Construction of almost harmonic matrix-vector pairs

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Let $x, y \in R^n$ be any two vectors and their inner product as $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$. For any $n \times n$ real matrix $A \in M_n(R)$ and for any real n-vector $u \in R^n$ which is non trivial, we consider the sequence of real numbers

$$w_k = \langle u, A^k u\rangle \tag{1}$$

for all $k = 0, 1, 2, \ldots$. If A is a symmetric matrix, then $w_k \geq 0$ for all even number $k = 2h$, since a symmetric matrix has the property of being self adjoint, that is $\langle x, Ay\rangle = \langle Ax, y\rangle$ for any n-vectors $x, y \in R^n$. If A is a symmetric matrix, then its h-th powers A^h are also symmetric. The condition $w_{2h} \geq 0$ is then clear since for any symmetric matrix, we see that

$$w_{2h} = \langle u, A^{2h}u\rangle = \langle u, A^hA^hu\rangle = \langle A^hu, A^hu\rangle \geq 0.$$

We remark that if $w_{2h} = 0$ holds then $A^hu = 0$. For the adjacency matrix A of a simple graph, the sequence w_k denotes the number of walks in the graph. In investigations in graph theory, in particular concerning the molecular structure of certain trees in organic chemistry, the sequence $w_{k-1}w_{k+1} - w_k^2$ has gained some importance and seems to enjoy various interesting properties.

Let A be a real symmetric $n \times n$ matrix and $u \in R^n$ be a nonzero real n-vector. Let $w_k = \langle A^ku, u\rangle$ for integer $k \geq 0$. Then w_k satisfy the inequality for all integer i, $j \geq 0$ as follows:

$$w_{2i}w_{2j} \geq w_{i+j}^2. \tag{2}$$

This is trivial by using the Cauchy Schwarz inequality:

$$\langle A^{2i}u, u\rangle \cdot \langle A^{2j}u, u\rangle \leq \langle A^iu, A^ju\rangle^2 \text{ by Cauchy Schwarz} \tag{3}$$
$$= \langle A^{i+j}u, u\rangle^2. \tag{4}$$

Let us assume now without loss of generality that $h \leq k$. In the case of equality $w_{2h}w_{2k} = w_{h+k}^2$ we see that

$$\langle A^hu, A^hu\rangle \cdot \langle A^ku, A^ku\rangle = \langle A^hu, A^ku\rangle^2, \tag{5}$$

it follows that the vectors A^hu, A^ku must be linearly dependent, that is there exists a real constant λ such that

$$A^ku = \lambda \cdot A^hu, \text{ i.e. } A^h(A^{k-h}u - \lambda u) = 0. \tag{6}$$

Let us denote the equality in (5) as a function of h, k to be $E(h, k)$. It is trivial that $E(h, h)$ always holds.
In the case of equality of (2), either of the five following alternatives holds:

(i) u is the eigenvector of A;

(ii) Au is the eigenvector of A;

(iii) u is the eigenvector of A^2;

(iv) Au is the eigenvector of A^2;

(v) None of the case above.

In the case of graphs this was observed in [3]. This follows by simple considerations using the equality case of the Cauchy Schwarz inequality. The proof is outlined in the next section. In the case of graph, the question of the existence of graphs with adjacency matrix A and all-one vector u satisfying equality in

$$\langle A^{k+1}u, A^k u\rangle \cdot \langle A^k u, A^{k-1}u\rangle = \langle A^k u, A^k u\rangle^2, \tag{7}$$

has been raised in [2] and [1].

In this thesis, the main result will be the construction of a real symmetric $n \times n$ matrix A and a $n-$vector $u = (u_1, u_2, \dots, u_n)$ with $n = 4m$, where m is positive integer, such that the equality of (7) holds for all $k > 0$, and such that $u_i > 0$ for all $i = 1, 2, \dots, n$.

The main theorem in the thesis is as follows:

Theorem 1 Let A =Diag$(\lambda^{2m}, \lambda^{2m-1}, -\lambda^{2m-1}, \lambda^{2m-2}, -\lambda^{2m-2}, \dots, \lambda, -\lambda, 1)$ for positive integer m and $\lambda > 1$. Let $u = (u_1, \dots, u_n)$ where $u_i = \sqrt{x_i}$ for all $i = 1, \dots, n$ where x_i is given as a function of λ from (44) to (48). Then $\langle u, A^{k-1}u\rangle \cdot \langle u, A^{k+1}u\rangle = \langle u, A^k u\rangle^2$ holds for $n = 4m$ and any positive even integer k.

The following sections will explain how to get the main theorem and some further results related to the theorem.

1 Symmetric matrices

Let $v \in R^n$ be a non zero real $n-$vector which has the property that it is not an eigenvector, but that some power of A^k produces an eigenvector. Hence we assume that there exists an integer $k \geq 1$ such that $A^k v$ is an eigenvector of A, i.e. that there exists a real number $\lambda \in R$ such that

$$A(A^k v) = \lambda \cdot A^k v \tag{8}$$

holds. We can determine k uniquely by adding the condition that $A^{k-1}v$ is not an eigenvector of A. In this case we can say that v is an *almost eigenvector of A* of level k.

Proposition 1 Let A be a real symmetric $n \times n$ matrix. Under the assumption that $v \neq 0$ is an almost eigenvector we have that $\det(A) = 0$, and that $k = 1$.

Let us assume for any $k \geq 1$ the existence of a real number λ with the conditions

$$A(A^k v) = \lambda \cdot A^k v \text{ and } A(A^{k-1} v) \neq \lambda \cdot A^{k-1} v. \tag{9}$$

Then we get that

$$A^k(Av - \lambda \cdot v) = 0. \tag{10}$$

Since v is not an eigenvector, i.e. $Av - \lambda v \neq 0$, this implies that $\det(A^k) = 0$, whence $\det(A) = 0$.

If $k > 1$ then we can rewrite the condition (10) as

$$A(A^k v - \lambda \cdot A^{k-1} v) = 0, \tag{11}$$

which means that the vector $z := A^k v - \lambda A^{k-1} v$ has the property

$$z \in Ker(A) \cap Im(A^{k-1}). \tag{12}$$

From $k \geq 2$ and for a symmetric matrix we see that for $y = Av - \lambda v$ the inner product can be rewritten as

$$\langle z, z \rangle = \langle z, A^{k-1} y \rangle = \langle Az, A^{k-2} y \rangle = 0. \tag{13}$$

Hence $z = 0$. This means $A(A^{k-1} v) = A^k v = \lambda \cdot A^{k-1} v$ contradicting the second part of (9).

Lemma 1 Assume that A is a symmetric matrix. If $E(0, k)$ holds for any odd positive integer k, then also $E(0, 1)$ holds, and in this case u is an eigenvector, and $E(h, k)$ holds for any pair $h, k \geq 0$ of non negative integers.

The proof is an immediate consequence of the Cauchy Schwarz inequality. Indeed if

$$\langle u, u \rangle \cdot \langle A^k u, A^k u \rangle = \langle u, A^k u \rangle^2$$

is true for some odd integer $k > 0$, then from the Cauchy Schwarz inequality we see that there exists a real scalar λ with $A^k u = \lambda \cdot u$. Thus u is an eigenvector for the eigenvalue λ.

Clearly any eigenvector of an arbitrary matrix A is also an eigenvector of A^k. Now consider the diagonal normal form of the matrix A, and a basis of eigenvectors of the underlying space. It is immediate that for k odd the converse holds: Any eigenvector of the matrix A^k must be an eigenvector of A, since two distinct components in that basis are only possible if the basis vectors belong to the same eigenspace. This proves lemma 1.

Lemma 2 Assume that A is a symmetric matrix. If $E(h, h+1)$ holds for some integer $h \geq 1$, then also $E(1, 2)$ holds, and consequently $E(h, k)$ holds for any pair of positive integers $0 < h < k$.

Proof: Assume that $E(h, h+1)$ holds for some positive integer h. Then for some constant real λ the matrix vector equation

$$A^h(Au - \lambda u) = 0$$

holds. This means that u is an almost eigenvector of level $\leq h$. By proposition 1 it follows that

$$A(Au - \lambda u) = 0$$

already holds, so that $E(1, 2)$ is valid.
It follows by repeated applications of A that $E(m, m+1)$ holds for all $m \geq 1$. Now consider the expression

$$A^h(A^r u - \lambda u)$$

for $h > 0$ and $r > 1$. From $A^h(Au - \lambda u) = 0$ it follows by induction that

$$A^{h-1}(A^2 u - \lambda Au) = 0, \text{ i.e. } A^{h-1}(A^2 u - \lambda u) = 0.$$

Lemma 3 Assume that A is a symmetric matrix. If $E(0, k)$ holds for any even integer k, then $E(0, 2)$ holds, and consequently $E(h, k)$ holds for any pair of non negative integers h, k with an even sum.

Lemma 4 Assume that A is a symmetric matrix. If $E(h, k)$ holds for any pair of distinct integers h, k with an even sum, then $E(1, 3)$ holds, and consequently $E(h, k)$ holds for all pairs of integers h, k with even sum.

2 Quadratic operators

Recall that for a symmetric matrix A and real n-vector u we had defined $w_k = \langle u, A^k u \rangle$. We restate equation (2) as follows:

Proposition 2 For any real symmetric matrix $A \in M_n(R)$ and for any non trivial vector $u \in R^n$, the numbers w_k in (1) have the following property:

$$q_{k,t} = w_{k-t} w_{k+t} - w_k^2 \geq 0$$

is a non negative number for all values of k, t with $0 < t < k$ and such that $k - t$ is even.

Given any real and symmetric $n \times n$ matrix A and a real n-vector u, the pair (A, u) is said to be *harmonic* if and only if the expressions $q_{k,1} = 0$ for all sufficiently large $k \geq k_0$. This condition has been discussed in the case of graphs by several authors (see e.g. [1], [2] and [3]). In particular, if u is an eigenvector of A then (A, u) is a harmonic pair. The pair (A, u) as above is *almost harmonic* if and only if the expressions $q_{k,1} = 0$ for all (sufficiently large) values of even k.
Since $q_{k,t}$ depends on A and u, we may write $q_{k,t}(A, u) = \langle A^{k-t} u, u \rangle \cdot \langle A^{k+t} u, u \rangle - \langle A^k u, u \rangle^2$. A nonzero vector $u \in R^n$ is *t-harmonic (w.r.t. the matrix A)* if and only if $q_{k,t}(A, u) = 0$ for all $k = t+1, t+2, \ldots$. The nonzero vector u is said to be *almost t-harmonic (w.r.t.*

the matrix A) if and only if $q_{k,t}(A,u) = 0$ for all sufficiently large values of k such that $k-t$ is odd.
Let us interpret the above construction in terms of quadratic mappings. If we let $x_i = c_i^2$ for $i = 1, 2, \ldots, n$ and if we set for any fixed n distinct and non zero reals $\lambda_1, \lambda_2, \ldots, \lambda_n$ their set of two fold products to be

$$S = \{|\lambda_i \lambda_j| : 1 \leq i < j \leq n\},$$

then define $r = |S|$. Assume that the set is enumerated as

$$S = \{\xi_1, \xi_2, \ldots, \xi_r\}.$$

We then let for $k = 1, 2, \ldots, r$

$$z_k = \sum_{i<j} \frac{\lambda_i \lambda_j}{|\lambda_i \lambda_j|} \cdot (\lambda_i - \lambda_j)^2 \cdot x_i x_j \quad (1 \leq k \leq r) \tag{14}$$

where the sum in (14) is taken over all pairs (i,j) with $1 \leq i < j \leq n$ such that $|\lambda_i \cdot \lambda_j| = \xi_k$.
We note that the first factor in the sum in (14) is a sign which is positive if and only if $\lambda_i \cdot \lambda_j = +\xi_k$, and which is negative if and only if $\lambda_i \cdot \lambda_j = -\xi_k$.
Note that we may interpret the formulas (14) as a quadratic transformation

$$z : R^n \to R^r. \tag{15}$$

We also note that the possible domain relevant for the above problem is the set of all non negative values $x = (x_1, x_2, \ldots, x_n)$ with $x_i \geq 0, i = 1, 2, \ldots, n$ and

$$x_1 + x_2 + \cdots + x_n = \langle u, u \rangle.$$

This is the simplex $\Delta_c(n)$ of size $\langle u, u \rangle = c$ in real $n-$space $\Delta_c(n) \subset R^n$. We remark that in [5], see also [4], a map (15) (with $r = n$) has been called a *system of quadratic stochastic operators* if for each $x \in \Delta_c(n)$ we get the image $z(x) \in \Delta_c^2(r)$.

2.1 Additive and Multiplicative Nasir Labelings of Complete Graphs

In the following, we shall describe the combinatorial structures that is needed to construct almost t-harmonic vectors for a given matrix A, via the quadratic operators defined above.
We are seeking labelings of the n vertex points of a complete graphs K_n by real numbers such that the edges receive the product of the labels of the vertices, and the absolute value of each such edge label appears at least twice, namely at least once with a positive sign and at least once with a negative sign.
An example of such a labeling for the case $n = 4$ is $4, 2, 1, -2$ and the edges receive the labels $8, -8, 4, -4, 2, -2$. It was shown that such labelings do not exist if $n = 2, 3$ or if $n = 5$. We call these labelings *Nasir Labelings* because of their relationship with the work of Nasir Ganikhodjaev [5] on the quadratic stochastic operators.
The question of construction of further $t-$harmonic pairs (A, u) can then be reduced to the question whether there exist Nasir labelings with associated quadratic operator which

have non trivial solutions. In particular one is interested in any solutions that have all their variables x_i non vanishing.
We will say that a system of n non negative integers

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \tag{16}$$

is an *additive Nasir labeling* if and only if it satisfies the following properties:

(i) $a_1 = 0$.

(ii) If there exists any integer i with $a_i = a_{i+1}$, then $a_{i-1} < a_i = a_{i+1} < a_{i+2}$. In particular $a_1 < a_2$ and $a_{n-1} < a_n$.

(iii) For any pair $1 \leq i < j \leq n$ there exists at least one other pair $1 \leq k < l \leq n$ with $(i, j) \neq (k, l)$ such that $a_i + a_j = a_k + a_l$.

If we think of the a_i as a vertex labeling of a complete graph, and the $a_i + a_j$ as the corresponding edge labels, then we may rephrase the condition (ii) by saying that no vertex label appears more than twice, and we may rephrase condition (iii) by saying that no value of an edge labeling will "stand alone."

We will say that an additive Nasir labeling is *reduced* if and only if the greatest common divisor of the (positive) values $a_2, a_3, \ldots, a_n$ is one:

$$g = g.c.d(a_2, a_3, \ldots, a_n) = 1. \tag{17}$$

We note that from condition (ii) and (iii) it follows that $a_1 < a_2 = a_3$ since $a_1 + a_2 \leq a_1 + a_3$ are the two smallest edge labels, and hence by (iii) they must be equal. Similarly $a_{n-2} = a_{n-1} < a_n$.
For any additive Nasir labeling we let n be the number of vertices, so that we have a vertex and edge labeling of the complete graph K_n. As explained, the vertex labels satisfy the inequalities

$$a_1 < a_2 = a_3 < a_4 \leq a_5 \leq \cdots \leq a_{n-1} < a_n, \tag{18}$$

and we let $M = a_n$ be the largest vertex label.

We note that any additive Nasir labeling gives rise to a unique reduced labeling by taking the greatest common divisor g of the positive numbers a_i and then dividing by this number g.

In this thesis we show that the number of distinct reduced additive Nasir labelings is finite if and only if the integer $n \leq 8$ or $n = 10$.
We will say that a system of n non negative integers

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \tag{19}$$

is a *normalized multiplicative Nasir labeling* iff it satisfies the following properties.

M(i) $\lambda_1 = 1$.

M(ii) If there exists any integer $i, 2 \leq i \leq n-2$ with $\lambda_i = \lambda_{i+1}$, then $\lambda_{i-1} < \lambda_i = \lambda_{i+1} < \lambda_{i+2}$.

M(iii) For any pair $1 \leq i < j \leq n$ there exists at least one other pair $1 \leq k < l \leq n$ with $\{i, j\} \neq \{k, l\}$ such that $\lambda_i \lambda_j = \lambda_k \lambda_l$.

If we think of the λ_i as a vertex labeling of a complete graph, and the products $\lambda_i \lambda_j$ as the corresponding edge labels, then we may rephrase the condition M(ii) by saying that no vertex label appears more than twice, and we may rephrase condition M(iii) by saying that no value of an edge labeling will "stand alone."
As an example we see that the sequence $\lambda_1 = 1 < \lambda_2 = \lambda_3 = 2 < \lambda_4 = 4$ is a normalized multiplicative Nasir labeling and that it gives rise to the first labeling in figure below.

Lemma 5 If $\{a_1, a_2, \ldots, a_n\}$ is an additive Nasir labeling, then for any positive real number $\lambda > 1$, the assignment $\lambda_i = \lambda^{a_i}$ defines a normalized multiplicative Nasir labeling.

For any additive Nasir labeling in terms of the complete graph K_n, we let n be the number of vertices, so that we have a vertex and edge labeling of K_n, and let the set of the vertex labels be

$$a_1 < a_2 = a_3 < a_4 \leq a_5 \leq \cdots \leq a_{n-4} \leq a_{n-3} < a_{n-2} = a_{n-1} < a_n. \tag{20}$$

We let $M = a_n$ be the largest vertex label. We say that the set of all values $a_i + a_j$ together with their multiplicities of occurence as edge labels constitutes the *spectrum* of the Nasir labelings.
We say that a multiplicative Nasir Labeling is *Nasir-signed* if the vertices have a collection of signs, and the edges carry the corresponding product sign, which is induced from the two associated vertex signs such that the following holds:

S(0) The unique vertex label 1 carries a positive sign $+$.

S(i) If two vertex labels are equal, they carry opposite signs.

S(ii) For each value of the multiplicative edge labels, there exists at least one edge having that value with a positive sign, and another edge label having that value with a negative sign.

We should like to show an example of a multiplicative Nasir labeling which has two distinct signings satisfying conditions S(0), S(i) and S(ii) above. The multiplicative Nasir labeling is

$$1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^3.$$

The two signings are given by

$$+1, +\lambda, -\lambda, +\lambda^2, -\lambda^2, +\lambda^3 \quad \text{and} \quad +1, +\lambda, -\lambda, +\lambda^2, -\lambda^2, -\lambda^3.$$

3 An Example of a Quadratic Operator and Its Zeroes

3.1 An Example with $n = 8$ variables with quasi symmetric solution

We should like to calculate the example (with $\lambda > 1$ real) corresponding to the first row of the table with $n = 8$, that is the multiplicative labeling

$$1, \pm\lambda, \pm\lambda^2, \pm\lambda^3, \lambda^4 \quad .$$

We associate the labels to the following variables:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1	λ	$-\lambda$	λ^2	$-\lambda^2$	λ^3	$-\lambda^3$	λ^4

In this case we obtain the following system of components of a quadratic operator:

$$\begin{aligned}
q_1 &= +(\lambda-1)^2x_1x_2 - (\lambda+1)^2x_1x_3; && (21)\\
q_2 &= +(\lambda^2-1)^2x_1x_4 - (\lambda^2+1)^2x_1x_5 - 4\lambda^2x_2x_3; && (22)\\
q_3 &= +(\lambda^3-1)^2x_1x_6 - (\lambda^3+1)^2x_1x_7 + (\lambda^2-\lambda)^2x_2x_4 && (23)\\
&\quad -(\lambda^2+\lambda)^2x_2x_5 - (\lambda^2+\lambda)^2x_3x_4 + (\lambda^2-\lambda)^2x_3x_5;\\
q_4 &= +(\lambda^4-1)^2x_1x_8 + (\lambda^3-\lambda)^2x_2x_6 - (\lambda^3+\lambda)^2x_2x_7 && (24)\\
&\quad -(\lambda^3+\lambda)^2x_3x_6 + (\lambda^3-\lambda)^2x_3x_7 - 4\lambda^4x_4x_5;\\
q_5 &= +(\lambda^4-\lambda)^2x_2x_8 - (\lambda^4+\lambda)^2x_3x_8 + (\lambda^3-\lambda^2)^2x_4x_6 && (25)\\
&\quad -(\lambda^3+\lambda^2)^2x_4x_7 - (\lambda^3+\lambda^2)^2x_5x_6 + (\lambda^3-\lambda^2)^2x_5x_7;\\
q_6 &= +(\lambda^4-\lambda^2)^2x_4x_8 - (\lambda^4+\lambda^2)^2x_5x_8 - 4\lambda^6x_6x_7; && (26)\\
q_7 &= +(\lambda^4-\lambda^3)^2x_6x_8 - (\lambda^4+\lambda^3)^2x_7x_8. && (27)
\end{aligned}$$

We want to find solutions with all the coordinates $x_i > 0$. From $x_1 > 0, x_8 > 0$ and equation (21),(27) we see that

$$\begin{aligned}
(\lambda-1)^2x_2 &= (\lambda+1)^2x_3, && (28)\\
(\lambda-1)^2x_6 &= (\lambda+1)^2x_7. && (29)
\end{aligned}$$

Hence we let for any $b > 0$,

$$\begin{aligned}
x_1 &= 2b\lambda^2, && (30)\\
x_2 &= (\lambda+1)^2, && (31)\\
x_3 &= (\lambda-1)^2, && (32)\\
x_4 &= \frac{2}{b} + b(\lambda^2+1)^2, && (33)\\
x_5 &= b(\lambda^2-1)^2, && (34)
\end{aligned}$$

and then

$$x_6 = x_2, x_7 = x_3, x_8 = x_1.$$

We can check by explicit calculations that all the equations $q_i = 0$ for $i = 1, 2, 3, 4, 5, 6, 7$ are satisfied.

3.2 A Non Quasi Symmetric Solution for $n = 8$

We want to exhibit a non quasi symmetric solution of (21) to (27) in case $m = 8$, $\lambda = 2$. In this case the system

$$\begin{aligned}
0 &= x_1x_2 - 9x_1x_3 && (35)\\
0 &= 9x_1x_4 - 25x_1x_5 - 16x_2x_3 && (36)\\
0 &= 49x_1x_6 - 81x_1x_7 + 4(x_2x_4 + x_3x_5) - 36(x_2x_5 + x_3x_4) && (37)\\
0 &= 225x_1x_8 + 36(x_2x_6 + x_3x_7) - 100(x_2x_7 + x_3x_6) - 64x_4x_5 && (38)\\
0 &= 49x_2x_8 - 81x_3x_8 + 4(x_4x_6 + x_5x_7) - 36(x_4x_7 + x_5x_6) && (39)\\
0 &= 9x_4x_8 - 25x_5x_8 - 16x_6x_7. && (40)\\
0 &= x_6x_8 - 9x_7x_8. && (41)
\end{aligned}$$

admits the solution

$$x_1 = 4, x_2 = 9, x_3 = 1, x_4 = 29, x_5 = 9, x_6 = 18, x_7 = 2, x_8 = 16.$$

This solution is clearly not quasi symmetric.

4 The solution for quadratic operators of standard Nasir Labeling with general even n and arbitrary λ.

A system of n non negative integers $\{a_1, \ldots, a_n\}$ as in (16) is said to be *the standard Nasir labeling* if and only if it satisfies the following properties:

(i) $\{a_1, \ldots, a_n\}$ is a reduced additive Nasir labeling,

(ii) $a_n = h$ if $n = 2h$.

In this section, we shall describe the common solution for the quadratic system associated to the standard Nasir labeling for the case of any real $\lambda > 1$. Let us assume that $n = 4m$. Then choose the variables x_i according to the signed labels as follows:

$$x_1 \leftrightarrow +1, \quad x_{2k} \leftrightarrow +2^k, x_{2k+1} \leftrightarrow -2^k \quad (1 \leq k \leq 2m-1), x_n \leftrightarrow +2^{2m}.$$

We assume throughout that we are in the quasi symmetric case, that is we have $x_{2i} = x_{4m-2i}$, and $x_{2i+1} = x_{4m-2i+1}$, and also $x_1 = x_{4m}$.

We describe the common solution for the quadratic system associated to the standard Nasir labelling for the case of any real $\lambda > 1$. We then choose the variables x_i according to the signed labels as follows:

$$x_1 \leftrightarrow +1, \quad x_{2k} \leftrightarrow +\lambda^k, x_{2k+1} \leftrightarrow -\lambda^k \quad (1 \leq k \leq 2m-1), x_n \leftrightarrow +\lambda^{2m}.$$

We assume throughout that we are in the *quasi symmetric* case, that is we have $x_{2i} = x_{4m-2i}$, and $x_{2i+1} = x_{4m-2i+1}$, and also $x_1 = x_{4m}$.

In order to describe the solution of the common case, we first inductively define the following two numerical functions s_k and t_k :

$$s_1 = 2\lambda^2 + 2, s_{k+1} = \lambda^2 s_k - 2(\lambda^2 - \lambda - 1) \ \ \forall\, k \geq 1. \tag{42}$$

$$t_1 = 2\lambda^2 + 2, t_{k+1} = \lambda^2 t_k - 2(\lambda^2 - 2) \ \ \forall\, k \geq 1. \tag{43}$$

It is clear that s_k, t_k are polynomials in λ of degree k with integer coefficients.
We now assume that $n = 4m$ is given and we describe the solution values of x_i for this particular choice of n as follows: First we assume that $1 \leq i \leq m-1$,

$$x_{2i} = \frac{1}{2} \cdot \lambda^{m-i-1} \cdot s_i + \lambda^{m-2} \cdot (2\lambda + i - 1), \tag{44}$$

$$x_{2i+1} = \frac{1}{2} \cdot \lambda^{m-i-1} \cdot s_i - \lambda^{m-2} \cdot (2\lambda + i - 1). \tag{45}$$

We have the special values

$$x_1 = 2\lambda^m, \tag{46}$$

$$x_{2m} = \frac{1}{2} t_m + \lambda^{m-2} \cdot (2\lambda^2 - 1 + m), \tag{47}$$

$$x_{2m+1} = \frac{1}{2} t_m - \lambda^{m-2} \cdot (2\lambda^2 - 1 + m). \tag{48}$$

So far we have defined all x_i for $i \leq 2m+1$. We then define the other values by quasi symmetry, that is we let

$$x_{2i} = x_{4m-2i}, \tag{49}$$

$$x_{2i+1} = x_{4m-2i+1}, \tag{50}$$

$$x_n = \lambda^{m+1}, \tag{51}$$

where i takes the remaining values $i = m+1, m+2, \ldots, 2m-1$.

The relations (42) can be solved explicitely by the following function

$$s_k = \frac{2\lambda^{2k-1}(\lambda^3 - \lambda + 1) + 2(\lambda^2 - \lambda - 1)}{(\lambda^2 - 1)}, \tag{52}$$

$$t_k = \frac{2\lambda^{2k-2}(\lambda^4 - \lambda^2 + 1) + 2(\lambda^2 - 2)}{(\lambda^2 - 1)}. \tag{53}$$

4.1 Some Summation Formulas

We recall some well known functional identities. By differentiating

$$1 + x + x^2 + \cdots + x^{d-1} + x^d = \frac{x^{d+1} - 1}{x - 1} \tag{54}$$

we get the polynomial equation

$$\sum_{k=1}^{d} kx^{k-1} = \frac{dx^{d+1} - (d+1)x^d + 1}{(x-1)^2}. \tag{55}$$

We can multiplicatively invert x in (55) to obtain

$$\sum_{k=1}^{d} (d+1-k)x^{k-1} = \frac{d-(d+1)x+x^{d+1}}{(x-1)^2} \tag{56}$$

We may multiply this equation (56) by a power of x to obtain

$$\sum_{k=1}^{d} (d+1-k)x^{k+c-1} = \frac{dx^c-(d+1)x^{c+1}+x^{c+d+1}}{(x-1)^2}. \tag{57}$$

The equation (57) can now be differentiated with respect to x as to obtain

$$\sum_{k=1}^{d} (c-1+k)\cdot(d+1-k)x^{k+c-2} = \tag{58}$$

$$\frac{(c+d-1)x^{c+d+1}-(c+d+1)x^{c+d}-(c-1)(d+1)x^{c+1}+(2dc+c-d+1)x^c-cdx^{c-1}}{(x-1)^3}.$$

We now assume that $c<d$ and e are positive integers. Then we have an expression for the sum

$$(c+1)(c+e)x^{c+e-1}+(c+2)(c+e+1)x^{c+e}+\cdots+d(d+e-1)x^{d+e-2} = \tag{59}$$

$$+\frac{d(d+e-1)x^{d+e+1}-c(c+e-1)x^{c+e+1}}{(x-1)^3}$$

$$-\frac{(2d^2+2de+e-2)x^{d+e}-(2c^2+2ce+e-2)x^{c+e}}{(x-1)^3}$$

$$+\frac{(d+1)(d+e)x^{d+e-1}-(c+1)(c+e)x^{c+e-1}}{(x-1)^3}.$$

This is proved in a similar way as above.
Again let us assume that positive integers $c<d$ and e are given. Then

$$(c+1)(e-c+d-1)x^{e-c+d-2}+(c+2)(e-c+d-2)x^{e-c+d-3}+\dots \tag{60}$$

$$+(d-1)(e+1)x^e+dex^{e-1} =$$

$$\frac{(c+1)(d+e-c-1)x^{d+e-c+1}-(2cd+2ce-2c^2+d-2c+e+1)x^{d+e-c}}{(x-1)^3}$$

$$-\frac{-c(d+e-c)x^{d+e-c-1}+(d+1)(e-1)x^{e+1}-(2de+e-d+1)x^e+dex^{e-1}}{(x-1)^3}$$

This is proved in a similar way as above.

4.2 The k-th equation, $k < m$

Assume first that $k = 2h + 1$ is odd, and that we are in the case $k < m$.
The numbers s_k, and t_k satisfy a number of identities which can be used to verify the corresponding quadratic equations.
Assume first that $k = 2h + 1$ is odd, and that we are in the case $k < m$. Then we get the relation

$$2\lambda^3 \cdot s_k + \lambda^2 \cdot \sum_{i=1}^{h} s_i s_{k-i} = 2 \cdot \lambda^2 \cdot (\lambda^{2k} + 1)(2\lambda + k - 1) + \tag{61}$$

$$2 \cdot [\sum_{j=1}^{h} (\lambda^{2k-2j} + \lambda^{2j}) \cdot (2\lambda + j - 1) \cdot (2\lambda + k - j - 1)\ ] \ .$$

We first verify that the relations (61) imply the validity of the first half of the equations for the associated quasi symmetric quadratic system.
For (61) we have to calculate in the quadratic equation that contributes to product labels which are odd powers of λ, indeed to λ^k with positive odd integral k with $k < m$, the corresponding terms involving

$$tm^+_{k,2i} = x_{2i}x_{2k-2i} + x_{2i+1}x_{2k-2i+1} \text{ and } tm^-_{k,2i} = x_{2i}x_{2k-2i+1} + x_{2i+1}x_{2k-2i}. \tag{62}$$

Using (44) and (45) we may write

$$x_{2i} = \frac{1}{2}\lambda^{m-i-1} \cdot s_i + \lambda^{m-2} \cdot (2\lambda + i - 1), \tag{63}$$

$$x_{2i+1} = \frac{1}{2}\lambda^{m-i-1} \cdot s_i - \lambda^{m-2} \cdot (2\lambda + i - 1), \tag{64}$$

and

$$x_{2k-2i} = \frac{1}{2}\lambda^{m-(k-i)-1} \cdot s_{k-i} + \lambda^{m-2} \cdot (2\lambda + k - i - 1), \tag{65}$$

$$x_{2k-2i+1} = \frac{1}{2}\lambda^{m-(k-i)-1} \cdot s_{k-i} - \lambda^{m-2} \cdot (2\lambda + k - i - 1). \tag{66}$$

Now we substitute (63), (64), (65) and (66) into the two terms of (62) as to find the following expressions:

$$tm^+_{k,2i} = \frac{1}{2}\lambda^{2m-k-2} s_i s_{k-i} + 2\lambda^{2m-4} \cdot (2\lambda + i - 1)(2\lambda + k - i - 1), \tag{67}$$

$$tm^-_{k,2i} = \frac{1}{2}\lambda^{2m-k-2} s_i s_{k-i} - 2\lambda^{2m-4} \cdot (2\lambda + i - 1)(2\lambda + k - i - 1). \tag{68}$$

The sum of the corresponding terms in the quadratic function that has a product label λ^k will be the expression

$$(\lambda^{k-i} - \lambda^i)^2 tm^+_{k,2i} - (\lambda^{k-i} + \lambda^i)^2 tm^-_{k,2i} = \tag{69}$$

$$2[2(\lambda^{2k-2i}+\lambda^{2i})\lambda^{2m-4}(2\lambda+i-1)(2\lambda+k-i-1)-\lambda^{2m-2}\cdot s_i s_{k-i}].$$

The remaining two special terms come from the contribution of

$$(\lambda^k-1)^2 x_1 x_{2k}-(\lambda^k+1)^2 x_1 x_{2k+1}$$

which can be worked out in a similar but slightly easier way. Hence,

$$(\lambda^k-1)^2 x_1 x_{2k}-(\lambda^k+1)^2 x_1 x_{2k+1}=4\lambda^m[(\lambda^{m-2+2k}+\lambda^{m-2})(2\lambda+k-1)-\lambda^{m-1}\cdot s_k]\ . \quad (70)$$

Therefore, we have

$$\sum_{i=1}^{h}[(\lambda^{k-i}-\lambda^i)^2 tm^+_{k,2i}-(\lambda^{k-i}+\lambda^i)^2 tm^-_{k,2i}]+(\lambda^k-1)^2 x_1 x_{2k}-(\lambda^k+1)^2 x_1 x_{2k+1}=0. \quad (71)$$

If we take out from these expressions the common factor of $2\lambda^{2m-4}$, we see that we obtain all the terms within the two summation signs of the equation (61).
This completes the proof of the equivalence of solvability of equations (61) and the quadratic equations contributing to the product λ^k for odd k.

We describe the right hand side of (61) as a rational function as follows:

$$\begin{aligned}\mathrm{RHS}(61)=2\lambda^2\cdot(\lambda^{2k}+1)(2\lambda+k-1)+4\lambda^4\cdot\frac{(\lambda^2-1)^2(\lambda^{2k-2}-1)}{(\lambda^2-1)^3} \quad (72)\\ +\frac{2(k-2)\lambda^3(\lambda^2-1)^2(\lambda^{2k-2}-1)}{(\lambda^2-1)^3}\\ +\frac{\lambda^4(k-3)\lambda^{2k-2}-(k-1)\lambda^{2k}+(k-1)\lambda^6-(k-3)\lambda^4}{(\lambda^2-1)^3}.\end{aligned}$$

This comes out of the splitting of the factor

$$(2\lambda+j-1)(2\lambda+k-j-1)=4\lambda^2+2(k-2)\lambda+(j-1)(k-j-1)$$

and then by using the formula (54) with $x=\lambda^2$ twice for the first two parts, yet the summation formula (58) with the parameters of (58) written as $c=1, d=k-3$ and again with $x=\lambda^2$ for the last part. More precisely we have for the first part that

$$4\lambda^2(\lambda^2+\lambda^4+\cdots+\lambda^{2k-2})=4\lambda^4\cdot\frac{\lambda^{2k-2}-1}{\lambda^2-1}; \quad (73)$$

and we have for the second part that

$$2(k-2)\lambda(\lambda^2+\lambda^4+\cdots+\lambda^{2k-2})=2(k-2)\lambda^3\cdot\frac{\lambda^{2k-2}-1}{\lambda^2-1}; \quad (74)$$

and finally for the third part we get

$$((k-3)\lambda^4+2(k-4)\lambda^6+m\cdots+(k-4)\cdot 2\lambda^{2k-6}+(k-3)\lambda^{2k-4})= \quad (75)$$

$$\lambda^4 \cdot \frac{(k-3)\lambda^{2(k-1)} - (k-1)\lambda^{2(k-2)} + (k-1)\lambda^2 - (k-3)}{(\lambda^2-1)^3}.$$

This shows equation (72).

We now substitute the expressions (52) for s_j into each term of the left hand side of (61).

$$2\lambda^3 \cdot s_k + \lambda^2 \cdot \sum_{i=1}^{h} s_i s_{k-i} =$$

$$\begin{aligned} \frac{\lambda^2}{(\lambda^2-1)^3} \cdot \quad & [4\lambda \cdot (\lambda^{2k-1}(\lambda^3-\lambda+1)\cdot(\lambda^2-1)^2 + (\lambda^2-\lambda-1)(\lambda^2-1)^2) + \\ & 4h(\lambda^2-1)\lambda^{2k-2}(\lambda^3-\lambda+1)^2 + 4h(\lambda^2-1)(\lambda^2-\lambda-1)^2 + \\ & 4(\lambda^3-\lambda+1)(\lambda^2-\lambda-1)\cdot\lambda\cdot(\lambda^{2k-2}-1) \ ] \ . \end{aligned} \tag{76}$$

Note that the first line after the equality sign in (76) is just the term $4\lambda^3 s_k$, while the second and third line are obtained by cross multiplying and then summing

$$s_i = \frac{2\lambda^{2i-1}(\lambda^3-\lambda+1) + 2(\lambda^2-\lambda-1)}{(\lambda^2-1)}$$

with

$$s_{k-i} = \frac{2\lambda^{2k-2i-1}(\lambda^3-\lambda+1) + 2(\lambda^2-\lambda-1)}{(\lambda^2-1)}.$$

Two of the four product terms can be summed trivially as they have no terms index with i, while the other two terms can be combined to form a geometric sum similar to (54) with $x = \lambda^2$.

We then work out the numerators of both rational functions on the right hand sides of (72) and of (76) as a polynomial in λ. Indeed we find that these two polynomials agree. More precisely it is easily seen that the resulting polynomials have some terms divisible by λ^{2k}, and some other terms independent of k. Thus we can write the answer for both calculations in the form

$$p(\lambda)\cdot\lambda^{2k} + q(\lambda),$$

where it can be checked that for both sides we get the same answer as

$$p(\lambda) = 4\lambda^9+4h\lambda^8-12\lambda^7-(12h-8)\lambda^6+(8h+8)\lambda^5+(12h-16)\lambda^4-(16h-4)\lambda^3+4\lambda^2+(8h-4)\lambda-4h,$$

$$q(\lambda) = 4\lambda^9+(4h-8)\lambda^8-(8h+8)\lambda^7-(8h-16)\lambda^6+(16h+4)\lambda^5+(8h-4)\lambda^4-8h\lambda^3-4h\lambda^2.$$

This completes the proof of equation (61).

Assume that $k = 2h$ is even. Then we get the relation

$$2\lambda^3 \cdot s_k + \lambda^2 \cdot \sum_{i=1}^{h-1} s_i s_{k-i} + \frac{1}{2}\lambda^2 \cdot s_h^2 = 2\cdot\lambda^2\cdot(\lambda^{2k}+1)(2\lambda+k-1) + \tag{77}$$

$$2\cdot\sum_{j=1}^{h-1}\ (\lambda^{2k-2j}+\lambda^{2j})\cdot(2\lambda+j-1)(2\lambda+k-j-1)+2\lambda^{k}(2\lambda+h-1)^2\ \ .$$

By using a similar way as what we did for $k<m$ equation and k is odd, thus we can write the answer for both calculations in the form

$$p(\lambda)\cdot\lambda^{2k}+q(\lambda),$$

where it can be checked that for both sides we get the same answer as

$$p(\lambda)=4\lambda^9+(2k-2)\lambda^8-12\lambda^7+(14-6k)\lambda^6+(4+4k)\lambda^5+(6k-22)\lambda^4+(12-8k)\lambda^3+4\lambda^2$$
$$+(4k-8)\lambda+2(1-k),$$
$$q(\lambda)=4\lambda^9+(2k-10)\lambda^8-(4k+4)\lambda^7+(20-4k)\lambda^6+(8k-4)\lambda^5+(4k-8)\lambda^4+(4-4k)\lambda^3+(2-2k)\lambda^2.$$

This completes the proof of equation (77).

4.3 The m-th equation

Assume that $k=2l+1$ is odd, and that we are in the case $k=m$.
Then we get the relation

$$2\lambda^4\cdot t_m+\lambda^2\cdot\sum_{i=1}^{l}\ s_is_{m-i}=2\cdot\lambda^2\cdot(\lambda^{2m}+1)(2\lambda^2+m-1)+ \tag{78}$$
$$2\cdot[\sum_{j=1}^{l}\ (\lambda^{2m-2j}+\lambda^{2j})\cdot(2\lambda+j-1)(2\lambda+m-j-1)\ ]\ \ .$$

By using a similar way as what we did for $k<m$ equation and k is odd, thus we can write the answer for both calculations in the form

$$p(\lambda)\cdot\lambda^{2m}+q(\lambda),$$

where it can be checked that for both sides we get the same answer as

$$p(\lambda)=4\lambda^{10}+(2m-14)\lambda^8+(26-6m)\lambda^6+(4m-8)\lambda^5+(6m-26)\lambda^4 \tag{79}$$
$$+(16-8m)\lambda^3+4\lambda^2+(4m-8)\lambda+(2-2m),$$
$$q(\lambda)=4\lambda^{10}+(2m-22)\lambda^8+(8-4m)\lambda^7+(32-4m)\lambda^6+(8m-16)\lambda^5 \tag{80}$$
$$+(4m-12)\lambda^4+(8-4m)\lambda^3+(2-2m)\lambda^2.$$

This completes the proof of equation (78).

Assume that $k=m=2l$ is even. Then we get the relation

$$2\lambda^4\cdot t_m+\lambda^2\cdot\sum_{i=1}^{l-1}\ s_is_{m-i}+\frac{1}{2}\lambda^2\cdot s_l^2=2\cdot\lambda^2\cdot(\lambda^{2m}+1)(2\lambda^2+m-1)+ \tag{81}$$

$$2 \cdot \sum_{j=1}^{l-1} \ (\lambda^{2m-2j} + \lambda^{2j}) \cdot (2\lambda + j - 1)(2\lambda + m - j - 1) + 2\lambda^m (2\lambda + l - 1)^2 \ .$$

By using a similar way as what we did for $k < m$ equation and k is odd, thus we can write the answer for both calculations in the form

$$p(\lambda) \cdot \lambda^{2m} + q(\lambda),$$

where it can be checked that for both sides we get the same answer as

$$p(\lambda) = 4\lambda^{10} + (2m - 14)\lambda^8 + (26 - 6m)\lambda^6 + (4m - 8)\lambda^5 + (6m - 26)\lambda^4 \quad (82)$$
$$+(16 - 8m)\lambda^3 + 4\lambda^2 + (4m - 8)\lambda - 2(m - 1),$$
$$q(\lambda) = 4\lambda^{10} + (2m - 22)\lambda^8 + (8 - 4m)\lambda^7 + (32 - 4m)\lambda^6 + (8m - 16)\lambda^5 + (4m - 12)\lambda^4 \quad (83)$$
$$+(8 - 4m)\lambda^3 + (2 - 2m)\lambda^2.$$

This completes the proof of equation (81).

5 Quadratic operators and almost t-harmonic matrices

We resume the discussion of the previous example in order to explain how the above result is applied numerically. Let us consider the case $n = 8$ with the standard Nasir Labeling. Then choose the variables x_i according to the signed labels as follows:

$$x_1 \leftrightarrow +1,\ x_2 \leftrightarrow \lambda, x_3 \leftrightarrow -\lambda,\ x_4 \leftrightarrow \lambda^2,\ x_5 \leftrightarrow -\lambda^2,\ x_6 \leftrightarrow \lambda^3,\ x_7 \leftrightarrow -\lambda^4,\ x_8 \leftrightarrow \lambda^4.$$

From the previous section, we obtain that the solution for quadratic operators with $\lambda = 2$ and $t = 1$ is $x_1 = 8$, $x_2 = 9$, $x_3 = 1$, $x_4 = 27$, $x_5 = 9$, $x_6 = 9$, $x_7 = 1$ and $x_8 = 8$.
Let

$$u = (\sqrt{8}, 3, 1, \sqrt{27}, 3, 3, 1, \sqrt{8}) \in M_{8,1}$$

and A=diag$(1, 2, -2, 4, -4, 8, -8, 16) \in M_8$. Note that the entries for the column vector u is the square root of the solution for quadratic operators. The multiplicative Nasir labeling will be the entries for the diagonal matrix A respectively. Then, we have $w_0 = \langle u, u\rangle$=72, $w_1 = \langle Au, u\rangle$=288 and in general $w_i = \langle A^i u, u\rangle$. For the case $t = 1$, we have $q_k = w_{k-1}w_{k+1} - w_k^2$. By subsititute A and u as above, we have $q_2 = q_4 = q_6 = 0$ but $q_i > 0$ for $i = 1, 3, 5, 7$. Hence, A is the almost 1-harmonic, as required.

References

[1] A. Dress, S. Grünewald, I. Gutman, M. Lepović and D. Vidović, On the number of walks in trees, 2002.

[2] A. Dress, I. Gutman, Asymptotic Results Regarding the Number of Walks in a Graph, preprint, University of Bielefeld, 2002.

[3] A. Dress, A. Gutman, On the number of walks in a graph, to appear in Appl. Math. Letters

[4] M.I. Zakharevitch, On behaviour of trajectories and the ergodic hypothesis for quadratic transformations of the simplex, Uspechi Matem. Nauk 33(1978) 207-208 (in Russian)

[5] N.N. Ganikhodjaev, On the regular direct product of quadratic stochastic operators, submitted paper.