivative $\partial U/\partial F = \alpha F^{\alpha-1} I^\beta$, which is infinite when $F = 0$ and $\alpha \le 1$. Choosing $\alpha > 1$ requires the assumption of increasing marginal psychic returns to F.

A.2 Utility from Fame in the Afterlife

Adding $h\,[F(t)]$ to V results in a changed perturbation for $\mathcal{L}$:

$$\mathcal{L} \to \mathcal{L} + \int_0^T e^{-rt} \left\{ \left.\frac{\partial U}{\partial F}\right|_t \delta F(t) + \left.\frac{\partial U}{\partial I}\right|_t \delta I(t) + \lambda(t)\,(s(t) + \epsilon(t)) + \mu(t)\,(s(t) + \epsilon(t)) \right\} + h\,[F(T) + \delta F(T)]$$

This results in the addition of the term $dh/dF|_T \delta F(T)$ to $\delta\mathcal{L}$ in equation (13). Thus, equation (11) obtains.

[15] The first variation of $x(t)$ is technically defined as its first derivative with respect to the perturbation, evaluated at zero perturbation, i.e. $\delta x(t) = d\,(x(t) + \epsilon(t))\,/d\epsilon(t)|_{\epsilon(t)=0}$.

[16] Giaquinta and Hildebrandt [3] p. 51–59.

A.3 Interpretation of Lagrange Multipliers

Whenever the constraint $s(t) \geq 0$ binds, $\lambda(t) \neq 0$. Likewise, $\mu(t) \neq 0$ whenever the constraint $s(t) \leq 24$ binds. Thus, at most one of the multipliers can differ from zero at any t. We solve equation (12) for $\lambda(t)$, noting $s(t) = 0$:

$$\lambda(t) = e^{-rt} \left.\frac{\partial U}{\partial I}\right|_t - \left.\frac{dh}{dF}\right|_T - \left.\frac{df}{ds}\right|_t \int_t^T e^{-r\tau} \left[\left.\frac{\partial U}{\partial F}\right|_\tau + \left.\frac{\partial U}{\partial I}\right|_\tau \cdot 24 \left.\frac{dg}{dF}\right|_\tau \right] d\tau \tag{14}$$

Similarly, we solve equation (8) for $\mu(t)$, noting $s(t) = 24$:

$$\mu(t) = e^{-rt} \left.\frac{\partial U}{\partial I}\right|_t - \left.\frac{dh}{dF}\right|_T - \left.\frac{df}{ds}\right|_t \int_t^T e^{-r\tau} \left.\frac{\partial U}{\partial F}\right|_\tau d\tau \tag{15}$$

The reader may observe that $\lambda(t)$, $\mu(t)$ become zero at the instants at which their constraints no longer bind.

A.4 Solution Properties around $t = T$

Given that $F(T)$ is determined, $s(T)$ can be found by maximizing $U(T)$ subject to conditions (2)–(6). This is done by substituting (4) and (5) into (1) at time T:

$$U(T) = U\left(F(T), (24 - s(T))\, w(T)\right) \tag{16}$$

The partial derivative with respect to $s(T)$ is

$$\left.\frac{\partial U}{\partial s}\right|_T = -w(T) \left.\frac{\partial U}{\partial I}\right|_T < 0$$

This indicates a corner solution with the maximizing $s(T) = 0$.[17] Any continuous solution for $s(t)$ must contain an interval $[t^*, T]$, $t^* \leq T$, on which $s(t) = 0$.[18]

From equation (14)

$$\lambda(T) = e^{-rT} \left.\frac{\partial U}{\partial I}\right|_T - \left.\frac{dh}{dF}\right|_T \tag{17}$$

For $dh/dF|_T < e^{-rT} \partial U/\partial I|_T$, $\lambda(t) > 0$ on (t^*, T), by continuity. Also from (14), $\partial\lambda(t)/\partial(dh/dF|_T) = -1$. Therefore, by continuity, t^* rises in $dh/dF|_T$, i.e., $dt^*/d(dh/dF|_T) > 0$. Essentially, the $\lambda(t)$ curve is displaced vertically, resulting in a shorter segment below the t-axis. $s(t)$ increases globally, and $F(T)$ rises. In short, increased utility from legacy causes increased scientific effort resulting in higher fame at the end of life.

When $dh/dF|_T > e^{-rT} \partial U/\partial I|_T$, equation (12) lacks a solution. This can be seen by noting that $\lambda(T) < 0$ and $d\lambda/dt|_T < 0$. Thus, there is no continuous function λ which has a point t^* at which $\lambda(t^*) = 0$.[19]

Finally, $dh/dF|_T = e^{-rT} \partial U/\partial I|_T$ implies $t^* = T$.

B Numerical Solution Technique

Equation (9) is a type of initial value problem, which is solved by grid search and bisection. A discrete version of (9) is solved on the grid $s(0) = 1, 2, \ldots, 23$. Total utility V is computed by trapezoidal rule integration for each value of $s(0)$. The values of V are searched to find a maximum, at which bisection is used to find the maximum value. This has the advantage of not having to recompute the grid when h_0 changes.

[17] Note that fame in the afterlife has no effect on this result, as it depends on the predetermined $F(T)$.

[18] Discontinuous solutions need not have this property, but they violate the marginal condition (9).

[19] One possibility is that the solution follows equation (12) until $t = T$, when it has a discontinuous jump to zero.

B.1 Discretizing Equation (9)

We begin by rewriting equation (9) in the form $G(t,F,F')=0$, where $F'(t)=dF(t)/dt=f(s(t))$ and substituting for $w(t)$:

$$e^{-rt}\left[\frac{\partial U}{\partial F}+\frac{\partial U}{\partial I}\left(24-f^{-1}(F')\right)\frac{dg}{dF}\right]+\frac{d}{dt}\left[\frac{e^{-rt}\frac{\partial U}{\partial I}\left(w_0+g(F)\right)}{\frac{df}{ds}}\right]=0 \tag{18}$$

Equation (9) is greatly simplified by assuming linear forms for f and g:

$$f\big(s(t)\big)=f_0 s(t),\; g(F)=g_0 F$$

A further simplification ensues from taking $g_0=1$ since the product $f_0 g_0$ determines $w(t)$, i.e. $w(t)=g(F(t))=g_0\int_0^t f_0 s(\tau)d\tau=f_0 g_0\int_0^t s(\tau)d\tau$:

$$e^{-rt}\left[\frac{\partial U}{\partial F}+\frac{\partial U}{\partial I}\left(24-\frac{F'}{f_0}\right)\right]+\frac{d}{dt}\left[\frac{e^{-rt}\frac{\partial U}{\partial I}\left(w_0+F\right)}{f_0}\right]=0 \tag{19}$$

Multiplying through by f_0 obtains

$$e^{-rt}\left[f_0\frac{\partial U}{\partial F}+\frac{\partial U}{\partial I}\left(24f_0-F'\right)\right]+\frac{d}{dt}\left[e^{-rt}\frac{\partial U}{\partial I}\left(w_0+F\right)\right]=0 \tag{20}$$

The derivative is then expanded, where the subscripts on U refer to partial derivatives:

$$\begin{aligned}&\frac{d}{dt}\left[e^{-rt}\frac{\partial U}{\partial I}\left(w_0+F\right)\right]\\ =\;& e^{-rt}\left\{(w_0+F)\left[U_{IF}F'+U_{II}\left(\left(24-\frac{F'}{f_0}\right)F'-\frac{(w_0+F)F''}{f_0}\right)-rU_I\right]+U_IF'\right\}\end{aligned} \tag{21}$$

Substituting the above in equation (20) and eliminating e^{-rt} obtains

$$f_0U_F+24f_0U_I+(w_0+F)\left[U_{IF}F'+U_{II}\left(\left(24-\frac{F'}{f_0}\right)F'-\frac{(w_0+F)F''}{f_0}\right)-rU_I\right]=0 \tag{22}$$

This can be solved for F'':

$$F''=\frac{f_0}{U_{II}(w_0+F)}\left\{f_0\frac{U_F+24U_I}{w_0+F}+F'U_{IF}+U_{II}\left[\left(24-\frac{F'}{f_0}\right)F'-rU_I\right]\right\} \tag{23}$$

Without knowing the values of the variables it is impossible to sign F'' and, therefore, s.

Having obtained an expression for F'', Euler's forward method can be used to generate paths. F' is obtained by integrating (23), then F is obtained by integrating F'. Formally, this is done by the matrix equation

$$\begin{bmatrix}F'(t+\Delta t)\\ F(t+\Delta t)\end{bmatrix}=\begin{bmatrix}F'(t)\\ F(t)\end{bmatrix}+\Delta t\left.\begin{bmatrix}\frac{f_0}{U_{II}(w_0+F)}\left\{f_0\frac{U_F+24U_I}{w_0+F}+F'U_{IF}+U_{II}\left[\left(24-\frac{F'}{f_0}\right)F'-rU_I\right]\right\}\\ F'\end{bmatrix}\right|_t \tag{24}$$

It should be noted that equation (22) contains some subtleties. If F equals a constant $\bar{F}$, $F'=F''=0$. Then (22) becomes quadratic in $\bar{F}$. If its roots are real, then it will have one or two singular solutions $F=\bar{F}$.[20] In practice, a numerical solution can converge to a singular one.

[20] For more information about singular solutions, see Zwillinger [9] pp. 540–542.

B.1.1 Null Lagrangians

The space of utility functions generating null Lagrangians is generated by taking $U_{II} \equiv 0$.[21] This corresponds to utility functions of the form $U(F,\, I) = m(F)I + C_1F + C_2I + C_3$.[22] This has partial derivatives

$$\begin{aligned} U_F &= Im'(F) + C_1 \\ U_I &= m(F) + C_2 \\ U_{IF} &= m'(F) \end{aligned}$$

Equation (22) becomes

$$f_0(Im'(F) + C_1) + 24f_0(m(F) + C_2) + (w_0 + F)\left[m'(F)F' - r(m'(F) + C_1)\right] = 0 \tag{25}$$

After expanding I, the terms in F' cancel, resulting in an equation for F with $F' \equiv 0$. This is like a singular solution, but with the difference that no other solutions are possible.

B.1.2 Logarithmic Utility

Let $U(F,\, I) = \log(\alpha F + \beta I)$. This has derivatives

$$\begin{aligned} U_F &= \frac{\alpha}{\alpha F + \beta I} = \alpha e^{-U} \\ U_I &= \frac{\beta}{\alpha F + \beta I} = \beta e^{-U} \\ U_{FI} &= -\frac{\alpha\beta}{(\alpha F + \beta I)^2} = -\alpha\beta e^{-2U} \\ U_{II} &= -\frac{\beta^2}{(\alpha F + \beta I)^2} = -\beta^2 e^{-2U} \end{aligned}$$

Equation (23) becomes

$$F'' = \frac{f_0}{\beta(w_0 + F)}\left\{\beta F'\left[\alpha + \beta\left(24 - \frac{F'}{f_0}\right)\right] - f_0\frac{(\alpha + 24\beta)(\alpha F + \beta I)}{w_0 + F} + \frac{\beta^3 r}{\alpha F + \beta I}\right\} \tag{26}$$

It should be noted that equation (26) has a pole at $F = -w_0$, or, equivalently, $w = 0$. That is, it diverges at this point. To avoid numerical difficulties, one should choose $w_0 \gg 0$.

B.2 Numerical Analysis of a Purely Amateur Profession

When fame does not influence the wage, $dg/dF = 0$. Assuming no exogenous wage growth, $w(t) = w_0$. This results in a modification of equation (19):[23]

$$e^{-rt}\left[\frac{\partial U}{\partial F} + \frac{\partial U}{\partial I}\left(24 - \frac{F'}{f_0}\right)\right] + \frac{d}{dt}\left[\frac{e^{-rt}\frac{\partial U}{\partial I}w_0}{f_0}\right] = 0 \tag{27}$$

Upon multiplication by f_0, this becomes

$$e^{-rt}\left[f_0\frac{\partial U}{\partial F} + \frac{\partial U}{\partial I}(24f_0 - F')\right] + \frac{d}{dt}\left[e^{-rt}\frac{\partial U}{\partial I}w_0\right] = 0 \tag{28}$$

The evaluation of the derivative is simplified:

$$\begin{aligned} &\frac{d}{dt}\left[e^{-rt}\frac{\partial U}{\partial I}w_0\right] \\ = \;& e^{-rt}w_0\left[U_{IF}F' + w_0U_{II}\left(24 - \frac{F''}{f_0}\right) - rU_I\right] \end{aligned} \tag{29}$$

[21] These are noted on in Appendix Subsubsection A.1.1

[22] m is taken not to include a constant. Including a constant eliminates the term C_2I from U, with no effect on the argument.

[23] This modification can obtained by the technique used in Appendix A

Substituting the above into equation (28) and eliminating e^{-rt} obtains

$$f_0 U_F + U_I\left(24 f_0 - F'\right) + w_0\left[U_{IF}F' + w_0 U_{II}\left(24 - \frac{F''}{f_0}\right) - rU_I\right] \tag{30}$$

This is solved for F'':

$$F'' = \frac{f_0}{w_0 U_{II}}\left[\frac{f_0 U_F + U_I(24 f_0 - F')}{w_0} + U_{IF}F' - rU_I + 24 w_0 U_{II}\right] \tag{31}$$

Euler's forward method is again used to obtain numeric solutions. The same provisos about null Lagrangians and singular solutions apply.

B.2.1 Logarithmic Utility

Assuming again that $U(F, I) = \log(\alpha F + \beta I)$ obtains

$$F'' = \frac{f_0}{w_0 \times -\beta^2 e^{-2U}}\left[\frac{f_0 \alpha e^{-U} + \beta e^{-U}(24 f_0 - F')}{w_0} + -\alpha\beta e^{-2U}F' - r\beta e^{-U} + 24 w_0 \times -\beta^2 e^{-2U}\right]$$

which simplifies to

$$F'' = \frac{f_0}{w_0 \times -\beta^2 e^{-U}}\left[\frac{f_0 \alpha + \beta(24 f_0 - F')}{w_0} + -\alpha\beta e^{-U}F' - r\beta + 24 w_0 \times -\beta^2 e^{-U}\right]$$

or

$$F'' = \frac{f_0}{w_0 \beta^2}\left[(\alpha F + \beta I)\left(\beta r - \frac{f_0(\alpha + 24\beta) - \beta F'}{w_0}\right) + \alpha\beta F' + 24 w_0 \beta^2\right] \tag{32}$$

B.3 Computation Details

The computations and graphics were performed in Quattro Pro for DOS v. 5.0 on an 80386SX.

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