8. More about Vectors

8.1 Dot Product of Vectors (Key Stage 4 and beyond)

8.1.1 Concept of Dot Products

8.1.2 Operation of Dot Products

8.1.3 Applications of Dot Products

8.1.4 Dot Products in 3-D Space

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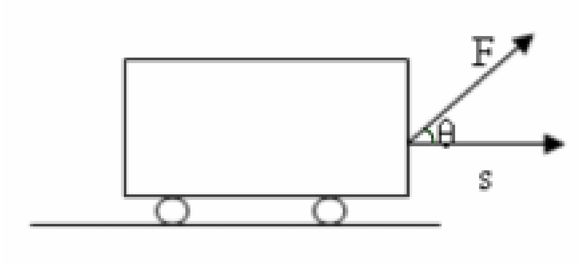
8.1.1 Concept of Dot Products

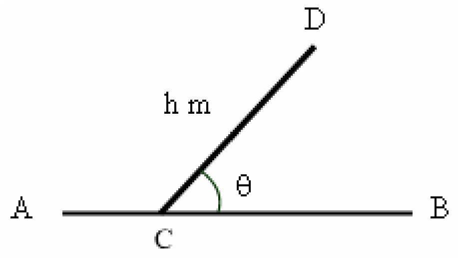


A force $\vec{F}$ applies on a car with displacement $\vec{s}$ and $\vec{F}$ makes an angle θ with the displacement $\vec{s}$ (as shown in the graph above).

By definition, the work done W is:
$W = Fs\cos\theta$.

The work done equals to the component of the force in the direction of $\vec{s}$ multiplied by the distance that the car has moved.
The work done depends on the magnitude and the direction of the two vectors: the force and the displacement.

Example 1



A stick was obliquely inserted on the ground, its length was h m and the included angle between the stick and the ground was θ. The length of the shadow of the stick is $h\cos\theta$ (m).

In this example, we have to find the projection of the stick CD on the ground AB. This kind of examples has a close relationship to dot product that will be discussed later.

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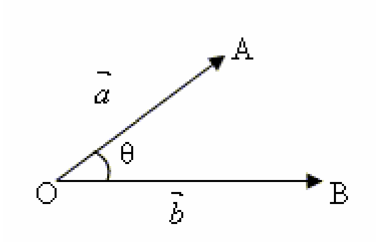


Given that the angle between the two vectors $\vec{a}$ and $\vec{b}$ is θ. The dot product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

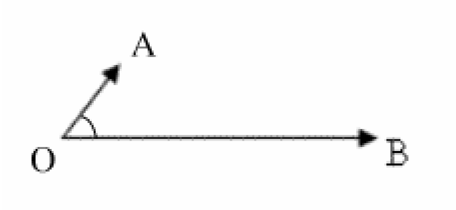
Dot product is one of the products of two vectors.

To calculate the dot product, we have to find out the angle between the two vectors.

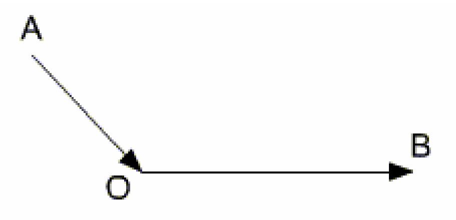
Example 2

Write down the angles between the two vectors.

1.



2.

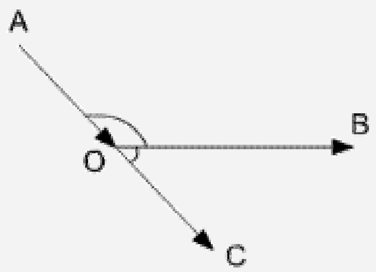


Answer

1. $\angle AOB$

2. Put $\overrightarrow{OC} = \overrightarrow{AO}$

$\angle BOC (= 180° - \angle AOB)$



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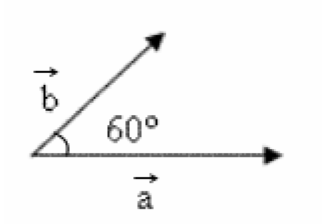
To find the included angle between the two vectors, we can simply translate the two vectors so that they have a common starting point. The angle θ ($0° \le \theta \le 180°$) made by the two vectors is the included angle.

Example 3

Given that $|\vec{a}| = 5$, $|\vec{b}| = 4$ and the angle θ between $\vec{a}$ and $\vec{b}$ is $60°$, find $\vec{a} \cdot \vec{b}$.



Answer

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= 5(4)\cos 60°$$
$$= 10$$

Example 4

Given that $|\vec{a}| = 2$, $|\vec{b}| = 3$, and the angle θ between $\vec{a}$ and $\vec{b}$ is $150°$, find $\vec{a} \cdot \vec{b}$ (correct to 3 significant figures).

Answer

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= 2(3)\cos 150°$$
$$= -5.20$$

Let $\vec{a}$ and $\vec{b}$ be non-zero vectors and the angle between them be θ.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

This is a commonly used formula for finding the angle between vectors.

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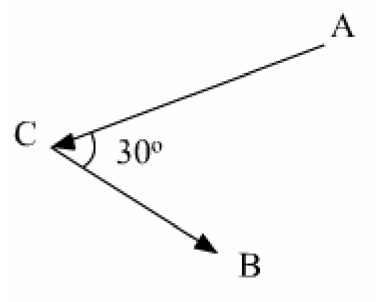
Example 5

In $\triangle ABC$, given that $\left|\overrightarrow{AC}\right| = 8$ and $\left|\overrightarrow{BC}\right| = 5$.

If $\angle ACB = 30°$, find $\overrightarrow{BC} \cdot \overrightarrow{AC}$.

If $\overrightarrow{BC} \cdot \overrightarrow{AC} = 20$, find $\angle BCA$.



Answer

1. $\overrightarrow{BC} \cdot \overrightarrow{AC}$

$= \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AC}\right| \cos 150°$

$= 8 \times 5 \times \left(\dfrac{-\sqrt{3}}{2} \right)$

$= -20\sqrt{3}$

2. $\overrightarrow{BC} \cdot \overrightarrow{AC} = \left|\overrightarrow{BC}\right| \left|\overrightarrow{AC}\right| \cos\theta$

$\cos\theta = \dfrac{\overrightarrow{BC} \cdot \overrightarrow{AC}}{\left|\overrightarrow{BC}\right| \left|\overrightarrow{AC}\right|}$

$= \dfrac{20}{8 \times 5}$

$= 0.5$

$\therefore \quad \angle BCA = 60°$

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$|\vec{b}|\cos\theta$ is the projection of $\vec{b}$ on $\vec{a}$.

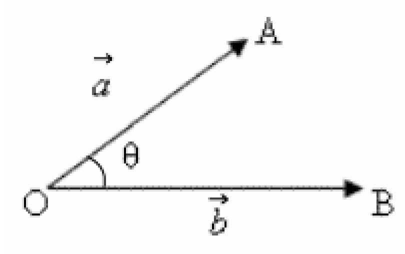
The Geometric Meaning of $\vec{a}\cdot\vec{b}$

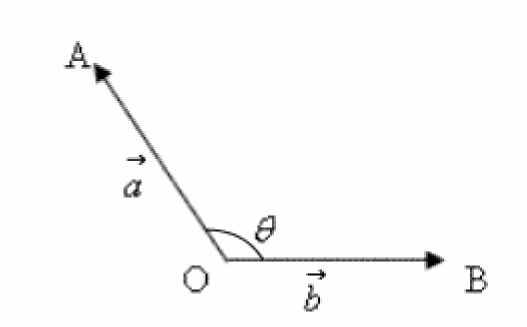
The dot product of $\vec{a}\cdot\vec{b}$ equals to the product of the magnitude of $\vec{a}$ ($|\vec{a}|$) and the projection of $\vec{b}$ on $\vec{a}$ ($|\vec{b}|\cos\theta$).

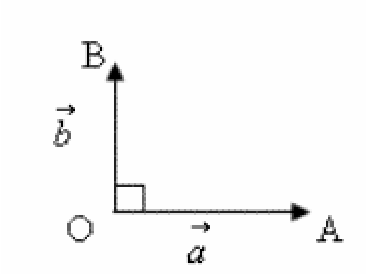
When θ is an acute angle, the value of $|\vec{b}|\cos\theta$ is positive.

When θ is an obtuse angle, the value of $|\vec{b}|\cos\theta$ is negative.

When $\theta = 90°$, the value of $|\vec{b}|\cos\theta$ is 0.







Example 6

Given that the projection of $\vec{b}$ on $\vec{a}$ is 4 and $|\vec{a}| = 5$, find $\vec{a}\cdot\vec{b}$.

Answer

$|\vec{b}|\cos\theta = 4$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= 5(4)$$
$$= 20$$

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Example 7

Given that $\vec{a}\cdot\vec{b} = 13$ and the projection of $\vec{a}$ on $\vec{b}$ is 4, find

1. the magnitude of vector $\vec{b}$.
2. the angle between the two vectors, if $|\vec{a}| = 8$.

Answer

1. $|\vec{a}|\cos\theta = 4$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad = 13$$

$$|\vec{b}|(4) = 13$$

$$|\vec{b}| = \frac{13}{4}$$

2. $|\vec{a}|\cos\theta = 4$

$$8\cos\theta = 4$$

$$\theta = 60°$$

Practice 8.1A

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8.1.2 Operation of Dot Products

Properties of dot products

The operation of a dot product is different from that of real numbers. Below are the laws for operations of dot products:

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors and λ is a real number, then

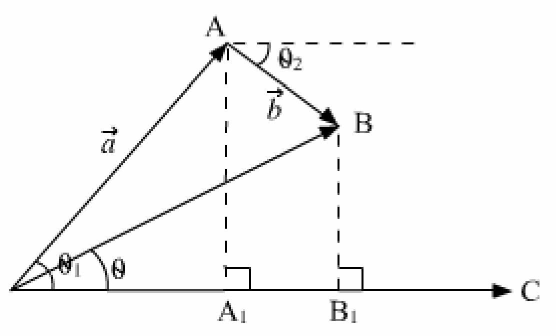
(1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

(2) $(\lambda\vec{a})\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{b})$

(3) $(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

Proofs of (1) and (2) may left for exercises. Let us prove (3).

Proof



Since $|\vec{a}+\vec{b}|\cos\theta = OB_1 = OA_1 + A_1B_1 = |\vec{a}|\cos\theta_1 + |\vec{b}|\cos\theta_2$,

so, $|\vec{a}+\vec{b}||\vec{c}|\cos\theta = |\vec{a}||\vec{c}|\cos\theta + |\vec{b}||\vec{c}|\cos\theta$.

i.e. $(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

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Example 1

Given that $|\vec{a}| = 6$, $|\vec{b}| = 4$ and the angle between $\vec{a}$ and $\vec{b}$ is $60°$, find $(\vec{a} + 2\vec{b}) \cdot (\vec{a} - 3\vec{b})$.

Answer

$$
\begin{aligned}
& (\vec{a} + 2\vec{b}) \cdot (\vec{a} - 3\vec{b}) \\
&= \vec{a} \cdot \vec{a} - 3\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{a} - 6\vec{b} \cdot \vec{b} \\
&= |\vec{a}|^2 - \vec{a} \cdot \vec{b} - 6|\vec{b}|^2 \\
&= |\vec{a}|^2 - |\vec{a}||\vec{b}|\cos\theta - 6|\vec{b}|^2 \\
&= 6^2 - 6 \times 4 \cos 60° - 6 \times 4^2 \\
&= -72
\end{aligned}
$$

If $\vec{a}$, $\vec{b}$ are non-zero vectors.

1. If $\vec{a} \perp \vec{b}$, then $\vec{a} \cdot \vec{b} = 0$.
2. If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$.

Example 2

Given that $\vec{a}$ and $\vec{c}$, $\vec{a} + \vec{b}$ and $\vec{c}$ are perpendicular to each other, prove that $\vec{b}$ and $\vec{c}$ are perpendicular to each other.

Answer

$\vec{a}$ and $\vec{c}$, $\vec{a} + \vec{b}$ and $\vec{c}$ are perpendicular to each other.

$\therefore \quad \vec{a} \cdot \vec{c} = 0$

$(\vec{a} + \vec{b}) \cdot \vec{c} = 0$

$\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = 0$

$\vec{b} \cdot \vec{c} = 0$

$\therefore \vec{b}$ and $\vec{c}$ are perpendicular to each other.

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Example 3

Given that $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 4$, if $\vec{a} + k\vec{b}$ and $\vec{a} - k\vec{b}$ are perpendicular to each other, find k.

Answer

$\vec{a} + k\vec{b}$ and $\vec{a} - k\vec{b}$ are perpendicular to each other

$$\therefore \quad \left(\vec{a} + k\vec{b}\right)\cdot\left(\vec{a} - k\vec{b}\right) = 0$$

$$\left(\vec{a}\cdot\vec{a}\right) - k^2\left(\vec{b}\cdot\vec{b}\right) = 0$$

$$\left|\vec{a}\right|^2 - k^2\left|\vec{b}\right|^2 = 0$$

$$9 - 16k^2 = 0$$

$$k = \pm\frac{3}{4}$$

Answer: When $k = \pm\frac{3}{4}$, $\vec{a} + k\vec{b}$ and $\vec{a} - k\vec{b}$ are perpendicular to each other.

Example 4

$\vec{e_1}$, $\vec{e_2}$ are unit vectors and the angle between them is $45°$. Find $\left(\vec{e_1} + \vec{e_2}\right)\cdot\left(\vec{e_1} - 3\vec{e_2}\right)$.

Answer

$$\left(\vec{e_1} + \vec{e_2}\right)\cdot\left(\vec{e_1} - 3\vec{e_2}\right)$$

$$= \left(\vec{e_1}\right)\cdot\left(\vec{e_1}\right) + \left(\vec{e_1}\right)\cdot\left(-3\vec{e_2}\right) + \left(\vec{e_2}\right)\cdot\left(\vec{e_1}\right) + \left(\vec{e_2}\right)\cdot\left(-3\vec{e_2}\right)$$

$$= \left|\vec{e_1}\right|^2 - 2\vec{e_1}\cdot\vec{e_2} - 3\left|\vec{e_2}\right|^2$$

$$= 1 - 2\cos 45^\circ - 3$$

$$= -2 - \sqrt{2}$$

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$$\vec{i}\cdot\vec{i}=1,\ \ \vec{j}\cdot\vec{j}=1,\ \ \vec{i}\cdot\vec{j}=0\ \ \text{and}\ \ \vec{j}\cdot\vec{i}=0.$$

When $\vec{a}=x_1\vec{i}+y_1\vec{j}$, $\vec{b}=x_2\vec{i}+y_2\vec{j}$

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (x_1\vec{i}+y_1\vec{j})\cdot(x_2\vec{j}+y_2\vec{j})\\ &= x_1x_2\vec{i}\cdot\vec{i}+x_1y_2\vec{i}\cdot\vec{j}+x_2y_1\vec{j}\cdot\vec{i}+y_1y_2\vec{j}\cdot\vec{j}\\ &= x_1x_2+y_1y_2\end{aligned}$$

$\therefore\ \vec{a}\cdot\vec{b}=x_1x_2+y_1y_2$.

The dot product of two vectors equals to the sum of the product of their corresponding x and y components.

Example 5

Given that $\vec{a}=5\vec{i}+7\vec{j}$, $\vec{b}=-6\vec{i}-4\vec{j}$, find $\vec{a}\cdot\vec{b}$.

Answer

$\vec{a}\cdot\vec{b}=5(-6)+(7)(-4)=-58$

Example 6

Given that $\vec{a}=2\vec{i}+3\vec{j}$, $\vec{b}=-2\vec{i}+4\vec{j}$ and $\vec{c}=-\vec{i}-\vec{j}$, find the following values

1. $\vec{a}\cdot\vec{b}$
2. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$
3. $\vec{a}\cdot(\vec{b}+\vec{c})$

Answer

1. $\vec{a}\cdot\vec{b}$

$=(2\vec{i}+3\vec{j})\cdot(-2\vec{i}+4\vec{j})$

$=2(-2)+3(4)$

$=-4+12$

$=8$

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2. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$

$= [(2\vec{i}+3\vec{j})+(-2\vec{i}+4\vec{j})]\cdot[(2\vec{i}+3\vec{j})-(-2\vec{i}+4\vec{j})]$

$= (7\vec{j})\cdot(4\vec{i}-\vec{j})$

$= 7(-1)$

$= -7$

3. $\vec{a}\cdot(\vec{b}+\vec{c})$

$= (2\vec{i}+3\vec{j})\cdot[(-2\vec{i}+4\vec{j})+(-\vec{i}-\vec{j})]$

$= (2\vec{i}+3\vec{j})\cdot(-3\vec{i}+3\vec{j})$

$= 2(-3)+3(3)$

$= -6+9$

$= 3$

Given that $\vec{a} = x_1\vec{i} + y_1\vec{j}$.

$\vec{a}\cdot\vec{a} = x_1^2 + y_1^2$

$\therefore\ |\vec{a}| = \sqrt{(x_1^2 + y_1^2)}$

Example 7

Find the magnitude of the following vectors.

1. $3\vec{i}+4\vec{j}$
2. $-5\vec{i}+12\vec{j}$
3. $-2\vec{i}-2\vec{j}$

Answer

1. $(3\vec{i}+4\vec{j})\cdot(3\vec{i}+4\vec{j}) = 3(3)+4(4) = 25$

$\therefore |3\vec{i}+4\vec{j}| = 5$

2. $(-5\vec{i}+12\vec{j})(-5\vec{i}+12\vec{j}) = (-5)(-5)+12(12) = 169$

$\therefore |-5\vec{i}+12\vec{j}| = 13$

3. $(-2\vec{i}-2\vec{j})\cdot(-2\vec{i}-2\vec{j}) = (-2)(-2)+(-2)(-2) = 8$

$\therefore |-2\vec{i}-2\vec{j}| = 2\sqrt{2}$

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$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^{\ 2}+y_1^{\ 2}}\sqrt{x_2^{\ 2}+y_2^{\ 2}}}$$

Example 8

Given that $\vec{a} = 3\vec{i} + 4\vec{j}$, $\vec{b} = -3\vec{i} + 4\vec{j}$, find the angle between $\vec{a}$ and $\vec{b}$ (correct to 1 decimal place).

Answer

$|\vec{a}| = \sqrt{3^2 + 4^2}$
$= 5$

$|\vec{b}| = \sqrt{(-3)^2 + 4^2}$
$= 5$

$\vec{a}\cdot\vec{b} = (3\vec{i} + 4\vec{j})\cdot(-3\vec{i} + 4\vec{j})$
$= 3(-3) + 4(4)$
$= -9 + 16$
$= 7$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$
$= \dfrac{7}{5\times 5}$
$= 0.28$
$\theta = 73.7°$

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Example 9

Given that $\vec{a} = 2\vec{i} - \vec{j}$, $\vec{b} = 3\vec{i} + 6\vec{j}$, find the angle between $\vec{a}$ and $\vec{b}$.

Answer

$$|\vec{a}| = \sqrt{2^2 + (-1)^2}$$
$$= \sqrt{5}$$

$$|\vec{b}| = \sqrt{3^2 + 6^2}$$
$$= 3\sqrt{5}$$

$$\vec{a} \cdot \vec{b} = (2\vec{i} - \vec{j}) \cdot (3\vec{i} + 6\vec{j})$$
$$= 2(3) - 1(6)$$
$$= 0$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$
$$= \frac{0}{\sqrt{5}(3\sqrt{5})}$$
$$= 0$$
$$\theta = 90°$$

Given that $\vec{a} = x_1\vec{i} + y_1\vec{j}$, $\vec{b} = x_2\vec{i} + y_2\vec{j}$

1. If $\vec{a} \perp \vec{b}$, then $x_1x_2 + y_1y_2 = 0$.
2. If $x_1x_2 + y_1y_2 = 0$, then $\vec{a} \perp \vec{b}$.

Example 10

Given that $\vec{a} = \vec{i} - 3\vec{j}$ and $\vec{b} = 6\vec{i} + 2\vec{j}$, prove that $\vec{a} \perp \vec{b}$.

Answer

$$\vec{a} \cdot \vec{b} = (\vec{i} - 3\vec{j}) \cdot (6\vec{i} + 2\vec{j})$$
$$= 1(6) - 3(2)$$
$$= 6 - 6$$
$$= 0$$
$$\therefore \vec{a} \perp \vec{b}$$

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Example 11

Given that A =(1, 2), B =(2, 3) and C =(−2, 5), prove that Δ ABC is a right-angled triangle.

Answer

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= 2\vec{i} + 3\vec{j} - (\vec{i} + 2\vec{j})$
$= 2\vec{i} + 3\vec{j} - \vec{i} - 2\vec{j}$
$= \vec{i} + \vec{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= -2\vec{i} + 5\vec{j} - (\vec{i} + 2\vec{j})$
$= -2\vec{i} + 5\vec{j} - \vec{i} - 2\vec{j}$
$= -3\vec{i} + 3\vec{j}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = (\vec{i} + \vec{j}) \cdot (-3\vec{i} + 3\vec{j})$
$= 1(-3) + 1(3)$
$= 0$

$\therefore \overrightarrow{AB} \perp \overrightarrow{AC}$
∴ ΔABC is a right-angled triangle.

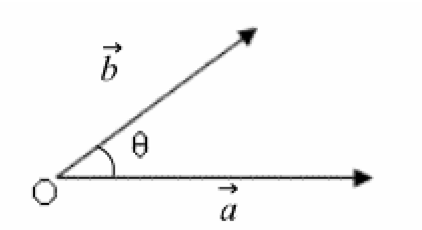
Practice 8.1B

Now, let us summarize the properties of the dot products in a 2-dimentional coordinate system.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$|\vec{b}|\cos\theta$ is the projection of vector $\vec{b}$ on $\vec{a}$.



Properties

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1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2. $(\lambda \vec{a}) \cdot \vec{b} = \lambda (\vec{a} \cdot \vec{b})$

3. $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

4. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

5. If $\vec{a} \perp \vec{b}$, then $\vec{a} \cdot \vec{b} = 0$.

6. Given that $\vec{a} = x_1 \vec{i} + y_1 \vec{j}$, $\vec{b} = x_2 \vec{i} + y_2 \vec{j}$.
   $\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2$
   $\vec{a} \cdot \vec{a} = x_1^2 + y_1^2$ and $|\vec{a}| = \sqrt{(x_1^2 + y_1^2)}$
   $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$$

7. If $\vec{a} \perp \vec{b}$, then $x_1 x_2 + y_1 y_2 = 0$

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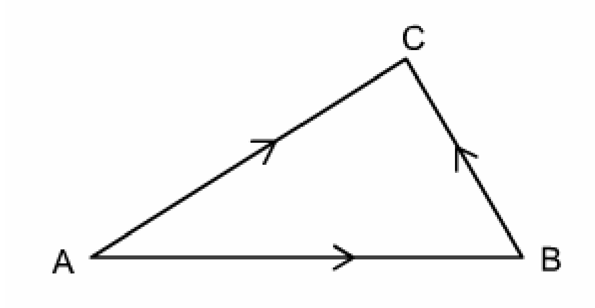
8.1.3 Applications of Dot Products

Example 1

Given that $\overrightarrow{AB} = 4\vec{i} + 2\vec{j}$, $\overrightarrow{AC} = 3\vec{i} + 4\vec{j}$,

1. prove that ΔABC be a right-angled triangle.
2. find the area of ΔABC.



Answer

1. $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= (3\vec{i} + 4\vec{j}) - (4\vec{i} + 2\vec{j})$

$= -\vec{i} + 2\vec{j}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (4\vec{i} + 2\vec{j}) \cdot (-\vec{i} + 2\vec{j})$

$= 4(-1) + 2(2)$

$= 4 - 4$

$= 0$

$\therefore$ ΔABC is a right-angled triangle.

2. Area

$= \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|$

$= \frac{1}{2}\sqrt{4^2 + 2^2} \times \sqrt{(-1)^2 + (2)^2}$

$= \frac{1}{2}\sqrt{20} \times \sqrt{5}$

$= 5$

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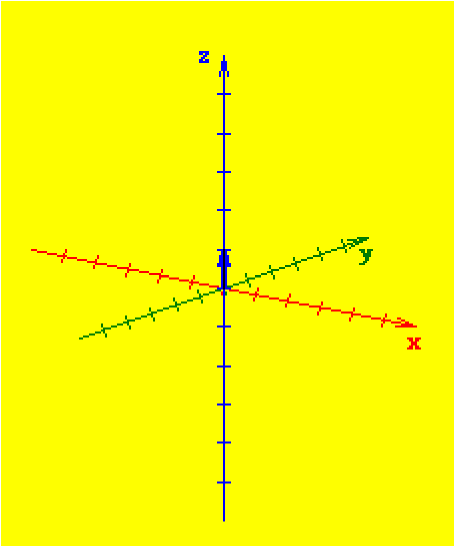
8.1.4 Dot Products in 3-D Space

$\vec{i}$ is a unit vector along the x-axis, $\vec{j}$ is a unit vector along the y-axis and $\vec{k}$ is a unit vector along the z-axis (as shown in the graph below).

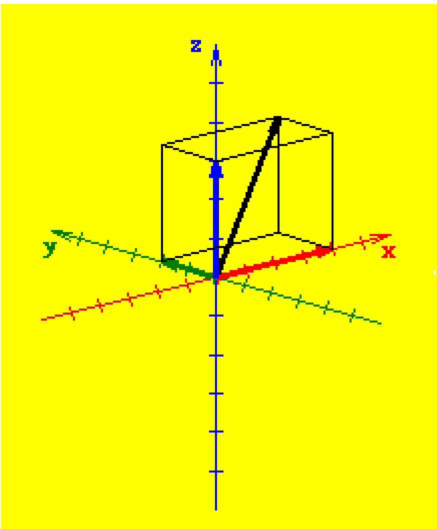


Hence, a 3 dimensional vector can be rewritten in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.

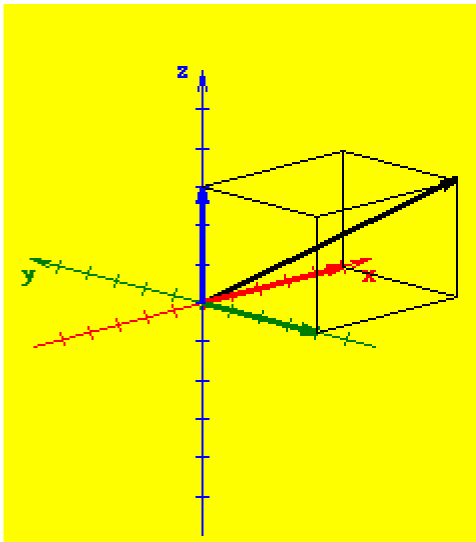
Example 1

Express the vector in black in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.

1.



2.



Answer

1. $4\vec{i} + 2\vec{j} + 3\vec{k}$
2. $5\vec{i} - 4\vec{j} + 3\vec{k}$

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$$\vec{i}\cdot\vec{i}=1,\ \vec{j}\cdot\vec{j}=1,\ \vec{k}\cdot\vec{k}=1$$
$$\vec{i}\cdot\vec{j}=0 \text{ and } \vec{j}\cdot\vec{i}=0,\ \vec{j}\cdot\vec{k}=0 \text{ and } \vec{k}\cdot\vec{j}=0,\ \vec{k}\cdot\vec{i}=0 \text{ and } \vec{i}\cdot\vec{k}=0.$$

Example 2

Evaluate

1. $(4\vec{i}+3\vec{j}-\vec{k})\cdot\vec{j}$
2. $(\vec{i}+3\vec{j}+2\vec{k})\cdot\vec{k}$
3. $\vec{i}\cdot(\vec{i}+2\vec{j}-\vec{k})$
4. $\vec{j}\cdot(2\vec{i}-\vec{j}+4\vec{k})$

Answer

1. $$\begin{aligned}&(4\vec{i}+3\vec{j}-\vec{k})\cdot\vec{j}\\ &=4\vec{i}\cdot\vec{j}+3\vec{j}\cdot\vec{j}-\vec{k}\cdot\vec{j}\\ &=4(0)+3-0\\ &=3\end{aligned}$$

2. $$\begin{aligned}&(\vec{i}+3\vec{j}+2\vec{k})\cdot\vec{k}\\ &=\vec{i}\cdot\vec{k}+3\vec{j}\cdot\vec{k}+2\vec{k}\cdot\vec{k}\\ &=0+3(0)+2\\ &=2\end{aligned}$$

3. $$\begin{aligned}&\vec{i}\cdot(\vec{i}+2\vec{j}-\vec{k})\\ &=\vec{i}\cdot\vec{i}+2\vec{i}\cdot\vec{j}-\vec{i}\cdot\vec{k}\\ &=1+2(0)-0\\ &=1\end{aligned}$$

4. $$\begin{aligned}&\vec{j}\cdot(2\vec{i}-\vec{j}+4\vec{k})\\ &=2\vec{j}\cdot\vec{i}-\vec{j}\cdot\vec{j}+4\vec{j}\cdot\vec{k}\\ &=2(0)-1+4(0)\\ &=-1\end{aligned}$$

If $\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ and $\vec{b}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$,
then $\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2+z_1z_2$.

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Example 3

Given that $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$, $\vec{b} = 4\vec{i} + 5\vec{j} - 2\vec{k}$, find $\vec{a} \cdot \vec{b}$.

Answer

$$\vec{a} \cdot \vec{b} = (\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} + 5\vec{j} - 2\vec{k})$$
$$= 1(4) + (-2)(5) + 4(-2)$$
$$= 4 - 10 - 8$$
$$= -14$$

Example 4

Given that $\vec{a} = 3\vec{i} - \vec{j} - 2\vec{k}$, $\vec{b} = 4\vec{i} + 6\vec{j} + (m+2)\vec{k}$ and $\vec{a} \cdot \vec{b} = 8$, find the value of *m*.

Answer

$$(3\vec{i} - \vec{j} - 2\vec{k}) \cdot [4\vec{i} + 6\vec{j} + (m+2)\vec{k}] = 8$$
$$3(4) - 1(6) - 2(m+2) = 8$$
$$12 - 6 - 2m - 4 = 8$$
$$2 - 2m = 8$$
$$2m = -6$$
$$m = -3$$

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

Example 5

Using the above result, find:

1. $|3\vec{i} + 2\vec{j} + \vec{k}|$
2. $|-\vec{i} + \vec{j} + 2\vec{k}|$
3. $|-\vec{i} - 3\vec{j} + 4\vec{k}|$

Answer

1. $(3\vec{i} + 2\vec{j} + \vec{k}) \cdot (3\vec{i} + 2\vec{j} + \vec{k})$
$$= 3(3) + 2(2) + 1(1)$$
$$= 9 + 4 + 1$$
$$= 14$$
$$\therefore |3\vec{i} + 2\vec{j} + \vec{k}| = \sqrt{14}$$

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2. $(-\vec{i}+\vec{j}+2\vec{k})\cdot(-\vec{i}+\vec{j}+2\vec{k})$
$= (-1)(-1)+1(1)+2(2)$
$= 1+1+4$
$= 6$
$\therefore \left|-\vec{i}+\vec{j}+2\vec{k}\right| = \sqrt{6}$

3. $(-\vec{i}-3\vec{j}+4\vec{k})\cdot(-\vec{i}-3\vec{j}+4\vec{k})$
$= (-1)(-1)+(-3)(-3)+4(4)$
$= 1+9+16$
$= 26$
$\therefore \left|-\vec{i}-3\vec{j}+4\vec{k}\right| = \sqrt{26}$

Practice 8.1C

Given $\vec{a} = x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ and $\vec{b} = x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$. If θ is the included angle between the two vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{x_1x_2+y_1y_2+z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}}.$$

Example 6

Given that $\vec{a} = 2\vec{i}+3\vec{j}+\vec{k}$, $\vec{b} = \vec{i}+3\vec{j}-2\vec{k}$, find the angle between $\vec{a}$ and $\vec{b}$ (correct to 3 significant figures).

Answer

$|\vec{a}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{1^2+3^2+(-2)^2} = \sqrt{14}$

$\vec{a}\cdot\vec{b} = (2\vec{i}+3\vec{j}+\vec{k})(\vec{i}+3\vec{j}-2\vec{k})$
$= 2(1)+3(3)+1(-2)$
$= 2+9-2$
$= 9$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

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$= \dfrac{9}{\sqrt{14}\sqrt{14}}$

$= \dfrac{9}{14}$

$\theta = 50.0°$

Let $\vec{a}$ and $\vec{b}$ be non-zero vectors.

1. If $\vec{a} \perp \vec{b}$, then $\vec{a} \cdot \vec{b} = 0$.
2. If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$.

Example 7

Given that $\vec{a} = 4\vec{i} + 3\vec{j} + 5\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 2\vec{k}$, prove that $\vec{a} \perp \vec{b}$.

Answer

$\vec{a} \cdot \vec{b} = (4\vec{i} + 3\vec{j} + 5\vec{k})(\vec{i} + 2\vec{j} - 2\vec{k})$

$= 4(1) + 3(2) + 5(-2)$

$= 4 + 6 - 10$

$= 0$

$\therefore \vec{a} \perp \vec{b}$

Example 8

Given that $\vec{a} = 4\vec{i} + 3\vec{j} + m\vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} + 2\vec{k}$ and $\vec{a} \perp \vec{b}$, find the value of *m*.

Answer

Since $\vec{a} \perp \vec{b}$

$\therefore \quad (4\vec{i} + 3\vec{j} + m\vec{k}) \cdot (2\vec{i} - 4\vec{j} + 2\vec{k}) = 0$

$4(2) + 3(-4) + m(2) = 0$

$8 - 12 + 2m = 0$

$m = 2$

Practice 8.1D

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Practice 8.1A

1. Given that $|\vec{a}| = 6$, $|\vec{b}| = 7$ and the included angle of the vectors = 45°. Find $\vec{a} \cdot \vec{b}$.

2. Given that $|\vec{b}| = 5$, the projection of $\vec{a}$ on $\vec{b}$ is 3.5. Find $\vec{a} \cdot \vec{b}$.

3. Given that $\vec{a} \cdot \vec{b} = -15$, $|\vec{a}| = 10$, $|\vec{b}| = 6$. Find the included angle of the vectors.

4. Given that $|\vec{a}| = 12$, $|\vec{b}| = 9$. If $\vec{a} \cdot \vec{b} = 54$. Find the included angle of $\vec{a}$ and $\vec{b}$.

5. Given that $|\vec{b}| = 3$, the projection of $\vec{a}$ on $\vec{b}$ is $0.5|\vec{b}|$. Find $\vec{a} \cdot \vec{b}$.

6. Given that $|\vec{a}| = 6$ and $\vec{e}$ is a unit vector. The included angle between them is 45°. Find the projection of $\vec{a}$ on $\vec{e}$.

Answer

1. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = 6(7)\cos 45° = 21\sqrt{2}$ (or 29.70)

2. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}|$ (The projection of $\vec{a}$ on $\vec{b}$) = 5(3.5) = 17.5

3.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore\ -15 = 10(6)\cos\theta$

$\cos\theta = -0.25$

$\theta = 104.5°$

4.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore\ 54 = 12(9)\cos\theta$

$\cos\theta = 0.5$

$\theta = 60°$

5.

The projection of $\vec{a}$ on $\vec{b}$ = $0.5|\vec{b}|$ = 0.5(3) = 1.5

$\vec{a} \cdot \vec{b} = |\vec{b}|$ (The projection of $\vec{a}$ on $\vec{b}$) = 3(1.5) = 4.5

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6.

$\vec{a}\cdot\vec{e} = |\vec{a}||\vec{e}|\cos\theta = |\vec{e}|$ (The projection of $\vec{a}$ on $\vec{b}$)

(6)(1)cos 45° = (1) (The projection of $\vec{a}$ on $\vec{b}$)

The projection of $\vec{a}$ on $\vec{e}$ = $3\sqrt{2}$ (4.24))

Practice 8.1B

1. $\overrightarrow{e_1}, \overrightarrow{e_2}$ are unit vectors and the included angle is $60°$. Find $\left(2\overrightarrow{e_1} - \overrightarrow{e_2}\right)\cdot\left(-3\overrightarrow{e_1} + 2\overrightarrow{e_2}\right)$.

2. $|\vec{a}| = 4$, $|\vec{b}| = 3$ and $\left(2\vec{a} - 3\vec{b}\right)\cdot\left(2\vec{a} + \vec{b}\right) = 61$. Find the included angle of $\vec{a}$ and $\vec{b}$.

3. Given $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = 4\vec{i} + 5\vec{j}$. Find

   (1) $|\vec{a}|$.

   (2) $|\vec{b}|$.

   (3) the included angle of $\vec{a}$ and $\vec{b}$.

4. $\vec{a} = 3\vec{i} + 4\vec{j}$, $\vec{b} = (x-1)\vec{i} + (3x-2)\vec{j}$ and $\vec{b}\perp\vec{a}$. Find *x*.

5. Given a right-angled triangle ABC, $\overrightarrow{AB} = 2\vec{i} + 3\vec{j}$, $\overrightarrow{AC} = \vec{i} + m\vec{j}$ and $\angle A = 90°$.

   Find the value of *m*.

Answer

1) $\because$ $\overrightarrow{e_1}, \overrightarrow{e_2}$ are unit vectors,

$\therefore |\overrightarrow{e_1}| = |\overrightarrow{e_2}| = 1$

$(2\overrightarrow{e_1} - \overrightarrow{e_2})\cdot(-3\overrightarrow{e_1} + 2\overrightarrow{e_2})$

$= \left(2\overrightarrow{e_1}\right)\cdot\left(-3\overrightarrow{e_1}\right) + \left(-\overrightarrow{e_2}\right)\cdot\left(-3\overrightarrow{e_1}\right) + \left(2\overrightarrow{e_1}\right)\cdot\left(2\overrightarrow{e_2}\right) + \left(-\overrightarrow{e_2}\right)\cdot\left(2\overrightarrow{e_2}\right)$

$= -6|\overrightarrow{e_1}|^2 + 3|\overrightarrow{e_1}||\overrightarrow{e_2}|\cos 60^{\circ} + 4|\overrightarrow{e_1}||\overrightarrow{e_2}|\cos 60^{\circ} - 2|\overrightarrow{e_2}|^2$

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$= -6 + 7\cos 60^{o} - 2$

$= -\dfrac{9}{2}$

2) Let the angle between $\vec{a}$ and $\vec{b}$ be θ。

$(2\vec{a} - 3\vec{b}) \cdot (2\vec{a} + \vec{b}) = 61$

$(2\vec{a}) \cdot (2\vec{a}) - (3\vec{b}) \cdot (2\vec{a}) + (2\vec{a}) \cdot (\vec{b}) - (3\vec{b}) \cdot (\vec{b}) = 61$

$4|\vec{a}|^2 - 6|\vec{b}||\vec{a}|\cos\theta + 2|\vec{a}||\vec{b}|\cos\theta - 3|b|^2 = 61$

$4(4)^2 - 4|\vec{a}||\vec{b}|\cos\theta - 3(3)^2 = 61$

$64 - 4(3)(4)\cos\theta - 27 = 61$

$\cos\theta = -\dfrac{1}{2}$

$\theta = 120^{o}$

3) (1) $|\vec{a}|^2 = |2\vec{i} - 3\vec{j}|^2 = 2^2 + (-3)^2 = 13$

$|\vec{a}| = \sqrt{13}$

(2) $|b|^2 = |4\vec{i} + 5\vec{j}|^2 = 4^2 + 5^2 = 41$

$|\vec{b}| = \sqrt{41}$

(3) Let the included angle of $\vec{a}$ and $\vec{b}$ be θ。

$\cos\theta$

$= \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$= \dfrac{(2)(4) + (-3)(5)}{\sqrt{13}\sqrt{41}}$

$= -0.303$

$\therefore \theta = 107.7°$

4)

Since $\vec{b} \perp \vec{a}$, $\vec{a} \cdot \vec{b} = 0$

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$(3\vec{i} + 4\vec{j}) \cdot [(x-1)\vec{i} + (3x-2)\vec{j}] = 0$

$3(x-1) + 4(3x-2) = 0$

$3x - 3 + 12x - 8 = 0$

$\therefore \quad x = \dfrac{11}{15}$

5)

$\angle A = 90^\circ$, AB⊥AC.

$(2\vec{i} + 3\vec{j}) \cdot (\vec{i} + m\vec{j}) = 0$

$2 + 3m = 0$

$\therefore \quad m = -\dfrac{2}{3}$

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Practice 8.1C

1. Find the dot product of the following vectors.

(1) $\vec{a} = 2\vec{i} + \vec{j} + 4\vec{k}$ and $\vec{b} = \vec{i} - 3\vec{j} + 2\vec{k}$

(2) $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = 6\vec{i} + 2\vec{j} - 2\vec{k}$

(3) $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 5\vec{j} + 3\vec{k}$

(4) $\vec{a} = \vec{i} + 4\vec{j} + 4\vec{k}$ and $\vec{b} = 7\vec{i} + \vec{j} - 4\vec{k}$

(5) $\vec{a} = 9\vec{i} + 5\vec{j} + \vec{k}$ and $\vec{b} = -3\vec{i} + 2\vec{j} + 7\vec{k}$

2. Find the magnitude of the following vectors.

(1) $2\vec{i} + 2\vec{j} - \vec{k}$

(2) $\vec{i} + 5\vec{j} - 2\vec{k}$

(3) $2\vec{i} + 3\vec{j} - 2\vec{k}$

(4) $4\vec{i} + 3\vec{j} - \vec{k}$

(5) $4\vec{i} + 7\vec{j} - 4\vec{k}$

Answer

1)

(1) $\vec{a} \cdot \vec{b}$

$= (2\vec{i} + \vec{j} + 4\vec{k}) \cdot (\vec{i} - 3\vec{j} + 2\vec{k})$

$= 2(1) + 1(-3) + 4(2)$

$= 7$

(2) $\vec{a} \cdot \vec{b}$

$= (\vec{i} + 2\vec{j} + \vec{k}) \cdot (6\vec{i} + 2\vec{j} - 2\vec{k})$

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$= 1(6) + 2(2) + 1(-2)$
$= 8$

(3) $\vec{a} \cdot \vec{b}$

$= (3\vec{i} - \vec{j} + 2\vec{k}) \cdot (\vec{i} + 5\vec{j} + 3\vec{k})$

$= (3)(1) + (-1)(5) + 2(3)$
$= 4$

(4) $\vec{a} \cdot \vec{b}$

$= (\vec{i} + 4\vec{j} + 4\vec{k}) \cdot (7\vec{i} + \vec{j} - 4\vec{k})$

$= 1(7) + 4(1) + 4(-4)$
$= -5$

(5) $\vec{a} \cdot \vec{b}$

$= (9\vec{i} + 5\vec{j} + \vec{k}) \cdot (-3\vec{i} + 2\vec{j} + 7\vec{k})$

$= 9(-3) + 5(2) + 1(7)$
$= -10$

2)

(1) $|2\vec{i} + 2\vec{j} - \vec{k}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

(2) $|\vec{i} + 5\vec{j} - 2\vec{k}| = \sqrt{1^2 + 5^2 + (-2)^2} = \sqrt{30}$

(3) $|2\vec{i} + 3\vec{j} - 2\vec{k}| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$

(4) $|4\vec{i} + 3\vec{j} - \vec{k}| = \sqrt{4^2 + 3^2 + (-1)^2} = \sqrt{26}$

(5) $|4\vec{i} + 7\vec{j} - 4\vec{k}| = \sqrt{4^2 + 7^2 + (-4)^2} = 9$

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Practice 8.1D

1. Given that $\vec{a} = \vec{i} + 7\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + \vec{j} - 6\vec{k}$. Find the included angle of $\vec{a}$ and $\vec{b}$ (correct to 3 significant figures).
2. Given that $\vec{a} = 3\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 3\vec{k}$. Find the included angle of $\vec{a}$ and $\vec{b}$ (correct to 3 significant figures).
3. Given that $\vec{a} = 5\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 4\vec{k}$. Prove $\vec{a} \perp \vec{b}$.
4. Given that $\vec{a} = \vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{b} = -2\vec{i} + m\vec{j} + 7\vec{k}$ and $\vec{a} \perp \vec{b}$. Find the value of *m*.
5. Given that $\vec{a} = 3\vec{i} + \vec{j} - 4\vec{k}$ and $\vec{b} = 7\vec{i} + (m-2)\vec{j} + 6\vec{k}$ and $\vec{a} \perp \vec{b}$. Find the value of *m*.

Answer

1.

$$|\vec{a}| = \sqrt{1^2 + 7^2 + 3^2} = \sqrt{59}$$

$$|\vec{b}| = \sqrt{1^2 + 1^2 + (-6)^2} = \sqrt{38}$$

$$\vec{a} \cdot \vec{b} = (\vec{i} + 7\vec{j} + 3\vec{k}) \cdot (\vec{i} + \vec{j} - 6\vec{k}) = 1(1) + (7)(1) + 3(-6) = -10$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$-10 = (\sqrt{59})(\sqrt{38})\cos\theta$$

$$\theta = 102°$$

2.

$$|\vec{a}| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$$

$$|\vec{b}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\vec{a} \cdot \vec{b} = (3\vec{i} - \vec{j} + 4\vec{k}) \cdot (-2\vec{i} + \vec{j} + 3\vec{k}) = 3(-2) + (-1)(1) + 4(3) = 5$$

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$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$5 = (\sqrt{26})(\sqrt{14})\cos\theta$

$\theta = 74.8°$

3.

$\vec{a} \cdot \vec{b} = (5\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (2\vec{i} + \vec{j} - 4\vec{k}) = 5(2) + 2(1) + 3(-4) = 0$

$\therefore \vec{a} \perp \vec{b}$

4.

$\because \vec{a} \perp \vec{b}$

$\therefore \vec{a} \cdot \vec{b} = 0$

$(\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (-2\vec{i} + m\vec{j} + 7\vec{k}) = 0$

$1(-2) + (-3)m + 2(7) = 0$
$12 - 3m = 0$
$m = 4$

5.

$\because \vec{a} \perp \vec{b}$

$\therefore \vec{a} \cdot \vec{b} = 0$

$(3\vec{i} + \vec{j} - 4\vec{k}) \cdot [7\vec{i} + (m - 2)\vec{j} + 6\vec{k}] = 0$

$3(7) + 1(m - 2) + (-4)(6) = 0$
$21 + m - 2 - 24 = 0$
$m = 5$

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