On the Dynamics of Solar Cell Optoelectronic Device Parameters with the Lambert W Function

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Abstract:

We present new views of solar cell optoelectronic device parameters, based on the standard one-diode real solar cell model with parasitic series and shunt resistance, using a systematic application of the Lambert W function. We also show that the one-diode real solar cell model is physically meaningless without the normalization condition $I_0 R_{\text{sh}} = AV_{\text{th}}$, where I_0 is the reverse saturation current, R_{sh} is the shunt resistance, A is the diode quality factor and V_{th} is the thermal voltage. A comparative analysis of the solutions obtained herein and those in standard literature is also presented. Graphs that contain all solutions for any possible combination of solar cell parameters are plotted, effectively adding an extra dimension to the standard theory of solar cells under both illuminated and dark conditions.

Keywords: Photovoltaics, Non-ideal solar cells, Diode quality factor, Current-voltage characteristics, Lambert W function, Wein's displacement law.

1 The standard one - diode real solar cell model

The standard equivalent model of an illuminated real solar cell consists of a diode and a current source that are switched in parallel, along with a shunt and series resistance as shown in figure 1 (a) [1-5]. The current source

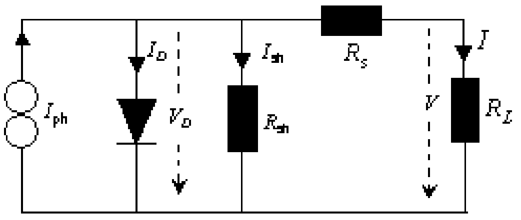

Figure 1 (a): Solar cell model under Illumination

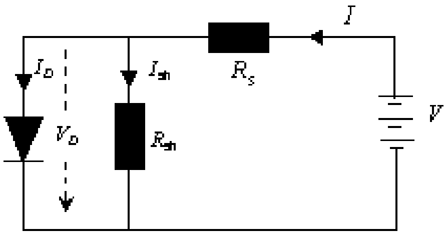

Figure 1 (b): Solar cell model under dark conditions with bias potential difference, V .

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generates a photocurrent, I_{ph}, due to incident radiant energy. The electric charge transition area of the solar cell is equivalent to a big diode in admissive orientation with a potential difference V_D across its terminals. A voltage drop is thought to occur on the way to the external circuit across the terminals of a (lumped) parasitic series resistance, R_s. Similarly, a leakage current could exist through a parallel (or shunt) resistor, R_sh. The output current, I, then passes through a load resistor, R_L, across which a potential difference, V, exists and is also the output voltage of the solar cell. Photogenerated electric charge per unit area per unit time must be conserved in accordance to the conservation law of electric charge. Hence, at every instant, the algebraic sum of currents per unit area flowing into any node of the solar cell model must be equal to zero (i.e. Kirchhoff's Current Law), namely

$$I = I_\mathrm{ph} - I_D - I_\mathrm{sh}\,. \tag{1}$$

Further, the energy required to move a unit positive charge between the terminals of the diode must be conserved at every instant, in accordance with the conservation law of energy. Since the electric potential difference between any two points is the amount of energy required to move a unit positive charge between the two points, it follows that for a short time interval, we must have

$$qV_D = \begin{cases} qI_\mathrm{sh}R_\mathrm{sh} \\ \\ q\left(V + IR_\mathrm{s}\right) \end{cases} = \text{constant} \tag{2}$$

where the electronic charge, q, is constant. Further since q is constant, we also have

$$V_D = \begin{cases} V + IR_\mathrm{s} \\ \\ I_\mathrm{sh}R_\mathrm{sh} \end{cases} = \text{constant}\quad , \tag{3}$$

from which the shunt current, I_sh, can be expressed as

$$I_\mathrm{sh} = \frac{V_D}{R_\mathrm{sh}} = \frac{V + IR_\mathrm{s}}{R_\mathrm{sh}}\,. \tag{4}$$

The current through the diode, I_D, is given by Shockley equation [5] as

$$I_D = I_\mathrm{o}\left(\exp\left[\frac{qV_D}{AkT}\right] - 1\right) \qquad \text{or} \tag{5}$$

$$I_D = I_\mathrm{o}\left(\exp\left[\frac{q\left(V + IR_s\right)}{AkT}\right] - 1\right) \tag{6}$$

where, I_o is the reverse saturation current, k is the Boltzmann constant, T is the average operating thermodynamic temperature of the solar cell and A is the diode quality factor defined such that $A \geq 1$. Substituting expressions for

I_{sh} and I_D in (4) and (6), respectively, into (1) results in the standard one-diode real solar cell equation for an illuminated solar cell given as

$$I = I_{\text{ph}} - I_{\text{o}}\left(\exp\left[\frac{q(V + IR_{\text{s}})}{AkT}\right] - 1\right) - \frac{V + IR_{\text{s}}}{R_{\text{sh}}}. \tag{7}$$

On the other hand, a solar cell under dark conditions does not have photogenerated charges (see figure 1(b)) and electric energy has to be supplied to the solar cell via a direct current bias potential, V , at its terminals. For this case it can be shown, in a similar manner, that the current passing through a solar cell is given by

$$I = I_{\text{o}}\left(\exp\left[\frac{q(V - IR_{\text{s}})}{AkT}\right] - 1\right) + \frac{V - IR_{\text{s}}}{R_{\text{sh}}}. \tag{8}$$

Hence, the current flowing into or out of a solar cell at every instant can be expressed, transcendentally, as

$$I = \begin{cases} I_{\text{ph}} - I_{\text{o}}\left(\exp\left[\dfrac{q(V + IR_{\text{s}})}{AkT}\right] - 1\right) - \dfrac{V + IR_{\text{s}}}{R_{\text{sh}}} & \text{under illumination} \\ I_{\text{o}}\left(\exp\left[\dfrac{q(V - IR_{\text{s}})}{AkT}\right] - 1\right) + \dfrac{V - IR_{\text{s}}}{AkT} & \text{in dark conditions} \end{cases}. \tag{9}$$

Usually, current-voltage curves for a solar cell under illuminated and dark conditions have shapes that are similar to those in figure 2(a) and (b).

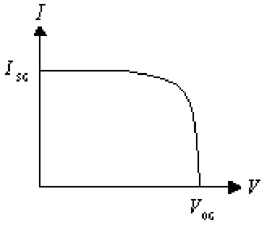

Figure 2(a): Current -voltage curve of a solar cell under illumination.

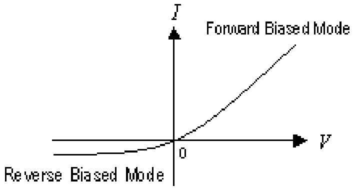

Figure 2(b): Current-voltage curve of a solar cell under dark conditions.

Measuring current – voltage characteristics of photovoltaic solar cells, under either illuminated or dark conditions or both, and extracting a set of solar cell optoelectronic device parameters, $\{I_{\text{ph}}, I_{\text{o}}, A, R_{\text{s}}, R_{\text{sh}}\}$, from the data is the conventional method of assessing these optoelectronic devices. However the theory relating to the application of (9) on real experimentally determined solar cell current – voltage curves, is still a subject of active research. Various authors have suggested different methods for the extraction of solar cell optpelectronic parameters, some of whose application result in (physically meaningless) negative values of series resistance R_{s} [8,9]. Inspection of literature shows that all methods that result in negative values of R_{s} employ two conventional approximations, namely $I_{\text{ph}} = I_{\text{sc}}$ and $R_{\text{sh}} = \infty$. In this paper, we attempt to address some of the problems faced in optoelectronic parameter extraction without making use of these approximations. Additionally, we also present some of the deterrent effects of using conventional approximations in solar cell parameter extraction.

2 The non-dimensional characteristic solar cell equation

The numerical physical restriction on temperature, namely $T > 0$, allows us to express (9) in a simple expression as

$$z = \exp[x] + xy \qquad \begin{cases} x = x_i \text{ and } z = z_i \text{ under illuminated conditions} \\ x = x_d \text{ and } z = z_d \text{ under dark conditions} \end{cases}, \tag{10}$$

where

$$x = \begin{cases} x_i = \dfrac{V + IR_\text{s}}{AV_\text{th}} & \text{under illuminated conditions} \\ x_d = \dfrac{V - IR_\text{s}}{AV_\text{th}} & \text{under dark conditions} \end{cases}, \tag{11}$$

$$y = \frac{AV_\text{th}}{I_\text{o} R_\text{sh}}, \quad \text{(i.e. } y \text{ has the same expression under both illuminated and dark conditions)} \tag{12}$$

$$z = \begin{cases} z_i = 1 + \eta_i & \text{under illuminated conditions} \\ z_d = 1 + \eta_d & \text{under dark conditions} \end{cases}, \tag{13}$$

$$\eta = \begin{cases} \eta_i = \dfrac{I_\text{ph} - I}{I_\text{o}}, & \text{under illuminated conditions} \\ \eta_d = I / I_\text{o}, & \text{under dark conditions} \end{cases} \quad \text{and} \tag{14}$$

$$V_\text{th} = kT/q \text{ (i.e. the thermal voltage).} \tag{15}$$

We will hereafter refer to (10) as the *characteristic non-dimensional real for the solar cell equation* for the model because the equation is valid for both illuminated and dark conditions. For an illuminated solar cell, we have $I \geq 0$, $V \geq 0$, $R_s > 0$, $R_\text{sh} > 0$ $A \geq 1$, $I_\text{ph} \geq 0$ and $I_\text{ph} >> I_\text{o} > 0$. Consequently, we must have $x \geq 0$, $y > 0$ and $z \geq 1$. On the other hand, a solar cell under dark conditions can be reverse biased, hence in this case we could have $-\infty < x < \infty$, $y > 0$ and $-\infty < z < \infty$. $x < 0$ when $IR_\text{s} > V$ while $z < 0$ when $I < -I_o$.

3 A necessary condition for the non-dimensional real solar cell equation

We can express the characteristic non-dimensional solar cell equation, (10), as

$$x = \ln|z - xy| \quad \text{or} \tag{16}$$

$$\ln(z - xy)^{1/x} = 1 \quad \text{or} \tag{17}$$

$$(z - xy)^{1/x} = e \approx 2.7183 = \text{constant}. \tag{18}$$

It is necessary for any consistent method used for the extraction of solar cell parameters, along with approximations used (if any), to be in agreement with (18) because (18) is an equivalent expression for (9) and (10).

4 Unveiling errors in the conventional methods of extracting solar cell optoelectronic parameters

During solar cell parameter extraction, it is common practice to make use of two highly conventionally accepted substitutions, namely $I_{\text{ph}} = I_{\text{sc}}$ and/or $R_{\text{sh}} = \infty$. Now suppose $I_{\text{ph}} = I_{\text{sc}}$ and $0 < R_{sh} < \infty$. Then (7) can be expressed as

$$I_{\text{sc}} - I = I_{\text{o}}\left(\exp\left[\frac{q(V + IR_{\text{s}})}{AkT}\right] - 1\right) + \frac{V + IR_{\text{s}}}{R_{\text{sh}}}\,. \tag{19}$$

Under short circuit conditions we have $I = I_{\text{sc}}$, so that

$$I_{\text{sc}} - I_{\text{sc}} = I_{\text{o}}\left(\exp\left[\frac{q(V + I_{\text{sc}}R_{\text{s}})}{AkT}\right] - 1\right) + \frac{V + I_{\text{sc}}R_{\text{s}}}{R_{\text{sh}}} \qquad \text{or} \tag{20}$$

$$0 = I_{\text{o}}\left(\exp\left[\frac{q(V + I_{\text{sc}}R_{\text{s}})}{AkT}\right] - 1\right) + \frac{V + I_{\text{sc}}R_{\text{s}}}{R_{\text{sh}}}\,. \tag{21}$$

Clearly, if $I_{\text{o}} \neq 0$, $0 < R_{\text{sh}} < \infty$ and $1 < A < \infty$, then the right hand side of (21) is equal to the left hand side if and only if

$$V + I_{\text{sc}}R_{\text{s}} = 0 \qquad \text{or} \tag{22}$$

$$R_{\text{s}} = -\frac{V}{I_{\text{sc}}}\,. \tag{23}$$

Since $V > 0$ and $I_{\text{sc}} > 0$, we see that values of R_{s} will be negative and that this condition is primarily due to the assertion $I_{\text{ph}} = I_{\text{sc}}$. Further, if we somehow manage to get $R_{\text{s}} > 0$ with $I_{\text{ph}} = I_{\text{sc}}$, then a comparative graph of I based on (7) and those of experimental values of I against V can result in two possibilities depending on the : either (a) a seemingly good matching between the computed and experimental of current-voltage curves when $I \to I_{\text{sc}}$ and $I \to 0$, but a mismatch on the "knee" of current – voltage curve or (b) the graphs of computed values o I using (7) will appear to match with those of experiment for $I \to I_{\text{sc}}$ but result in a mismatch as $I \to 0$ with computed values of I approaching zero earlier than those of experiment. The graph of computed values of I will effectively appear as though it is shifted by amount $I_{\text{sc}}R_{\text{s}}$ to the left of V_{oc}.

Now consider the case in which $I_{\text{ph}} = I_{\text{sc}}$ and $R_{\text{sh}} = \infty$. Then from (12) and (13), we see that $y = 0$ and $z = 1 + \frac{I_{\text{sc}} - I}{I_{\text{o}}}$, so that for the case of illuminated solar cells, (18) becomes

$$\left(1 + \frac{I_{\text{sc}} - I}{I_{\text{o}}}\right)^{\frac{AV_{\text{th}}}{V + IR_{\text{s}}}} = e\,. \tag{24}$$

Under short circuit conditions, experiments require that $(V, I) = (0, I_{\text{sc}})$, so that (24) would reduce into

$$\left(1+\frac{I_{\text{sc}}-I_{\text{sc}}}{I_{\text{o}}}\right)^{\frac{AV_{\text{th}}}{I_{\text{sc}}R_{\text{s}}}}=e \quad \text{or} \tag{25}$$

$$1=e\ . \tag{26}$$

Clearly, the conventionally accepted assertions $I_{\text{ph}}=I_{\text{sc}}$ and $R_{\text{sh}}=\infty$ result to a complete breakdown of the constant form of the characteristic non-dimensional real solar cell equation, (18). The percentage difference in the violation of the left hand side of (18) can easily be calculated as

$$\text{error}=\frac{\text{correct value - approximated value}}{\text{correct value}}\times 100\% \quad \text{or} \tag{27}$$

$$\text{error}=\frac{e-1}{e}\times 100\% \quad \text{or} \tag{28}$$

$$\text{error } \approx 63.2\%\ . \tag{29}$$

Although there is such an error in (18), it should be noted that the conventionally accepted expression for the open circuit voltage, V_{oc}, stems from (24) by taking logarithms on both side then first solving for V as

$$V=AV_{\text{th}}\ln\left|1+\frac{I_{\text{sc}}-I}{I_{\text{o}}}\right|-IR_{\text{s}} \tag{30}$$

so that as $I\rightarrow 0$ and $V\rightarrow V_{\text{oc}}$, we get the conventional expression for V_{oc} [5] as

$$V_{\text{oc}}=AV_{\text{th}}\ln\left|1+\frac{I_{\text{sc}}}{I_{\text{o}}}\right| \tag{31}$$

Hence, while (31) is conventionally used to analyze V_{oc}, its origins are mathematically inconsistent.

5 Solvability of x in the problem $z-\exp[x]-xy=0$ using the Lambert W function

We can express (10) as

$$\frac{z}{y}=\frac{\exp[x]}{y}+x, \qquad y\neq 0 \tag{32}$$

which upon exponentiation and dividing both sides of by y gives

$$\frac{\exp\left[z/y\right]}{y}=\frac{\exp[x]}{y}\exp\left[\frac{\exp[x]}{y}\right] \quad , \qquad y\neq 0 \quad . \tag{33}$$

Now (33) can be solved for $\exp[x]/y$ using the Lambert W function as

$$\frac{\exp[x]}{y}=W_0\left[\frac{\exp\left[z/y\right]}{y}\right] \quad , \qquad y\neq 0, \tag{34}$$

where W_0 is the principal branch of the Lambert W function[11-16]. Solving for $\exp[x]$ in (34) gives

$$\left.\begin{array}{l}\exp[x]\\ \\ z-xy\end{array}\right\}=yW_0\left[\frac{\exp\left[{}^{z}\!/\!_{y}\right]}{y}\right] \quad , \qquad y\neq 0\,, \tag{35}$$

from which the solution of x is obtained in two equivalent forms as

$$x=\left\{\begin{array}{l}\ln\left|yW_0\left[\dfrac{\exp\left[{}^{z}\!/\!_{y}\right]}{y}\right]\right| \\ \\ \dfrac{z}{y}-W_0\left[\dfrac{\exp\left[{}^{z}\!/\!_{y}\right]}{y}\right]\end{array}\right. \quad , \qquad y\neq 0\,. \tag{36}$$

We also see, from (36), that even though z is not transcendental in (10), it has its transcendental equivalent given by

$$z=y\ln\left|yW_0\left[\frac{\exp\left[{}^{z}\!/\!_{y}\right]}{y}\right]\exp\left[W_0\left[\frac{\exp\left[{}^{z}\!/\!_{y}\right]}{y}\right]\right]\right| \qquad y\neq 0 \tag{37}$$

6 Analytic solutions of V for the standard real solar cell equation based on Lambert W function

Analytic solutions of V in (9) for both illuminated and dark conditions are obtained by replacing corresponding expressions of x, y and z in (11), (12) and (13) into (36), from which the solution for V is obtained. It is easy to show that, in the case of an illuminated solar cell, the solution for V has two equivalent expressions, namely

$$V=\left\{\begin{array}{l}AV_{\mathrm{th}}\ln\left|\dfrac{AV_{\mathrm{th}}}{I_{\mathrm{o}}R_{\mathrm{sh}}}W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}}\exp\left[\dfrac{R_{\mathrm{sh}}\left(I_{\mathrm{ph}}+I_{\mathrm{o}}-I\right)}{AV_{\mathrm{th}}}\right]\right]\right|-IR_{\mathrm{s}} \\ \\ \left(I_{\mathrm{ph}}+I_{\mathrm{o}}\right)R_{\mathrm{sh}}-I\left(R_{\mathrm{s}}+R_{\mathrm{sh}}\right)-AV_{\mathrm{th}}W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}}\exp\left[\dfrac{R_{\mathrm{sh}}\left(I_{\mathrm{ph}}+I_{\mathrm{o}}-I\right)}{AV_{\mathrm{th}}}\right]\right]\end{array}\right. . \tag{38}$$

Similarly, the solution for V under dark conditions can also be obtained in two equivalent forms as

$$V=\left\{\begin{array}{l}AV_{\mathrm{th}}\ln\left|\dfrac{AV_{\mathrm{th}}}{I_{\mathrm{o}}R_{\mathrm{sh}}}W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}}\exp\left[\dfrac{R_{\mathrm{sh}}\left(I+I_{\mathrm{o}}\right)}{AV_{\mathrm{th}}}\right]\right]\right|+IR_{\mathrm{s}} \\ \\ I_{\mathrm{o}}R_{\mathrm{sh}}+I\left(R_{\mathrm{sh}}+R_{\mathrm{s}}\right)-AV_{\mathrm{th}}W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}}\exp\left[\dfrac{R_{\mathrm{sh}}\left(I+I_{\mathrm{o}}\right)}{AV_{\mathrm{th}}}\right]\right]\end{array}\right. . \tag{39}$$

The lower expression of (22) and (23) have been reported in [14] and [15], respectively. However the upper equivalent expressions, which tell us that V in under both illuminated and dark conditions, is strictly logarithmic are (at least to the author's knowledge) undocumented.

7 Normalization of V using boundary conditions from current – voltage curves

The typical shapes of current – voltage curves for a solar cell under illuminated and dark conditions are as shown in figures 2(a) and 2(b), respectively. In the case of an illuminated solar cell we have $V \to V_{\mathrm{oc}}$ (i.e. the open circuit voltage) as $I \to 0$ and $V \to 0$ as $I \to I_{\mathrm{sc}}$ (i.e. the short circuit current). On the other hand the current-voltage curve for a solar cell under dark conditions is such that $V \to 0^{\pm}$ (i.e. $V \to 0$ from the positive and/or negative values) as $I \to 0^{\pm}$. Although these basic boundary/critical conditions on solar cells may not be sufficient to completely transform (38) and (39) into normalized expressions, they are certainly necessary. Hence we seek for the condition(s) in (38) and (39) for which $V \to 0^{\pm}$ as $I \to 0^{\pm}$ for a solar cell under dark conditions, while $V \to V_{\mathrm{oc}}$ (where $V_{\mathrm{oc}} > 0$) as $I \to 0^{\pm}$ under illuminated conditions. Firstly, observe that as $I \to 0$, (38) and (39) become

$$V \to \begin{cases} AV_{\mathrm{th}} \ln\left| \dfrac{AV_{\mathrm{th}}}{I_{\mathrm{o}}R_{\mathrm{sh}}} W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \exp\left[\dfrac{R_{\mathrm{sh}}\left(I_{\mathrm{ph}} + I_{\mathrm{o}}\right)}{AV_{\mathrm{th}}} \right]\right]\right| \\ \left(I_{\mathrm{ph}} + I_{\mathrm{o}}\right)R_{\mathrm{sh}} - AV_{\mathrm{th}} W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \exp\left[\dfrac{R_{\mathrm{sh}}\left(I_{\mathrm{ph}} + I_{\mathrm{o}}\right)}{AV_{\mathrm{th}}} \right]\right] \end{cases}, \quad \text{and} \tag{40}$$

$$V \to \begin{cases} AV_{\mathrm{th}} \ln\left| \dfrac{AV_{\mathrm{th}}}{I_{\mathrm{o}}R_{\mathrm{sh}}} W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \exp\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \right]\right]\right| \\ I_{\mathrm{o}}R_{\mathrm{sh}} - AV_{\mathrm{th}} W_0\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \exp\left[\dfrac{I_{\mathrm{o}}R_{\mathrm{sh}}}{AV_{\mathrm{th}}} \right]\right] \end{cases}, \tag{41}$$

respectively. For (41) to be meaningful, we must have $V \to 0^{\pm}$ as $I \to 0^{\pm}$ while $AV_{\mathrm{th}} > 0$. Clearly this condition is obtained if and only if

$$\frac{AV_{\mathrm{th}}}{I_{\mathrm{o}}R_{\mathrm{sh}}} = y = 1 = \text{Constant}, \quad \text{or} \tag{42}$$

$$I_{\mathrm{o}}R_{\mathrm{sh}} = AV_{\mathrm{th}}. \tag{43}$$

This condition reduces the argument of the natural logarithm in (41) to $W_0[e] = 1$ so that $\ln|W_0[e]| = \ln 1 = 0$ and we have $V = 0$ when $I = 0$ as required in the current – voltage curve of a solar cell under dark conditions. It is easy to see that the lower expression for (41) also vanishes when $I_{\mathrm{o}}R_{\mathrm{sh}} = AV_{\mathrm{th}}$. Further, since the expression for y in (12), is the same under both illuminated and dark conditions, it follows that (43) should be valid for both the illuminated and dark conditions. While our claim that $y = \text{constant}$ may appear new, it has been in existence – albeit not pronounced – in standard literature whenever the assumption [2,6] $R_{\mathrm{sh}} = \infty$ is used because when this is replaced in (12), we get $y = 0 = \text{constant}$. Conditions for which $y = 0$, however, leads to divisions by zero and thus eliminated from form the possible set of solutions for solar cell parameters.

Now, setting $AV_{\text{th}} = I_{\text{o}} R_{\text{sh}}$ into (38) and (39), and simplifying leads to

$$V = \begin{cases} AV_{\text{th}} \ln\left| W_0\left[\exp\left[\dfrac{I_{\text{ph}} + I_{\text{o}} - I}{I_{\text{o}}}\right]\right]\right| - IR_{\text{s}} \\ AV_{\text{th}}\left(\dfrac{I_{\text{ph}} + I_{\text{o}} - I}{I_{\text{o}}} - W_0\left[\exp\left[\dfrac{I_{\text{ph}} + I_{\text{o}} - I}{I_{\text{o}}}\right]\right]\right) - IR_{\text{s}} \end{cases} \quad \text{and} \qquad (44)$$

$$V = \begin{cases} AV_{\text{th}} \ln\left| W_0\left[\exp\left[\dfrac{I + I_{\text{o}}}{I_{\text{o}}}\right]\right]\right| + IR_{\text{s}} \\ AV_{\text{th}}\left(\dfrac{I + I_{\text{o}}}{I_{\text{o}}} - W_0\left[\exp\left[\dfrac{I + I_{\text{o}}}{I_{\text{o}}}\right]\right]\right) + IR_{\text{s}} \end{cases}, \qquad (45)$$

respectively. In the case of a solar cell under illumination, as $I \to 0$, we have $V \to V_{\text{oc}} > 0$, where

$$V_{\text{oc}} = \begin{cases} AV_{\text{th}} \ln\left| W_0\left[\exp\left[\dfrac{I_{\text{ph}} + I_{\text{o}}}{I_{\text{o}}}\right]\right]\right| \\ AV_{\text{th}}\left(\dfrac{I_{\text{ph}} + I_{\text{o}}}{I_{\text{o}}} - W_0\left[\exp\left[\dfrac{I_{\text{ph}} + I_{\text{o}}}{I_{\text{o}}}\right]\right]\right) \end{cases}. \qquad (46)$$

As can be seen in (44) to (45), the potential difference V across the terminals of a solar cell under either illuminated or dark conditions is independent of the shunt resistance, R_{sh}. Consequently, V_{oc} is also independent of R_{sh}. Since $AV_{\text{th}} > 0$, it follows that $V_{\text{oc}} \to 0$ as $I_{\text{ph}} \to 0$ and $V_{\text{oc}} \to \infty$ and $I_{\text{ph}} \to \infty$.

8 Comparative analysis of the real solar cell model with and without shunt resistance

Consider the simplified model of a solar cell as in figures 3(a) and (b).

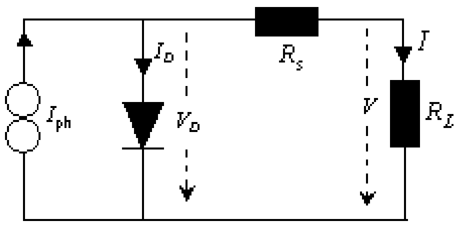

Figure 3(a): Simplified solar cell model under illumination

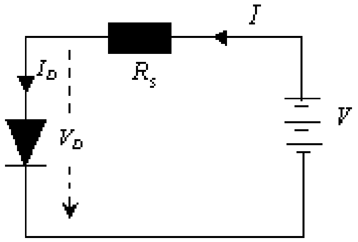

Figure 3(b): Simplified solar cell model under dark conditions

It is easy to show that for this approximation of a real solar cell, we have

$$I = \begin{cases} I_{\text{ph}} - I_{\text{o}}\left(\exp\left[\dfrac{V + IR_{\text{s}}}{AV_{\text{th}}}\right] - 1\right) & \text{under illuminated conditions} \\ I_{\text{o}}\left(\exp\left[\dfrac{V - IR_{\text{s}}}{AV_{th}}\right] - 1\right) & \text{under dark conditions} \end{cases}, \qquad (47)$$

which can be expressed in dimensionless form as

$$z_{ap} = \exp\left[x_{ap}\right] \quad \begin{cases} x_{ap} = x_i \text{ and } z_{ap} = z_i & \text{under illuminated conditions} \\ x_{ap} = x_d \text{ and } z_{ap} = z_d & \text{under dark conditions} \end{cases} \tag{48}$$

so that

$$x_{ap} = \ln\left|z_{ap}\right| \tag{49}$$

Suppose we define the different between x_{ap} in (49) and x in (36) as $\Psi(y,\eta) = x_{ap} - x$, or

$$\Psi(y,\eta) = \ln\left|\frac{1+\eta}{yW_0\left[\exp\left[\frac{1+\eta}{y}\right]\Big/ y\right]}\right|, \tag{50}$$

where η is as defined in (14). The graphs of $\Psi(y,\eta)$ against η for $0 \le \eta \le 700$ and $y = 1$, 5, 10, 50, 100, 500 are shown in fig 4. In fig 4, we have included values of y other than $y = 1$ (which is the normalized and physical value) in order to make a comparison between graphs for our normalized solutions and those that are not normalized. As can be seen in fig 4, the graph of $\Psi(y,\eta)$ is closer to zero for $y = 1$ than other curves, indicating that the difference between x and x_{ap} is practically negligible for higher values of η.

For an alternative view, we also plot the graphs of x_{ap} and x against η for different values of y are also plotted in figure 5. Here, we clearly see that for small values of y and $\eta >> 0$, there is no significant difference between the graphs of x and x_{ap} against η. For the case of an illuminated solar cell in which $I_{\text{ph}} >> I_{\text{o}}$, we have $\eta >> 0$, implying that we may employ use the solar cell model without R_{sh}. For such cases, solving for V in (47) gives

$$V = \begin{cases} AV_{\text{th}} \ln\left|\dfrac{I_{\text{ph}} + I_{\text{o}} - I}{I_{\text{o}}}\right| - IR_{\text{s}} & \text{under illuminated conditions} \\ AV_{\text{th}} \ln\left|\dfrac{I + I_{\text{o}}}{I_{\text{o}}}\right| + IR_{\text{s}} & \text{under dark conditions} \end{cases}. \tag{51}$$

Here, we see that as $I \to 0$, we have

$$V \to \begin{cases} AV_{\text{th}} \ln\left|\dfrac{I_{\text{ph}} + I_{\text{o}}}{I_{\text{o}}}\right| = V_{\text{oc}} & \text{under illuminated conditions} \\ AV_{\text{th}} \ln\left|\dfrac{I_{\text{o}}}{I_{\text{o}}}\right| = AV_{\text{th}} \ln 1 = 0 & \text{under dark conditions} \end{cases}, \tag{52}$$

which also satisfy the conditions $V \to V_{\text{oc}}$ (where $V_{\text{oc}} \geq 0$) for illuminated solar cells and $V \to 0$ as $I \to 0$ for solar cells under dark conditions. The expression for V_{oc} in (52) is in agreement with the conventional expression for the open circuit voltage [5], although we herein emphasize that $I_{\text{ph}} > I_{\text{sc}}$ for reasons already discussed in section 4.

Under reverse biased mode $I < 0$, and we see that the lower expression of (51) cannot be used to model a solar cell current-voltage curve whenever $I \leq -I_{\text{o}}$ because $\ln z$ is undefined for $z \leq 0$. On the other hand, since $e^{z} > 0$ for $-\infty < z < +\infty$ so we have $0 < W_0\left[e^{z}\right] < +\infty$ and $\ln W_0\left[e^{z}\right]$ is well defined for all real values of z. Consequently our expression in (45) is defined for all real values of I, and thus comparatively more elegant.

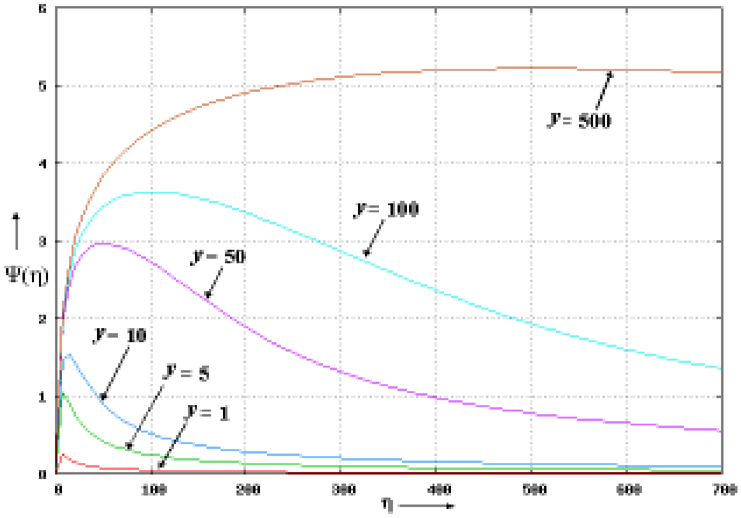

Figure 4 *Graphs of* $\Psi(y,\eta)$ *for* $y = 1$ *, 5, 10, 50, 100 and 500 with* $0 \leq \eta \leq 700$.

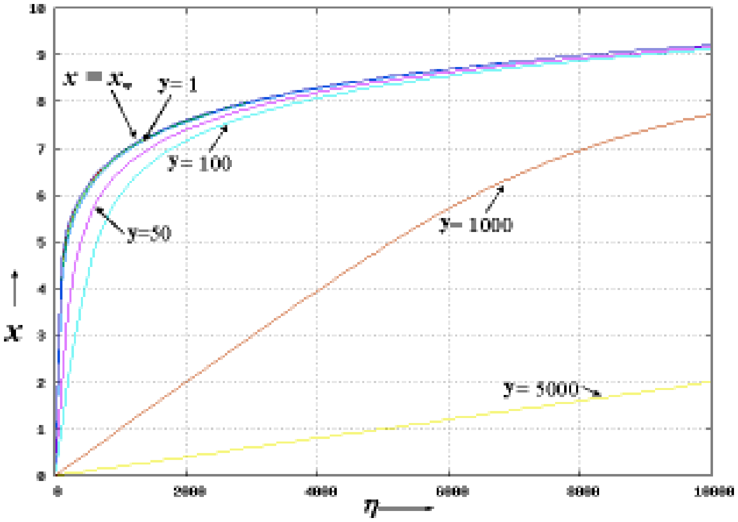

Figure 5: The graphs of x and x_{ap} against η for $y = 1$, 5, 10, 50, 100, 1000 and 5000 .

9 Power Transfer Analysis for Illuminated Photovoltaics

Measurements of I and V from illuminated solar cells involve the variation of the load resistance, R_L. At every instant, the potential difference, V, across the output terminals of the solar cell under study is equal to that across the terminals of the load resistor, R_L (refer to fig 1(a)). The output current, I, from and into the solar cell is equal to the current passing through R_L. If the load resistor, R_L, is strictly ohmic, then

$$V = IR_L \tag{53}$$

Replacing (53) into (3) transforms the later into

$$V_D = \begin{cases} I(R_L + R_\text{s}) \\ \\ I_\text{sh} R_\text{sh} \end{cases} = \text{Constant}\,. \tag{54}$$

We can then use (54) to express the output current, I, in terms of V_D as

$$I = \frac{V_D}{R_L + R_\text{s}};\ \ (R_L + R_\text{s}) \neq 0 \qquad \text{or} \tag{55}$$

$$I = \frac{V_D}{R_\text{s}(1+r)}\ , \qquad \text{where} \tag{56}$$

$$r = {R_L}\big/{R_\text{s}} \tag{57}$$

is the ratio of load resistance to the series resistance. The output power, P_out, transmitted to the load resistor is equal to the product of the output current, I, and the potential difference, V, across the terminals of the solar cell. Thus

$$P_\text{out} = IV \tag{58}$$

or equivalently,

$$P_\text{out} = I^2 R_L\ . \tag{59}$$

Substituting (56) into (59) leads to

$$P_\text{out} = \frac{V_D^2}{R_\text{s}}\left(\frac{r}{(1+r)^2}\right) \qquad \text{or} \tag{60}$$

$$P_\text{out} = \frac{V_D^2}{R_\text{s}}\left(\frac{1}{r + 2 + {1}\big/{r}}\right) \qquad \text{or} \tag{61}$$

$$P_\text{out} = \frac{V_D^2}{R_\text{s}} f[r], \tag{62}$$

where the (dimensionless resistive) function $f[r]$ is defined as

$$f[r] = \frac{P_{out} R_s}{V_D^2} = \left(r + 2 + {1}\big/{r}\right)^{-1} \qquad \text{or} \tag{63}$$

10 Impedance matching and maximum power transfer for solar cells under constant irradiation

Solar cells may operate over a wide range of voltages, V , and currents, I . By increasing the resistive load, R_L , on an illuminated cell continuously from zero (a short circuit) to a very high value (an open circuit) one can determine the maximum-power point, the point that maximizes , that is, the load for which the cell can deliver maximum electrical power at that level of irradiation. This process also gives us data that can be used to plot the current - voltage curve of a solar cell. The maximum-power point of a photovoltaic varies with incident illumination. Hence, it is important to maintain the incident irradiation on a solar cell under study.

Since most current-voltage curve tracers can obtain experimental measurements of I and V within short time intervals (typically a thousand I –V data points within sixty seconds), the potential difference across the terminals of the diode, V_D , can be assumed to be a constant within this time interval, provided they are subjected to constant irradiation. Additionally, since the time within which data is obtained is small, we can also assume that R_s is constant within this time interval. Therefore, for a single current–voltage curve, V_D and R_s are constant. This assumption also implies that variations in P_out are mainly due to variations in the resistance ratio, r . Meanwhile since R_s is constant for a single current-voltage curve, it follows that variations in r are only due to variations in R_L . Now, when R_s and V_D are constant, then variations in P_{out} depend only on $f[r]$ and indeed r . Hence, the point of maximum values of P_out corresponds to the point of maximum values for $f[r]$.

Firstly, we observe that

$$\lim_{r\to 0} f[r] = (0+2+\infty)^{-1} = 0 \qquad \text{and} \tag{64}$$

$$\lim_{r\to \infty} f[r] = (\infty+2+0)^{-1} = 0 \tag{65}$$

which indicates that $f[r]\to 0$ when either $r\to 0$ or $r\to\infty$. Hence $P_\text{out}\to 0$ when either $r\to 0$ or $r\to\infty$. Equivalently, we can say $P_\text{out}\to 0$ when either $R_L\to 0$ or $R_L\to\infty$.

Now we seek to obtain the value of the resistive ratio, r , which leads to maximum values of $f[r]$, which corresponds to maximum values of P_{out} . This requires maximization of $f[r]$, or equivalently minimization of $g[r]$, where

$$g[r] = r + 2 + 1/r \ . \tag{66}$$

Differentiating $g[r]$ with respect to r gives

$$\frac{d}{dr} g[r] = 1 - r^{-2} \tag{67}$$

Minimization of $g[r]$ requires that its derivative vanish and its second derivative at the minimum point be positive. Setting equation (67) to zero leads to

$$r^{-2} = 1 \tag{68}$$

which (after discarding the negative root of r since negative ratio of resistance is physically meaningless), leads to

$$r = 1 \tag{69}$$

The second derivative of $g[r]$ at $r = 1$ is

$$\left.\frac{d^2}{dr^2} g[r]\right|_{r=1} = \left.2r\right|_{r=1} = 2 > 0\,, \tag{70}$$

which is positive as required. The graph of $f[r]$ for $0 \le r \le 20$ are plotted in fig 6.

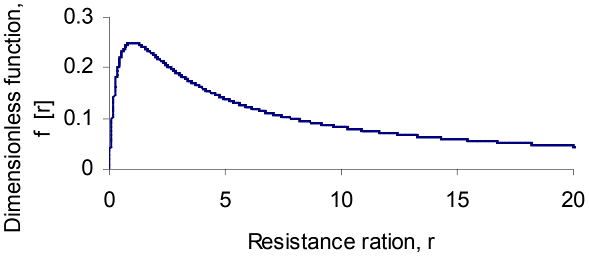

Fig 6: Graphs of $f[r]$ *against* r *for* $0 \le r \le 20$

As can be seen in fig 6, the maximum value of $f[r]$ is skewed towards lower values of the resistive ratio r. By the same token we expect the maximum power output from a solar cell, $P_{\text{out}}^{\text{m}}$ to be skewed towards lower values of the resistive ratio, r (or indeed lower values of the load resistor, R_L. In particular, $f[r]$ attains its maximum at $f[r = 1] = 1/4$. Maximum power is thus given by

$$P_{\text{out}}^{\text{m}} = \frac{V_D^2}{4R_{\text{s}}} \tag{71}$$

The condition for maximum power transfer between an illuminated solar cell and the load resistor is obtained by substituting (69) into (57), and results in

$$R_L = R_{\text{s}}\,. \tag{72}$$

At the point of maximum power point $(V, I) = (V_{\text{m}}, I_{\text{m}})$, so that equation (57) becomes

$$R_L = \frac{V_{\text{m}}}{I_{\text{m}}}; \quad I_{\text{m}} \neq 0\,. \tag{73}$$

By virtue of (72) and (73), we have

$$R_{\text{s}} = V_{\text{m}} / I_{\text{m}}\,. \tag{74}$$

Now, replacing (74) into (54) gives

$$V_D = 2V_{\text{m}} = \text{Constant} \qquad \text{or} \tag{75}$$

$$V + IR_{\text{s}} = 2V_{\text{m}} = \text{Constant}\,. \tag{76}$$

Replacing (75) into (71) gives

$$P_{\text{out}}^{\text{m}} = \frac{4V_{\text{m}}^2}{4R_{\text{s}}} \qquad \text{or} \qquad (77)$$

$$P_{\text{out}}^{\text{m}} = V_{\text{m}}^2 \Big/ \left(\frac{V_{\text{m}}}{I_{\text{m}}}\right) \qquad \text{or} \qquad (78)$$

$$P_{\text{out}}^{\text{m}} = I_{\text{m}} V_{\text{m}}\,, \qquad (79)$$

which is the common expression for the output power from a solar cell at the maximum power point [9]. Substituting (75) into (62) gives an expression for the output power at any given point as

$$P_{\text{out}} = 4 I_{\text{m}} V_{\text{m}} f[r]. \qquad (80)$$

11 The modified Moritz Von Jacobi's law for illuminated solar cells.

As seen in the preceding section, a solar cell subjected to constant irradiation and operating at the point of maximum power transfer, has the series and load resistances related by $R_{\text{s}} = R_L$, a relationship that is synonymous to the Moritz Von Jacobi's Law. According to Jacobi's Law, Maximum power is transferred when the internal resistance of the source equals the resistance of the load, when the external resistance can be varied, and the internal resistance is constant.

In the case of illuminated solar cells (see fig 1(a)) the total internal resistance is not only due to R_{s} but involves R_{sh} as well. However as seen in fig. 4, the effect of the presence of R_{sh} in the model is significant only when values of η are small. This is equivalent to saying that the effect of R_{sh} is not significant when $I_{\text{ph}} >> I_{\text{o}}$. In fact, we see in fig.4 that the presence of R_{sh} is insignificant for about $I_{\text{ph}} > 200\, I_{\text{o}}$. Jacobi's law can thus be modified to suit illuminated solar cells as follows:

> *Provided constant irradiation falls on a solar cell and that $I_{\text{ph}} >> I_{\text{o}}$, maximum power is transferred from the solar cell to a variable ohmic load resistance when the series resistance, R_{s}, of the solar cell is equal to the load resistance, R_L.*

12 Modeling the photocurrent

Suppose the radiant energy that is incident on a solar cell per unit area per unit time is P_{in}. Then the energy per unit time per unit area that is absorbed, P_{abs}, by the solar cell will be given by

$$P_{\text{abs}} = P_{\text{in}} - P_{\text{out}}\ . \qquad (81)$$

Hence the fraction of absorbed power, a, by the solar cell is

$$a = \begin{cases} \dfrac{P_{\text{in}} - P_{\text{out}}}{P_{\text{in}}} \\ \\ 1 - \dfrac{P_{\text{out}}}{P_{\text{in}}} \end{cases} \quad ; \quad P_{\text{in}} \neq 0 \ , \tag{82}$$

where $0 < a < 1$. From fig 1(a), we should have

$$I_{\text{ph}} V_D = a P_{\text{in}} \qquad \text{or} \tag{83}$$

$$I_{\text{ph}} = \frac{P_{\text{in}} - P_{\text{out}}}{V_D} \qquad \text{or} \tag{84}$$

$$I_{\text{ph}} = \frac{P_{\text{in}} - P_{\text{out}}}{2V_{\text{m}}} \quad ; \quad V_{\text{m}} \neq 0 \tag{85}$$

If in an experiment the incident illumination is kept constant, then P_{in} and V_{m} will be constant. Then in such conditions we see that I_{ph} will vary depending on the values of P_{out}. Since P_{out} is a function of r, it follows that I_{ph} is also dependent on r.

13 Synergy of solar cell parameters with Wein's displacement law

Under thermal equilibrium, the spectral distribution of the radiant energy of objects can be approximated to that of a black body radiator, and depends on the absolute temperature, T, of the object and not on its internal nature or structure [19-21]. As the temperature increases, the wavelength at which the energy emitted per second is a maximum decreases. The maximum emitted energy density per given wavelength satisfies the Wein's displacement law [19], namely

$$n\lambda_{\text{m}} T = 2898 \times 10^{-6} \text{ m.K} \tag{86}$$

where λ_{m} is the wavelength of maximum radiance and n is the average phase velocity refractive index of the medium in which radiation is propagating. For the case where the medium is air, n is close to unity [19]. Solar cells, under thermal equilibrium and of average phase velocity refractive index n, also obey Wein's displacement law, as in (53). Now, replacing (15) into (43) then solving for T gives

$$T = \frac{qI_{\text{o}} R_{\text{sh}}}{Ak} . \tag{87}$$

Substituting (87) into (86) gives

$$n\lambda_{\text{m}} \left(\frac{qI_{\text{o}} R_{\text{sh}}}{Ak} \right) = 2898 \times 10^{-6} \text{ m.K} \qquad \text{or} \tag{88}$$

$$\frac{2898 \times 10^{-6} \text{ m.K}}{n\lambda_{\text{m}}} = \frac{qI_{\text{o}} R_{\text{sh}}}{Ak} . \tag{89}$$

But λ_{m} can be expressed as

$$\lambda_{\mathrm{m}} = \frac{hc}{\mathrm{E_m}} \tag{90}$$

where $\mathrm{E_m}$ is the energy carried by monochromatic radiation of wavelength λ_{m}, $h = 6.63\times10^{-34}\ \mathrm{J.s}$ is the Plank's constant and $c \approx 3\times10^{8}\ \mathrm{m.s^{-1}}$ is the speed of light in a vacuum. Noting that $q = 1.60\times10^{-19}$ C and $k = 1.38\times10^{-23}$ $\mathrm{m^2.Kg.s^{-2}.K^{-1}}$ then substituting (90) into (89) and simplifying gives

$$\frac{I_o R_{sh}}{A} = \wp\frac{\mathrm{E_m}}{n} \tag{91}$$

where

$$\wp = \frac{2898\times10^{-6}\times k}{q\times hc} = 1\,256\,674\,208144\,796\,380.09\ \mathrm{V.J^{-1}} \tag{92}$$

is a constant. We observe in (91) that

$$I_{\mathrm{o}} R_{\mathrm{sh}} = b\,\wp\,\mathrm{E_m} \tag{93}$$

and

$$A = b\,n \tag{94}$$

where b is a positive dimensionless constant of proportionality. Since physical restrictions on A and n are that $A \geq 1$ and $n \geq 1$, then (94) implies that $b = 1$, so that (93) and (94) reduce to

$$I_{\mathrm{o}} R_{\mathrm{sh}} = \wp\,\mathrm{E_m} \qquad \text{or} \tag{95}$$

$$\frac{I_{\mathrm{o}} R_{\mathrm{sh}}}{\mathrm{E_m}} = \text{constant} \tag{96}$$

and

$$A = n\ . \tag{97}$$

Hence the diode quality factor, A, is equivalent to the phase refractive index, n, of a given solar cell.

14 Conclusion

Problems involving the extraction of solar cell optoelectronic device parameters based on the one-diode real solar cell model can be analyzed using in closed form using the recently defined Lambert W function. The conventional approximations that are aimed at simplifying the analysis of solar cell parameters often lead to inconsistent expressions relating to solar cell parameters.

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