7. Vectors

7.2 The Addition, Subtraction and Scalar Product of Vectors (Key Stage 4)

7.2.1 Addition of Vectors

7.2.2 Subtraction of Vectors

7.2.3 Scalar Product of Vectors

7.2.4 The Resolution of Vectors

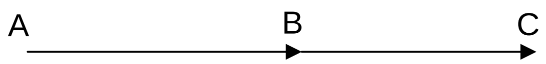
7.2.5 The Vector Operation of Rectangular Coordinates

7.2.1 Addition of Vectors

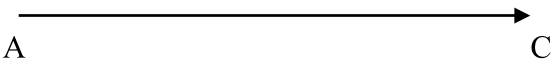
Addition and subtraction can be carried out on both real numbers and vectors. It is better to learn the addition and subtraction of vectors by going through the following examples.

Daily Application 1

If a person travelled from places A to B and then to C in the same direction, the diagram should be as displayed as follows:



Therefore, the route from A to C can be shown as:



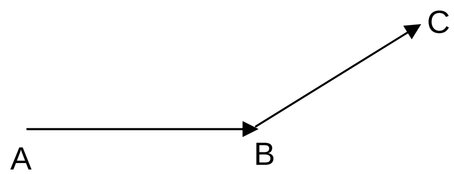
The addition of 2 vectors can be expressed as:

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Daily Application 2

If an aeroplane flies from places A to B, then changes its direction and flies to C. The route from A to C can be shown as:

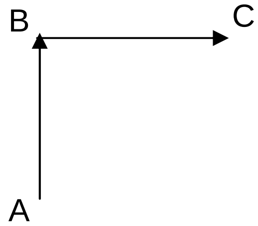
$\overrightarrow{AC}$



and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Daily Application 3

The velocity of the ship is $\overrightarrow{AB}$ while the velocity of water is $\overrightarrow{BC}$. So, what is the resultant velocity of the ship?

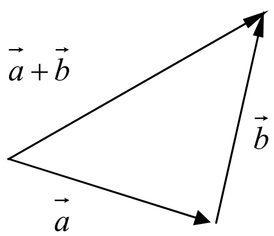


Answer

The resultant velocity of the ship = $\overrightarrow{AB} + \overrightarrow{BC}$.

Triangle Law of Addition

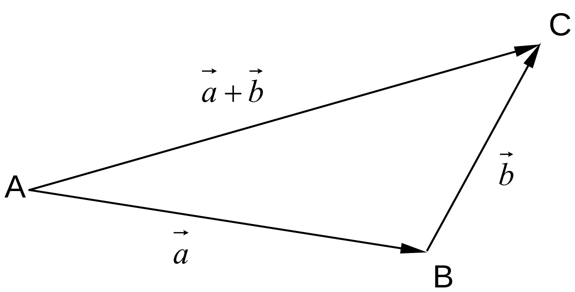
Given vectors $\vec{a}$ and $\vec{b}$, the sum of vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} + \vec{b}$.



This method is known as **Triangle Law of Addition**.

If $\vec{b}$ is translated so that its initial point coincides with the terminal point of $\vec{a}$, then the vector formed by joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ will be $\vec{a} + \vec{b}$.

For a triangle with vertices A, B and C, if the vector $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, then $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

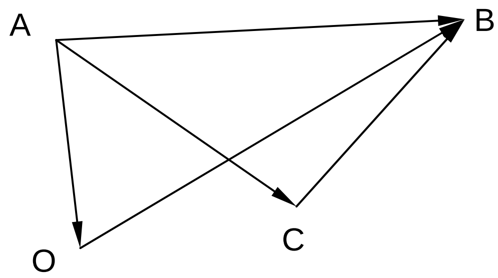


Example 1

Simplify

$\overrightarrow{AC}+\overrightarrow{CB}$.

$\overrightarrow{AO}+\overrightarrow{OB}$.



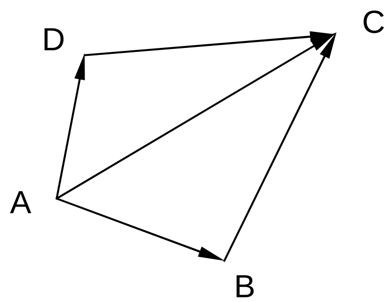
Answer

1. $\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$
2. $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{AB}$

Example 2

Simplify

1. $\overrightarrow{AB}+\overrightarrow{BC}$.
2. $\overrightarrow{AD}+\overrightarrow{DC}$.



Answer

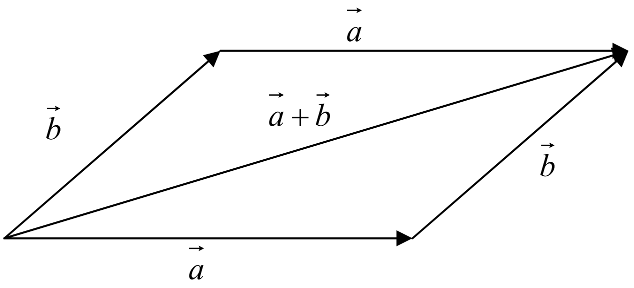
1. $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
2. $\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}$

Practice 7.2A

Parallelogram Law of Addition

Besides the Triangle Law of Addition, there is another addition law: **Parallelogram Law of Addition**. This Law can be illustrated as follows:

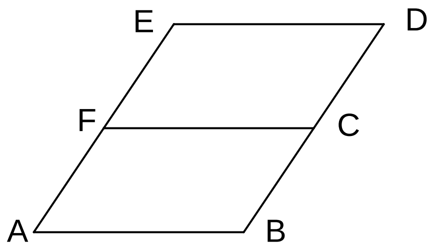
We can draw a parallelogram with $\vec{a}$ and $\vec{b}$ as its sides and the diagonal of this parallelogram is the vector $\vec{a}+\vec{b}$.



Example 3

C and F are the midpoints of BD and AE of the parallelogram ABDE. Simplify

1. $\overrightarrow{AB}+\overrightarrow{AF}$
2. $\overrightarrow{AB}+\overrightarrow{AE}$
3. $\overrightarrow{FC}+\overrightarrow{FE}$
4. $\overrightarrow{DE}+\overrightarrow{DB}$



Answer

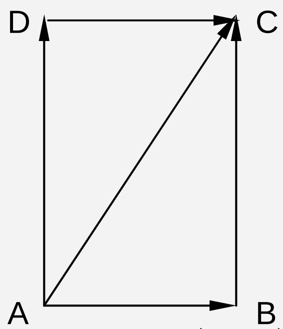
1. $\overrightarrow{AC}$
2. $\overrightarrow{AD}$
3. $\overrightarrow{FD}$
4. $\overrightarrow{DA}$

Example 4

A ship travels in the direction perpendicular to the opposite coast at the speed of $2\sqrt{3}$ km/h, the speed of water is 2 km/h. Find the resultant speed and direction of the ship.

Answer

Let $\overrightarrow{AD}$ be the velocity of the ship and $\overrightarrow{AB}$ be the velocity of water.
AD and AB are the sides of the parallelogram.



The resultant velocity of the ship = $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$

In the right-angled triangle ABC,

$\left|\overrightarrow{AB}\right| = 2$ km/h

$\left|\overrightarrow{BC}\right| = 2\sqrt{3}$ km/h

$$\begin{aligned}\left|\overrightarrow{AC}\right| &= \sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2} \\ &= \sqrt{2^2 + \left(2\sqrt{3}\right)^2} \\ &= 4\end{aligned}$$

$\because \tan\angle CAB = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore \angle CAB = 60°$

Therefore, the resultant speed of the ship is 4km/h which makes an angle 60° with the direction of the water.

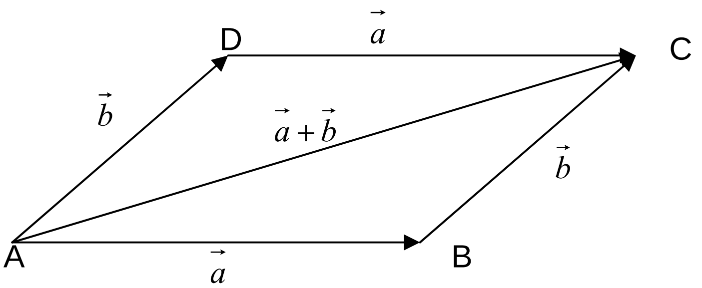
Commutative Law and Associative Law

Commutative Law

Given a parallelogram ABCD, $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$; then $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{DC} = \vec{a}$.

$\because$ $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$
$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$

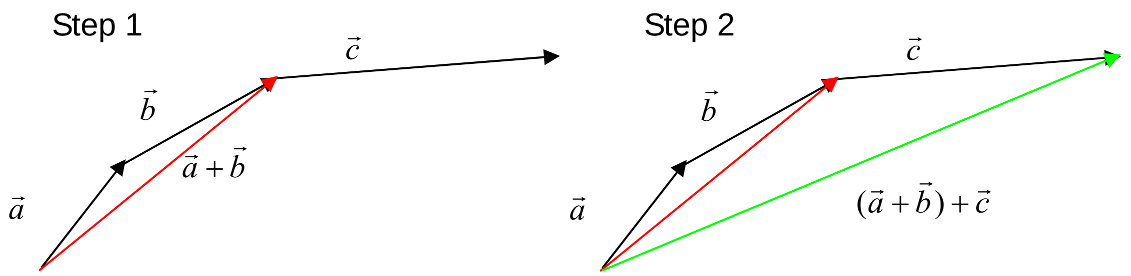
$\therefore$ $\vec{a} + \vec{b} = \vec{b} + \vec{a}$



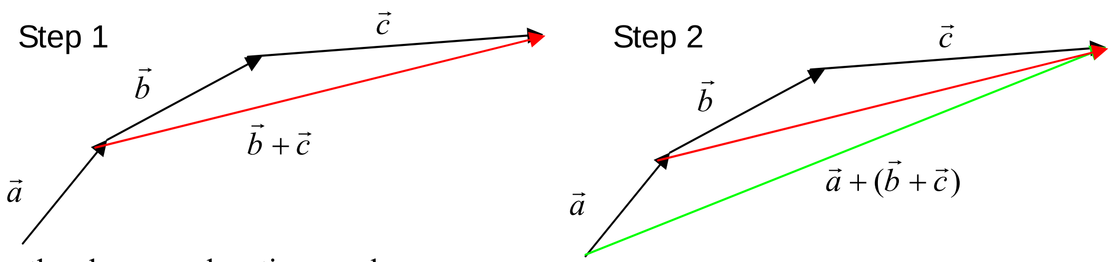
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ is known as **commutative law**.

Associative Law

If we resolve the vector $(\vec{a} + \vec{b}) + \vec{c}$ using the vector diagram, we will come up with the following diagram:



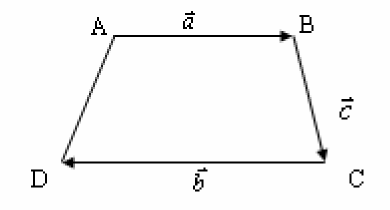
And if we resolve the vector $\vec{a} + (\vec{b} + \vec{c})$, we will get the same result as shown below :



From the above explanation, we know:
$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ is known as **commutative law**.

Example 5

Simplify $\vec{a}+\vec{b}+\vec{c}$.



Answer

$$\begin{aligned} & \vec{a}+\vec{b}+\vec{c} \\ = \ & \vec{a}+(\vec{b}+\vec{c}) \\ = \ & \vec{a}+(\vec{c}+\vec{b}) \\ = \ & (\vec{a}+\vec{c})+\vec{b} \\ = \ & \overrightarrow{AC}+\overrightarrow{CD} \\ = \ & \overrightarrow{AD} \end{aligned}$$

Example 6

Simplify $\overrightarrow{AB}+\overrightarrow{EF}+\overrightarrow{CE}+\overrightarrow{BC}$.

Answer

$$\begin{aligned} & \overrightarrow{AB}+\overrightarrow{EF}+\overrightarrow{CE}+\overrightarrow{BC} \\ = \ & \overrightarrow{AB}+\overrightarrow{EF}+(\overrightarrow{BC}+\overrightarrow{CE}) \\ = \ & \overrightarrow{AB}+\overrightarrow{EF}+\overrightarrow{BE} \\ = \ & \overrightarrow{AB}+(\overrightarrow{BE}+\overrightarrow{EF}) \\ = \ & \overrightarrow{AB}+\overrightarrow{BF} \\ = \ & \overrightarrow{AF} \end{aligned}$$

Practice 7.2B

7.2.2 Subtraction of Vector

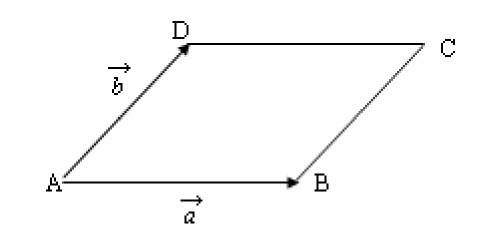
Negative Vector

Definition

Vector with the same magnitude but opposite in direction of $\vec{a}$ is known as the negative vector of $\vec{a}$. We mark it as $-\vec{a}$.

Example 1

In a parallelogram ABCD, $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, determine whether the following equations are correct or not.



1. $\overrightarrow{BA} = -\vec{a}$
2. $\overrightarrow{CB} = \vec{b}$
3. $\overrightarrow{CD} = -\vec{a}$
4. $\overrightarrow{BC} = \vec{b}$

Answer

1.	$\overrightarrow{BA} = -\vec{a}$	Correct
2.	$\overrightarrow{CB} = \vec{b}$	Incorrect ($\overrightarrow{CB} = -\vec{b}$)
3.	$\overrightarrow{CD} = -\vec{a}$	Correct
4.	$\overrightarrow{BC} = \vec{b}$	Correct

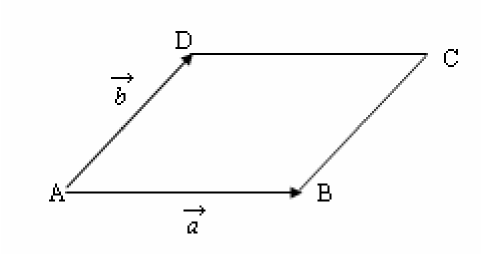
Properties

(1) $-\left(-\vec{a}\right) = \vec{a}$

(2) $\vec{a} + \left(-\vec{a}\right) = \left(-\vec{a}\right) + \vec{a} = \vec{0}$

Example 2

For a parallelogram ABCD with $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, express the following expressions in terms of $\vec{a}$ and $\vec{b}$.



1. $\overrightarrow{AB} + \overrightarrow{CD}$
2. $\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{DC}$

Answer

1. $\overrightarrow{AB} + \overrightarrow{CD}$

$= \vec{a} + \left(-\vec{a}\right)$

$= \vec{0}$

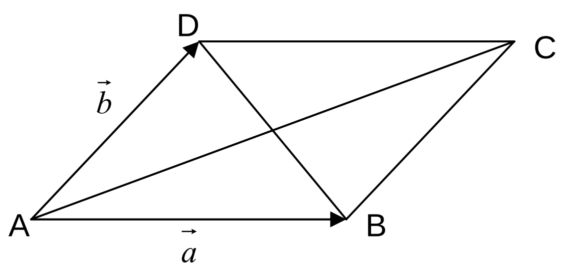
2. $\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{DC}$

$= -\vec{a} + \left(-\vec{b}\right) + \vec{a}$

$= -\vec{b}$

Example 3

For parallelogram ABCD with $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, express $\overrightarrow{CA}$ and $\overrightarrow{BD}$ in terms of $\vec{a}$ and $\vec{b}$.



Answer

By the parallelogram law of addition:

$\because\ \overrightarrow{\mathrm{AC}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{AD}} = \vec{\mathrm{a}} + \vec{\mathrm{b}}$

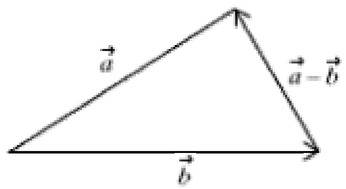
$\therefore\ \overrightarrow{CA} = -\overrightarrow{AC} = -(\vec{a} + \vec{b})$

$\because\ \overrightarrow{\mathrm{DB}} = \overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{AD}} = \vec{\mathrm{a}} - \vec{\mathrm{b}}$

$\therefore\ \overrightarrow{BD} = -\overrightarrow{DB} = -(\vec{a} - \vec{b})$

Subtraction

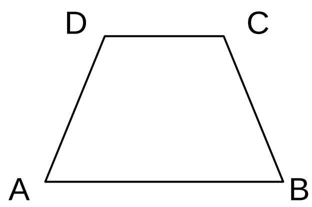
Vector $\vec{a}$ plus the negative vector of $\vec{b}$ is known as the difference of $\vec{a}$ and $\vec{b}$, written as: $\vec{a}-\vec{b}=\vec{a}+\left(-\vec{b}\right)$.



Example 4

With reference to the diagram, simplify

1. $\overrightarrow{AB}-\overrightarrow{AD}$
2. $\overrightarrow{AD}+\overrightarrow{CB}-\overrightarrow{CD}$



Answer

1. $\overrightarrow{AB}-\overrightarrow{AD}$

$= \overrightarrow{AB}+(-\overrightarrow{AD})$

$= \overrightarrow{AB}+\overrightarrow{DA}$

$= \overrightarrow{DA}+\overrightarrow{AB}$

$= \overrightarrow{DB}$

2. $\overrightarrow{AD}+\overrightarrow{CB}-\overrightarrow{CD}$

$= \overrightarrow{AD}+\overrightarrow{CB}+(-\overrightarrow{CD})$

$= \overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{DC}$

$= \overrightarrow{AD}+\overrightarrow{DC}+\overrightarrow{CB}$

$= \overrightarrow{AC}+\overrightarrow{CB}$

$= \overrightarrow{AB}$

Example 5

Simplify the following expressions

1. $\overrightarrow{PB}+\overrightarrow{OP}-\overrightarrow{OB}$
2. $\overrightarrow{OB}-\overrightarrow{OA}-\overrightarrow{OC}-\overrightarrow{CO}$

Answer

1.
$$\begin{aligned}&\overrightarrow{PB}+\overrightarrow{OP}-\overrightarrow{OB}\\=&\overrightarrow{OP}+\overrightarrow{PB}+\left(-\overrightarrow{OB}\right)\\=&\overrightarrow{OB}+\overrightarrow{BO}\\=&\vec{0}\end{aligned}$$

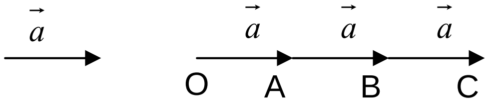
2.
$$\begin{aligned}&\overrightarrow{OB}-\overrightarrow{OA}-\overrightarrow{OC}-\overrightarrow{CO}\\=&\overrightarrow{OB}+\left(-\overrightarrow{OA}\right)-\left(\overrightarrow{OC}+\overrightarrow{CO}\right)\\=&\overrightarrow{OB}+\overrightarrow{AO}-\vec{0}\\=&\overrightarrow{AB}\end{aligned}$$

Practice 7.2C

7.2.3 Scalar Product

The operation of scalar product

Given vector $\vec{a}$ ($\vec{a} \neq \vec{0}$), what is $\vec{a}+\vec{a}+\vec{a}$?

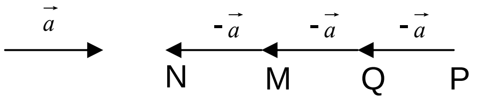


$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \vec{a}+\vec{a}+\vec{a}$

The direction of $\overrightarrow{OC}$ is the same as $\vec{a}$ and its magnitude is three times of $\vec{a}$.

Therefore, $\vec{a}+\vec{a}+\vec{a}$ is written as $3\vec{a}$.

Similarly, what is $(-\vec{a})+(-\vec{a})+(-\vec{a})$?



$\overrightarrow{PN} = \overrightarrow{PQ} + \overrightarrow{QM} + \overrightarrow{MN} = (-\vec{a})+(-\vec{a})+(-\vec{a})$

$(-\vec{a})+(-\vec{a})+(-\vec{a})$ is written as $3(-\vec{a}) = -3\vec{a}$,
then $\overrightarrow{PN} = -3\vec{a}$.

The direction of $-3\vec{a}$ is opposite to $\vec{a}$ and its magnitude is three times of $\vec{a}$.

That is $|-3\vec{a}| = 3|\vec{a}|$.

Definition

The product of a real number λ and a vector $\vec{a}$ is a vector, which is written as $\lambda\vec{a}$, and:

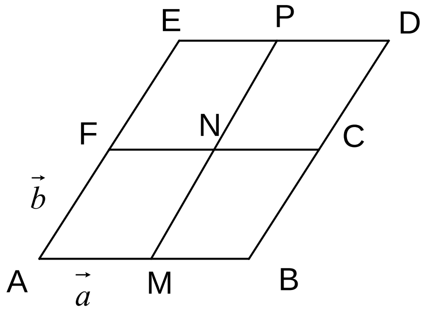
(1) $|\lambda\vec{a}| = |\lambda||\vec{a}|$

(2) When $\lambda > 0$, the direction of $\lambda\vec{a}$ and $\vec{a}$ are the same;
When $\lambda < 0$, the direction of $\lambda\vec{a}$ and $\vec{a}$ are opposite;
When $\lambda = 0$, $\lambda\vec{a} = \vec{0}$.

Example 1

In parallelogram ABDE, F, M, C and P are the midpoints of AE, AB, BD and DE respectively, $\overrightarrow{AM} = \vec{a}$ and $\overrightarrow{AF} = \vec{b}$. Express the following vectors in terms of $\vec{a}$ and $\vec{b}$.

1. $\overrightarrow{AN}$
2. $\overrightarrow{AC}$
3. $\overrightarrow{BF}$
4. $\overrightarrow{DF}$



Answer

1. $\overrightarrow{AN}$
$= \overrightarrow{AM} + \overrightarrow{MN}$
$= \vec{a} + \vec{b}$

2. $\overrightarrow{AC}$
$= \overrightarrow{AB} + \overrightarrow{BC}$
$= 2\overrightarrow{AM} + \overrightarrow{BC}$
$= 2\vec{a} + \vec{b}$

3. $\overrightarrow{BF}$
$= \overrightarrow{BA} + \overrightarrow{AF}$
$= -\overrightarrow{AB} + \overrightarrow{AF}$
$= -2\overrightarrow{AM} + \overrightarrow{AF}$
$= -2\vec{a} + \vec{b}$

4. $\overrightarrow{DF}$
$= \overrightarrow{DC} + \overrightarrow{CF}$
$= \overrightarrow{FA} + \overrightarrow{BA}$
$= -\overrightarrow{AF} - \overrightarrow{AB}$
$= -\vec{b} - 2\vec{a}$

Two vectors are treated as equal when the magnitude and direction of two vectors are the same, regardless of their initial points. Therefore, in the fourth part of the example above, we can put "$\overrightarrow{DC} = \overrightarrow{FA}$". This kind of vector is called **free vector**.

Properties

$\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a} = \mu(\lambda\vec{a})$

$(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$

$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$

λ, μ are real numbers.

Example 2

Simplify the following expressions:

1. $-3 \times (4\vec{a})$
2. $3(\vec{a} + \vec{b}) - 2(\vec{a} - \vec{b}) - \vec{a}$
3. $(2\vec{a} + 3\vec{b} - \vec{c}) - (3\vec{a} - 2\vec{b} + \vec{c})$

Answer

1. $-3 \times (4\vec{a}) = (-3 \times 4)\vec{a} = -12\vec{a}$

2. $$\begin{aligned} & 3(\vec{a} + \vec{b}) - 2(\vec{a} - \vec{b}) - \vec{a} \\ = & \ 3\vec{a} + 3\vec{b} - 2\vec{a} + 2\vec{b} - \vec{a} \\ = & \ 3\vec{a} - 2\vec{a} - \vec{a} + 3\vec{b} + 2\vec{b} \\ = & \ (3 - 2 - 1)\vec{a} + (3 + 2)\vec{b} \\ = & \ 5\vec{b} \end{aligned}$$

3. $$\begin{aligned} & (2\vec{a} + 3\vec{b} - \vec{c}) - (3\vec{a} - 2\vec{b} + \vec{c}) \\ = & \ 2\vec{a} + 3\vec{b} - \vec{c} - 3\vec{a} + 2\vec{b} - \vec{c} \\ = & \ 2\vec{a} - 3\vec{a} + 3\vec{b} + 2\vec{b} - \vec{c} - \vec{c} \\ = & \ (2 - 3)\vec{a} + (3 + 2)\vec{b} - (1 + 1)\vec{c} \\ = & \ -\vec{a} + 5\vec{b} - 2\vec{c} \end{aligned}$$

Practice 7.2D

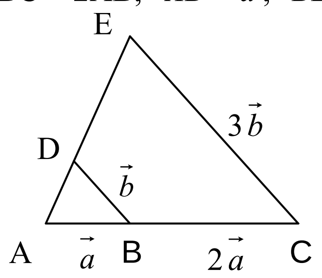
The Conditions for Three Points to be Collinear

If $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ ($\overrightarrow{AB} \neq \vec{0}$) where λ is a real number, then A, B and C are collinear.

On the other hand, given A, B and C are collinear, there is a real number λ such that $\overrightarrow{AB} = \lambda\overrightarrow{BC}$.

Example 3

Given a straight line ABC with BC = 2AB, $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BD} = \vec{b}$ and $\overrightarrow{CE} = 3\vec{b}$. Are A, D and E collinear?



Answer

$$\overrightarrow{BC} = 2\overrightarrow{AB} = 2\vec{a}$$

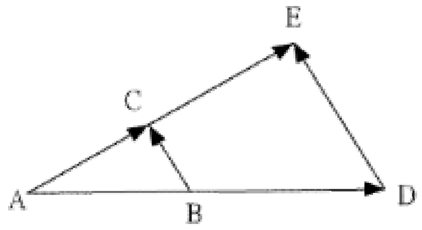
$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \vec{a} + 2\vec{a} \\ &= 3\vec{a}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AE} &= \overrightarrow{AC} + \overrightarrow{CE} \\ &= 3\vec{a} + 3\vec{b} \\ &= 3(\vec{a} + \vec{b}) \\ &= 3\overrightarrow{AD}\end{aligned}$$

∴ A, D and E are collinear.

Example 4

$\overrightarrow{AD} = 3\overrightarrow{AB}$, $\overrightarrow{DE} = 3\overrightarrow{BC}$. Prove that A, C and E are collinear.



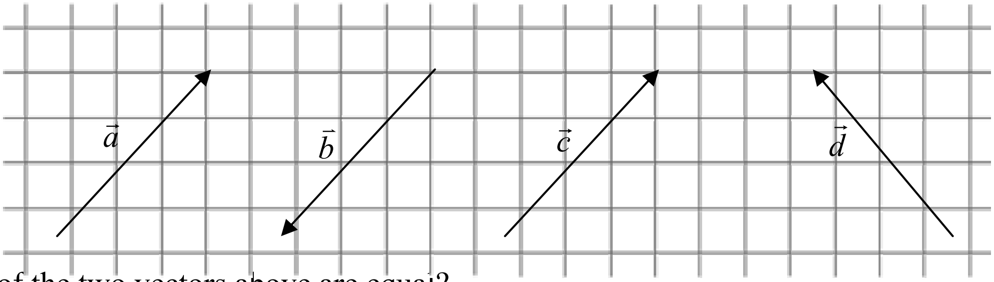
Answer

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$
$= 3\overrightarrow{AB} + 3\overrightarrow{BC}$
$= 3(\overrightarrow{AB} + \overrightarrow{BC})$
$= 3\overrightarrow{AC}$

$\therefore$ A, C and E are collinear.

Equal Vectors

Example 5



Which of the two vectors above are equal?

Answer

$\vec{a} = \vec{c}$
When the two vectors are equal in direction and magnitude, the two vectors are equal.
Therefore, $\vec{a}$ and $\vec{c}$ are equal.

Example 6

Find the value of k such that

1. $3\vec{a} = k\vec{a}$
2. $(3k-2)\vec{b} = (6+k)\vec{b}$

Answer

1. $3\vec{a} = k\vec{a}$
   $3\vec{a} - k\vec{a} = \vec{0}$
   $(3-k)\vec{a} = \vec{0}$
   $3 - k = 0$
   $k = 3$
2. $(3k-2)\vec{b} = (6+k)\vec{b}$
   $3k - 2 = 6 + k$
   $3k - k = 6 + 2$
   $2k = 8$
   $k = 4$

If $\vec{a}$ and $\vec{b}$ are not parallel and $\vec{a} = \vec{b}$, then $\vec{a} = \vec{b} = \vec{0}$.
In conclusion, if the two vectors are not parallel but equal, then the two vectors are zero vectors.

Example 7

Given vectors $\vec{a}$ and $\vec{b}$ are not parallel, $4\vec{a}+(m-2)\vec{b}=(n+1)\vec{a}-3\vec{b}$, find the value of m and n.

Answer

$4\vec{a}+(m-2)\vec{b}=(n+1)\vec{a}-3\vec{b}$

$4\vec{a}-(n+1)\vec{a}=-(m-2)\vec{b}-3\vec{b}$

$(4-(n+1))\vec{a}=((2-m)-3)\vec{b}$

$\because$ $\vec{a}$ and $\vec{b}$ are not parallel.

$\therefore$ $4-(n+1)=0$ and $(2-m)-3=0$

$4=n+1$ and $2-m=3$

$n=3$ and $m=-1$

Example 8

1. Given vectors $\vec{e_1}$ and $\vec{e_2}$ are not parallel, $\overrightarrow{AB}=2\vec{e_1}+k\vec{e_2}$, $\overrightarrow{CB}=\vec{e_1}+3\vec{e_2}$ and $\overrightarrow{CD}=2\vec{e_1}-\vec{e_2}$. If A, B and D are collinear, find the value of k.

Answer

$\because$ $\overrightarrow{BD}=\overrightarrow{CD}-\overrightarrow{CB}$

$=(2\vec{e_1}-\vec{e_2})-(\vec{e_1}+3\vec{e_2})$

$=\vec{e_1}-4\vec{e_2}$

$\because$ A, B and D are collinear.

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are parallel.

Let $\overrightarrow{AB}=\lambda\overrightarrow{BD}$.

$2\vec{e_1}+k\vec{e_2}=\lambda\vec{e_1}-4\lambda\vec{e_2}$

$\therefore$ $\begin{cases} 2=\lambda \\ k=-4\lambda \end{cases}$

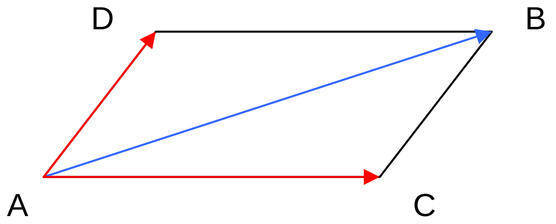
$k=-8$

Therefore, when A, B and D are collinear, the value of k is -8.

Practice 7.2E

7.2.4 The Resolution of Vectors

Given parallelogram ABCD,

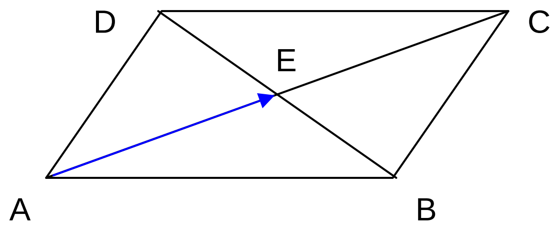


by using the parallelogram law of addition, we get $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{AD}$.

Thus, a vector can be rewritten as the sum of two vectors. This is known as the **resolution of vectors**. In the above example, we can say that $\overrightarrow{AB}$ is resolved into $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

Example 1

Given parallelogram ABCD, E is the midpoint of AC, resolve vector $\overrightarrow{AE}$.



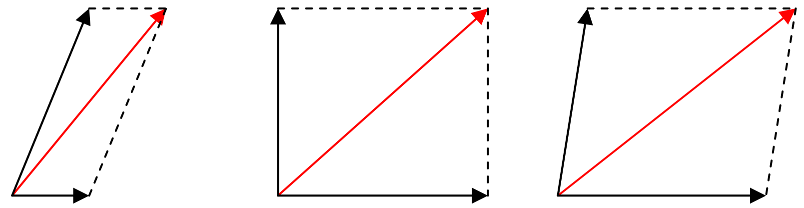
Answer

Based on the given information, vector $\overrightarrow{AE}$ can be resolved into:

$$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$$

$$\overrightarrow{AE} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AD}\right)$$

As a matter of fact, a vector can be resolved into infinite numbers of components. As the figure shown below, the vector in red can be resolved in many different ways.

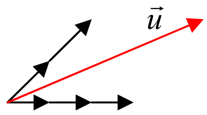


In general, we may say that a vector can be resolved into the addition of the multiples of two vectors, i.e. $\vec{u} = m\vec{a} + n\vec{b}$.

Example 2

Resolve the vectors into $\vec{a}$ and $\vec{b}$.

Figure 1



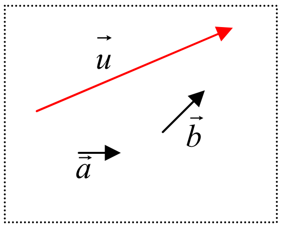
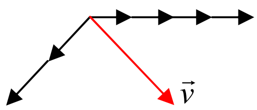


Figure 2



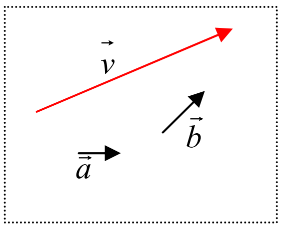
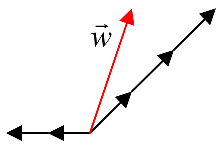
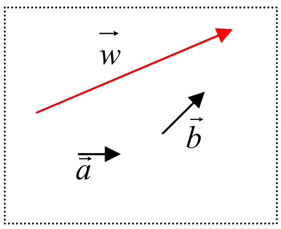


Figure 3





Answer

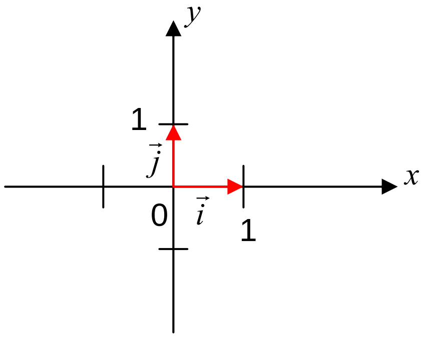
$\vec{u} = 3\vec{a} + 2\vec{b}$
$\vec{v} = 4\vec{a} - 2\vec{b}$
$\vec{w} = -2\vec{a} + 3\vec{b}$

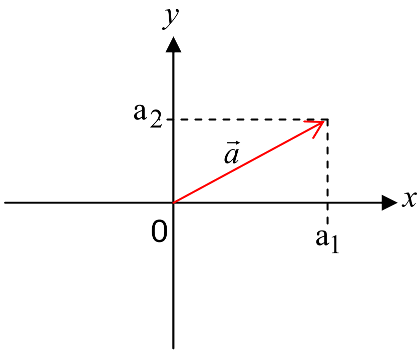
7.2.5 Vector Operation in Rectangular Coordinates

The coordinates of two dimensional (2-D) vectors

Vector $\vec{i}$ and $\vec{j}$ are the unit vectors along *x*-axis and *y*-axis respectively.

The magnitudes of $\vec{i}$ and $\vec{j}$ are one and they are called **unit base vector**.



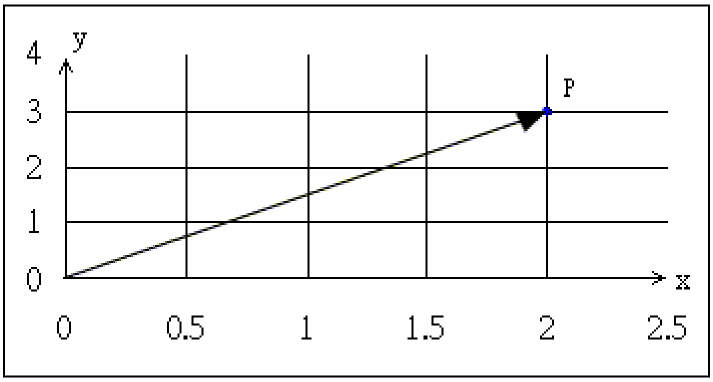


Thus, vector $\vec{a}$ can be written as:
$\vec{a} = a_1\vec{i} + a_2\vec{j}$.

Besides using symbols '→' over the alphabet like $\vec{a}$, vectors can be denoted by bold letters such as **a**, **b**, **c**.

Position Vector

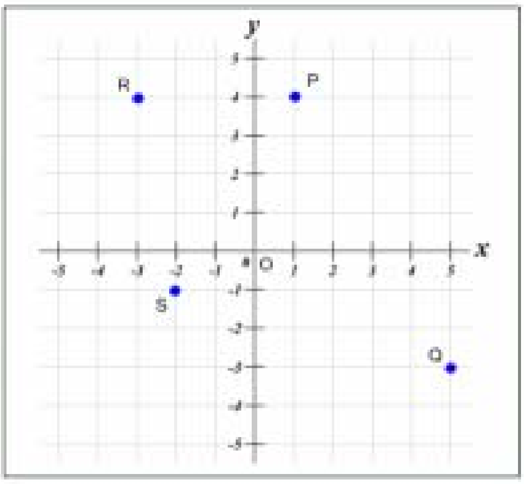
The coordinate of P is (2, 3). $\overrightarrow{OP}$ is known as the **position vector** of P.



$\overrightarrow{OP} = 2\vec{i} + 3\vec{j}$.

Example 1

Write down the position vectors of P, Q, R and S.



Answer

$\overrightarrow{OP} = \vec{i} + 4\vec{j}$

$\overrightarrow{OQ} = 5\vec{i} - 3\vec{j}$

$\overrightarrow{OR} = -3\vec{i} + 4\vec{j}$

$\overrightarrow{OS} = -2\vec{i} - \vec{j}$

Example 2

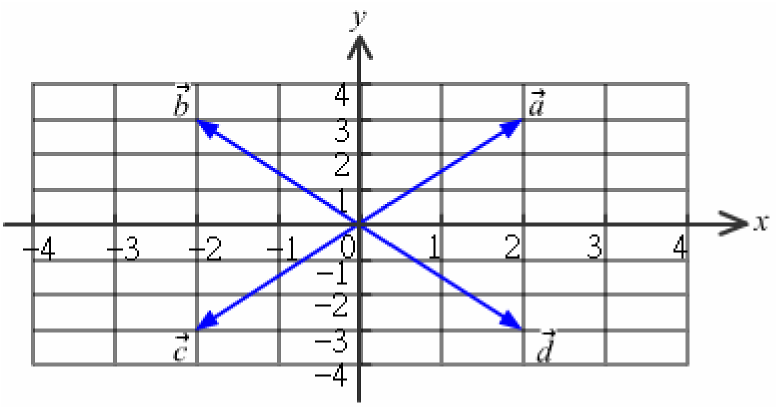
Given A(0,4), B(–3,6) and C(4,5), write down the position vector of these three points.

Answer

$\overrightarrow{OA} = 4\vec{j}$; $\overrightarrow{OB} = -3\vec{i} + 6\vec{j}$; $\overrightarrow{OC} = 4\vec{i} + 5\vec{j}$

Example 3

Express $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ in terms of $\vec{i}$ and $\vec{j}$.



Answer

$\vec{a} = 2\vec{i} + 3\vec{j}$

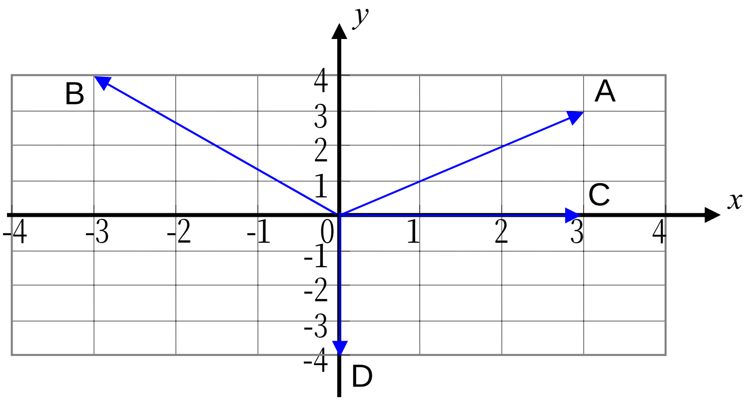
$\vec{b} = -2\vec{i} + 3\vec{j}$

$\vec{c} = -2\vec{i} - 3\vec{j}$

$\vec{d} = 2\vec{i} - 3\vec{j}$

Example 4

Express the following vectors in terms of $\vec{i}$ and $\vec{j}$.



Answer

$\overrightarrow{OA} = 3\vec{i} + 3\vec{j}$, $\overrightarrow{OB} = -3\vec{i} + 4\vec{j}$, $\overrightarrow{OC} = 3\vec{i}$ and $\overrightarrow{OD} = -4\vec{j}$.

The Operation of 2-D Vectors in a Coordinate Plane

Addition and Subtraction

Given $\vec{a} = x_1\vec{i} + y_1\vec{j}$, $\vec{b} = x_2\vec{i} + y_2\vec{j}$

$\vec{a}+\vec{b} = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j}$

$\vec{a}-\vec{b} = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j}$

Example 5

Given $\vec{a} = 2\vec{i} + \vec{j}$ and $\vec{b} = -3\vec{i} + 4\vec{j}$, find $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Answer

$\vec{a}+\vec{b}$
$= 2\vec{i} + \vec{j} + (-3\vec{i} + 4\vec{j})$
$= 2\vec{i} - 3\vec{i} + \vec{j} + 4\vec{j}$
$= -\vec{i} + 5\vec{j}$

$\vec{a}-\vec{b}$
$= 2\vec{i} + \vec{j} - (-3\vec{i} + 4\vec{j})$
$= 2\vec{i} + 3\vec{i} + \vec{j} - 4\vec{j}$
$= 5\vec{i} - 3\vec{j}$

Example 6

Given A(2, 3), B(4, 5), C(0, 2) and D(–1, –3), find $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Answer

$\overrightarrow{AB}$
$= \overrightarrow{OB} - \overrightarrow{OA}$
$= (4\vec{i} + 5\vec{j}) - (2\vec{i} + 3\vec{j})$
$= 4\vec{i} - 2\vec{i} + 5\vec{j} - 3\vec{j}$
$= 2\vec{i} + 2\vec{j}$

$\overrightarrow{CD}$
$= \overrightarrow{OD} - \overrightarrow{OC}$
$= (-\vec{i} - 3\vec{j}) - (2\vec{j})$
$= -\vec{i} - 3\vec{j} - 2\vec{j}$
$= -\vec{i} - 5\vec{j}$

Equal Vector

Given $\vec{a} = x_1\vec{i} + y_1\vec{j}$, $\vec{b} = x_2\vec{i} + y_2\vec{j}$.

If $\vec{a} = \vec{b}$, then $x_1 = x_2$, $y_1 = y_2$

Example 7

$4\vec{i} + (m-2)\vec{j} = (n+2)\vec{i} + 6\vec{j}$. Find *m* and *n*.

Answer

$4\vec{i} + (m-2)\vec{j} = (n+2)\vec{i} + 6\vec{j}$

$4 = n + 2$
$n = 4 - 2$
$n = 2$

$m - 2 = 6$
$m = 6 + 2$
$m = 8$

Therefore: $m = 8$, $n = 2$.

Example 8

Given the coordinates of the three vertices A, B and C of a parallelogram ABCD are (−2, 1), (−1, 3) and (3, 4) respectively, find the coordinates of vertex D.

Answer

Let the coordinates of vertex D be (x, y).

$\because \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 3\vec{j} - (-2\vec{i} + \vec{j}) = \vec{i} + 2\vec{j}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = 3\vec{i} + 4\vec{j} - (x\vec{i} + y\vec{j}) = (3-x)\vec{i} + (4-y)\vec{j}$

$\overrightarrow{AB} = \overrightarrow{CD}$

$\vec{i} + 2\vec{j} = (3-x)\vec{i} + (4-y)\vec{j}$

$1 = 3 - x$
$x = 2$

$2 = 4 - y$
$y = 2$

$\therefore$ the coordinates of vertex D is (2, 2).

Practice 7.2F

Scalar Product

Given $\vec{a} = x\vec{i} + y\vec{j}$.

$\lambda\vec{a} = \lambda(x\vec{i} + y\vec{j}) = \lambda x\vec{i} + \lambda y\vec{j}$, λ is a real number.

Example 9

Simplify:

1. $3(2\vec{i} - \vec{j}) + 4(\vec{i} + 2\vec{j})$
2. $2(\vec{i} + \vec{j}) - 3(\vec{j} - 2\vec{i})$

Answer

1.

$$\begin{aligned} & 3(2\vec{i} - \vec{j}) + 4(\vec{i} + 2\vec{j}) \\ &= 6\vec{i} - 3\vec{j} + 4\vec{i} + 8\vec{j} \\ &= 6\vec{i} + 4\vec{i} - 3\vec{j} + 8\vec{j} \\ &= 10\vec{i} + 5\vec{j} \end{aligned}$$

2.

$$\begin{aligned} & 2(\vec{i} + \vec{j}) - 3(\vec{j} - 2\vec{i}) \\ &= 2\vec{i} + 2\vec{j} - 3\vec{j} - 3(-2\vec{i}) \\ &= 2\vec{i} + 2\vec{j} - 3\vec{j} + 6\vec{i} \\ &= 2\vec{i} + 6\vec{i} + 2\vec{j} - 3\vec{j} \\ &= 8\vec{i} - \vec{j} \end{aligned}$$

Example 10

Given $\vec{a} = 2\vec{i} + \vec{j}$ and $\vec{b} = -3\vec{i} + 4\vec{j}$, find $3\vec{a} + 4\vec{b}$.

Answer

$$\begin{aligned} & 3\vec{a} + 4\vec{b} \\ &= 3(2\vec{i} + \vec{j}) + 4(-3\vec{i} + 4\vec{j}) \\ &= 6\vec{i} + 3\vec{j} - 12\vec{i} + 16\vec{j} \\ &= -6\vec{i} + 19\vec{j} \end{aligned}$$

Unit Vector

The magnitude of $\vec{a}$ is denoted by $|\vec{a}|$.

A **unit vector** is a vector with magnitude equals to **1**.

Notation for vector

The notation for a unit vector can be written as a " ^ " sign above the vector symbol, for instance, $\hat{a}$.

Example 11

Given $|\vec{u}| = 3$, find the unit vector of $\hat{u}$.

Answer

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}}{3}$$

If $\vec{v} = a\vec{i} + b\vec{j}$, its magnitude is $|\vec{v}| = \sqrt{a^2 + b^2}$, and the unit vector is $\hat{v} = \dfrac{a\vec{i} + b\vec{j}}{\sqrt{a^2 + b^2}}$.

Example 12

Find the unit vector of the following:

1. $3\vec{i} + 4\vec{j}$
2. $10\vec{i} + 10\vec{j}$

Answer

1. $\left|3\vec{i} + 4\vec{j}\right| = \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$

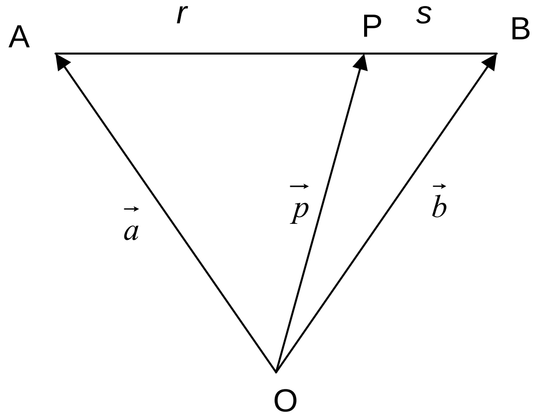
$$\hat{u} = \frac{3\vec{i} + 4\vec{j}}{5} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

2. $\left|10\vec{i} + 10\vec{j}\right| = \sqrt{10^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$

$$\hat{u} = \frac{10\vec{i} + 10\vec{j}}{10\sqrt{2}} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$$

Section Formula

Let the position vector of A and B be $\vec{a}$ and $\vec{b}$ respectively. If P is one of the points on the line segment AB such that $\frac{AP}{PB} = \frac{r}{s}$ and r and s are real numbers, then



the position vector of P is:

$$\vec{p} = \frac{s\vec{a} + r\vec{b}}{s + r}$$

Proof

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

$$\overrightarrow{AP} = \frac{r}{r+s}\overrightarrow{AB} = \frac{r}{r+s}(\vec{b} - \vec{a})$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \frac{r}{r+s}(\vec{b} - \vec{a}) = \frac{s\vec{a} + r\vec{b}}{s + r}$$

Example 13

Given P is a point on the line segment AB, $3\overrightarrow{AP} = 2\overrightarrow{PB}$. If the coordinates of A and B are (–2, –6) and (8, 14), find the position vector of P.

Answer

$$\left|\overrightarrow{AP}\right| : \left|\overrightarrow{PB}\right| = 2 : 3$$

$$\overrightarrow{OA} = -2\vec{i} - 6\vec{j}$$

$$\overrightarrow{OB} = 8\vec{i} + 14\vec{j}$$

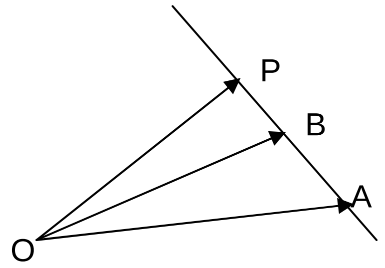
$$\overrightarrow{OP} = \frac{3(-2\vec{i} - 6\vec{j}) + 2(8\vec{i} + 14\vec{j})}{3 + 2}$$

$$= \frac{-6\vec{i} + 16\vec{i} - 18\vec{j} + 28\vec{j}}{5}$$

$$= 2\vec{i} + 2\vec{j}$$

Example 14

Given that $\overrightarrow{OA}$, $\overrightarrow{OB}$ are not parallel, if $\overrightarrow{AP} = t\overrightarrow{AB}$, express $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



Answer

Method 1

$\because \overrightarrow{AP} = t\overrightarrow{AB}$

$$\begin{aligned}\therefore \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \overrightarrow{OA} + t\overrightarrow{AB}\\ &= \overrightarrow{OA} + t(\overrightarrow{OB} - \overrightarrow{OA})\\ &= \overrightarrow{OA} - t\overrightarrow{OA} + t\overrightarrow{OB}\\ &= (1-t)\overrightarrow{OA} + t\overrightarrow{OB}\end{aligned}$$

Method 2

$$\left|\overrightarrow{AP}\right| : \left|\overrightarrow{BP}\right| = t : 1-t$$

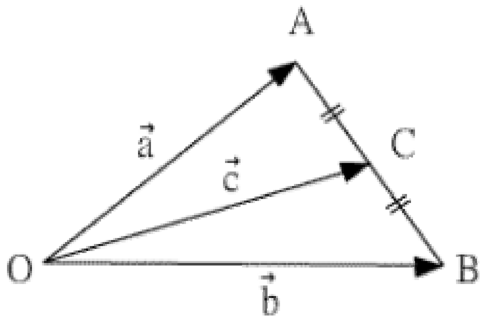
By section formula

$$\begin{aligned}\overrightarrow{OP} &= \frac{(1-t)\overrightarrow{OA} + t\overrightarrow{OB}}{t + (1-t)}\\ &= (1-t)\overrightarrow{OA} + t\overrightarrow{OB}\end{aligned}$$

Mid-point Formula

Mid-point Theorem

Given that the position vector of point A and point B are $\vec{a}$ and $\vec{b}$ respectively. If C is the mid-point on line AB, Vector $\vec{c} = \frac{1}{2}(\vec{a} + \vec{b})$.



Proof

$\because$ C is the mid-point of straight line AB
$\therefore$ $r = s = 1$
By section formula:

$$\overrightarrow{OC} = \frac{\vec{a} + \vec{b}}{1+1} = \frac{\vec{a} + \vec{b}}{2}$$

$$\therefore \quad \vec{c} = \frac{1}{2}(\vec{a} + \vec{b})$$

Example 15

Given that point P is the mid-point on line AB. If A and B are (–3, 7) and (5, –19) respectively, find the position vector of P.

Answer

$$\overrightarrow{OA} = -3\vec{i} + 7\vec{j}$$

$$\overrightarrow{OB} = 5\vec{i} - 19\vec{j}$$

$$\overrightarrow{OP} = \frac{(-3\vec{i} + 7\vec{j}) + (5\vec{i} - 19\vec{j})}{2}$$

$$= \frac{-3\vec{i} + 5\vec{i} + 7\vec{j} - 19\vec{j}}{2}$$

$$= \vec{i} - 6\vec{j}$$

Example 16

Given that P is the mid-point of the line AB. If the position vector of A and P are $4\vec{i}+5\vec{j}$ and $3\vec{i}+3\vec{j}$, find the position vector of B.

Answer

$$\overrightarrow{OP}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

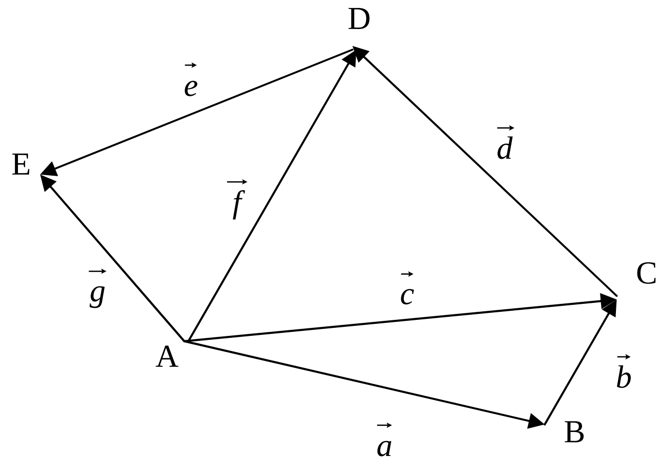
$$3\vec{i}+3\vec{j}=\frac{4\vec{i}+5\vec{j}+\overrightarrow{OB}}{2}$$

$$6\vec{i}+6\vec{j}=4\vec{i}+5\vec{j}+\overrightarrow{OB}$$

$$\overrightarrow{\mathrm{OB}}=2\vec{i}+\vec{j}$$

Practice 7.2G

Practice 7.2A



For questions 1 and 2, the given diagram is relevant.

1. Simplify

(1) $\vec{a}+\vec{b}$

(2) $\vec{c}+\vec{d}$

(3) $\vec{f}+\vec{e}$

2. Simplify

(1) $\overrightarrow{AB}+\overrightarrow{BC}$

(2) $\overrightarrow{BD}+\overrightarrow{DC}$

(3) $\overrightarrow{DA}+\overrightarrow{AC}$

Answer

1. (1) $\vec{c}$ (2) $\vec{f}$ (3) $\vec{g}$

2. (1) $\overrightarrow{AC}$ (2) $\overrightarrow{BC}$ (3) $\overrightarrow{DC}$

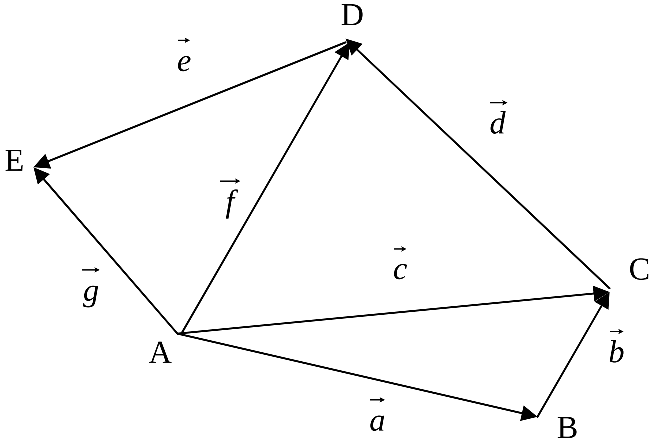
Practice 7.2B

1. Let $\vec{a}$ be the vector of "walk in the east direction for 10 km", $\vec{b}$ be the vector of "walk in the west direction for 5 km", $\vec{c}$ be the vector of "walk in the north direction for 10km" and $\vec{d}$ be the vector of "walk in the south direction for 5 km". Write down the magnitude and direction of the following vectors.

(1) $\vec{a}+\vec{a}$
(2) $\vec{a}+\vec{b}$
(3) $\vec{a}+\vec{c}$
(4) $\vec{b}+\vec{d}$
(5) $\vec{b}+\vec{c}+\vec{d}$
(6) $\vec{d}+\vec{a}+\vec{d}$

For questions 2 and 3, the given diagram is relevant.

2. Simplify



(1) $\vec{a}+\vec{b}+\vec{d}$
(2) $\vec{c}+\vec{d}+\vec{e}$
(3) $\vec{a}+\vec{b}+\vec{d}+\vec{e}$

3. Simplify

(1) $\overrightarrow{AB}+\overrightarrow{DC}+\overrightarrow{BD}$
(2) $\overrightarrow{AB}+\overrightarrow{DA}+\overrightarrow{BC}$

Answer

1.

(1) walk in the east direction for 20 km

(2) walk in the east direction for 5 km

(3) walk in the northeast direction for $10\sqrt{2}$ km

(4) walk in the southwest direction for $5\sqrt{2}$ km

(5) walk in the northwest direction for $5\sqrt{2}$ km

(6) walk in the southeast direction for $10\sqrt{2}$ km

2. (1) $\vec{f}$ (2) $\vec{g}$ (3) $\vec{g}$

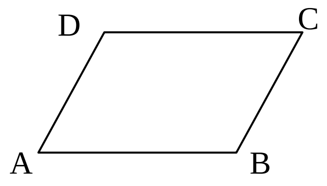
3. (1) $\overrightarrow{AC}$ (2) $\overrightarrow{DC}$

Practice 7.2C

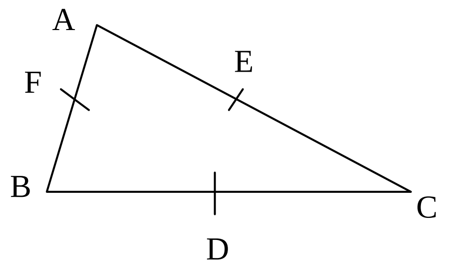
1.

(1) In ΔABC, if $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{CA} = \vec{b}$. Express $\overrightarrow{AB}$ in terms of $\vec{a}$ and $\vec{b}$.

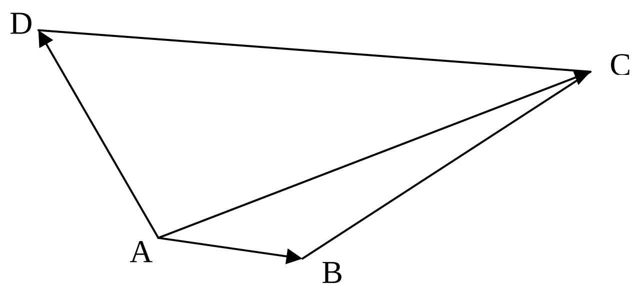
(2) Given the parallelogram ABCD, simplify $\overrightarrow{BC} - \overrightarrow{CD} + \overrightarrow{BA}$.



(3) In ΔABC, D, E and F are the midpoints of BC, CA and AB respectively.
Simplify $\overrightarrow{BF} - \overrightarrow{DC} + \overrightarrow{FD}$.



2. Given parallelogram ABCD, simplify $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AD}$.

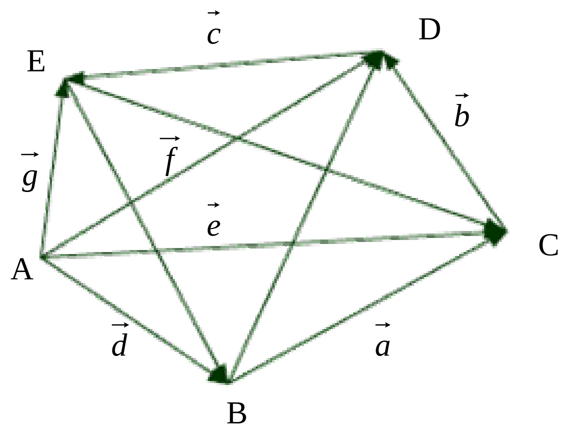


3. Express the following vector in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

(1) $\vec{e}-\vec{g}$

(2) $\vec{f}-\vec{d}$

(3) $\vec{d}-\vec{g}$



Answer

1. (1) $-\vec{a}-\vec{b}$ (2) $\overrightarrow{BC}$ (3) $\vec{0}$
2. $\overrightarrow{DC}$
3. (1) $-\vec{b}-\vec{c}$ (2) $\vec{a}+\vec{b}$ (3) $-\vec{a}-\vec{b}-\vec{c}$

Practice 7.2D

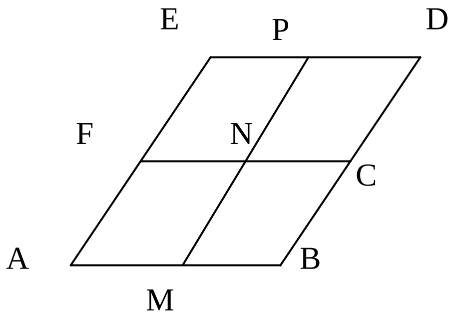
1. In parallelogram ABCD, F, M, C and P are the midpoints of AE, AB, BD and DE respectively, and $\overrightarrow{AM} = \vec{a}$ and $\overrightarrow{AF} = \vec{b}$. Express the following vectors in terms of $\vec{a}$ and $\vec{b}$.

   (1) $\overrightarrow{AD}$

   (2) $\overrightarrow{FD}$

   (3) $\overrightarrow{BP}$

   (4) $\overrightarrow{CA}$



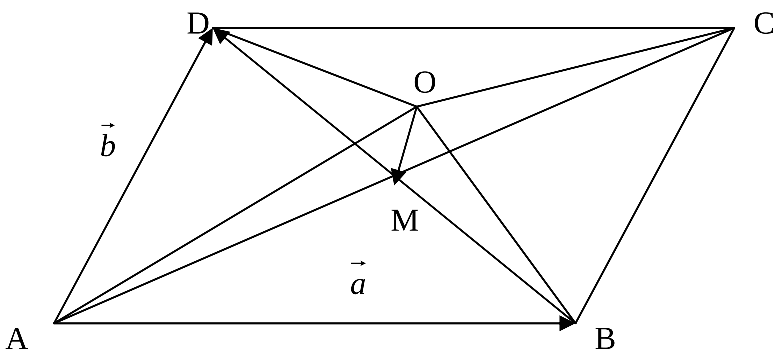
2. Simplify

   (1) $2(\vec{c} + \vec{d}) - \vec{c}$

   (2) $2(3\vec{a} + \vec{b}) + 3(\vec{a} - 2\vec{b})$

   (3) $4(\vec{a} + \vec{c}) + 3(\vec{a} + 2\vec{b} - \vec{c})$

3. Given a parallelogram ABCD, M is the point of intersection of the diagonals, $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$.



   (1) Express $\overrightarrow{MA}$, $\overrightarrow{MB}$, $\overrightarrow{MC}$ and $\overrightarrow{MD}$ in terms of $\vec{a}$ and $\vec{b}$.

   (2) Given that O is any point, prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OM}$.

Answer

1. (1) $2\vec{a}+2\vec{b}$ (2) $2\vec{a}+\vec{b}$ (3) $-\vec{a}+2\vec{b}$ (4) $-2\vec{a}-\vec{b}$

2. (1) $\vec{c}+2\vec{d}$ (2) $9\vec{a}-4\vec{b}$ (3) $7\vec{a}+6\vec{b}+\vec{c}$

3. (1) $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}=\vec{a}+\vec{b}$

$\overrightarrow{MA}=\frac{1}{2}\overrightarrow{CA}=-\frac{1}{2}\overrightarrow{AC}$

$\therefore\ \overrightarrow{MA}=-\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b}$

$\overrightarrow{DB}=\overrightarrow{DA}+\overrightarrow{AB}=-\vec{b}+\vec{a}$

$\overrightarrow{MB}=\frac{1}{2}\overrightarrow{DB}$

$\therefore\ \overrightarrow{MB}=\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b}$

$\overrightarrow{MC}=\overrightarrow{AM}=-\overrightarrow{MA}$

$\therefore\ \overrightarrow{MC}=\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$

$\overrightarrow{MD}=\overrightarrow{BM}=-\overrightarrow{MB}$

$\therefore\ \overrightarrow{MD}=-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}$

(2) $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}-4\overrightarrow{OM}$

$=\overrightarrow{OA}-\overrightarrow{OM}+\overrightarrow{OB}-\overrightarrow{OM}+\overrightarrow{OC}-\overrightarrow{OM}+\overrightarrow{OD}-\overrightarrow{OM}$

$=\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}+\overrightarrow{MD}$

$=(-\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b})+(\frac{1}{2}\vec{a}-\frac{1}{2}\vec{b})+(\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b})+(-\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b})$

$=0$

$\therefore\ \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OM}$

Practice 7.2E

1. Given A(−1, −1), B(1, 3), C(2, 5), prove that A, B and C are collinear.
2. Given $2\vec{a}+3\vec{b}=(k+1)\vec{b}+2\vec{a}$, $\vec{a}$ and $\vec{b}$ are parallel, find the value of *k*.
3. If $\vec{a}$ and $\vec{b}$ are non-parallel vectors, find the values of the unknowns.

   $(3-m)\vec{a}-(4-n)\vec{b}+5\vec{c}=2\vec{a}+(2n-1)\vec{b}-(3-p)\vec{c}$

4. Given $\vec{a}$ and $\vec{b}$ are non-parallel and $3\vec{a}+(10-y)\vec{b}=(2+x)\vec{b}+(4y+7)\vec{a}$. Find the values of *x* and *y*.

Answer

1. $\overrightarrow{OA}=-\vec{i}-\vec{j}$

   $\overrightarrow{OB}=\vec{i}+3\vec{j}$

   $\overrightarrow{OC}=2\vec{i}+5\vec{j}$

   $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(\vec{i}+3\vec{j})-(-\vec{i}-\vec{j})=2\vec{i}+4\vec{j}$

   $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=2\vec{i}+5\vec{j}-(\vec{i}+3\vec{j})=\vec{i}+2\vec{j}$

   $\overrightarrow{AB}=2\overrightarrow{BC}$

   ∴ A, B and C are collinear.

2. $2\vec{a}+3\vec{b}=(k+1)\vec{b}+2\vec{a}$

   $2\vec{a}+3\vec{b}=2\vec{a}+(k+1)\vec{b}$

   ∴ $3=k+1$

   $k=2$

3. $(3-m)\vec{a}-(4-n)\vec{b}+5\vec{c}=2\vec{a}+(2n-1)\vec{b}-(3-p)\vec{c}$

$3 - m = 2$
$m = 1$

$-(4 - n) = 2n - 1$
$-4 + n = 2n - 1$
$n = -3$

$5 = -(3 - p)$
$5 = -3 + p$
$p = 8$

$\therefore$ $m = 1$, $n = -3$ and $p = 8$.

4. $3\vec{a}+(10-y)\vec{b}=(2+x)\vec{b}+(4y+7)\vec{a}$

$3\vec{a}+(10-y)\vec{b}=(4y+7)\vec{a}+(2+x)\vec{b}$

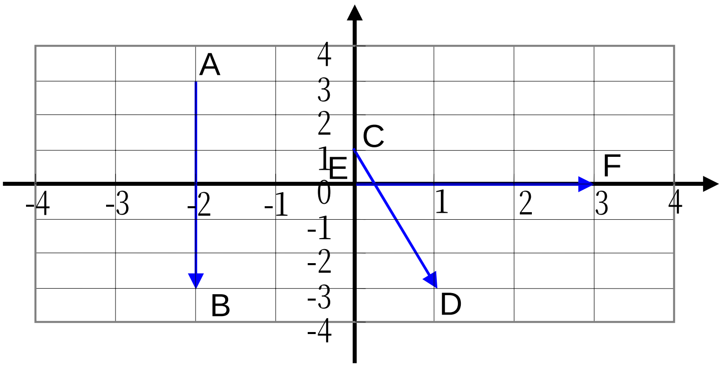
$3 = 4y + 7$
$4y = -4$
$y = -1$

$10 - y = 2 + x$
$10 - (-1) = 2 + x$
$x = 11 - 2 = 9$

$\therefore x = 9, y = -1$

Practice 7.2F

1. Express the vectors in terms of $\vec{i}$ and $\vec{j}$.



2. Simplify

   (1) $3(2\vec{i} + 3\vec{j}) - (\vec{i} - 4\vec{j})$

   (2) $4(\vec{i} - 3\vec{j}) + 3(2\vec{i} + \vec{j})$

   (3) $6(\vec{i} + 3\vec{j}) - 4(2\vec{i} + 4\vec{j})$

3. Find the values of the unknowns.

   (1) $(n + 2)\vec{i} + 4\vec{j} = 5\vec{i} + m\vec{j}$

   (2) $(4 - m)\vec{i} - (5 - n)\vec{j} = 6\vec{j}$

   (3) $(20 - 3m)\vec{i} + (30 - 5n)\vec{j} = (m - 4)\vec{i} + 10\vec{j}$

4. Given $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = x\vec{i} + \vec{j}$ and $\vec{a} + 2\vec{b}$ is parallel to $2\vec{a} - \vec{b}$. Find the value of x.

5. Given $\vec{a} = 2\vec{i} + 3\vec{i}$, $\vec{b} = x\vec{i} - 6$ and $\vec{a} // \vec{b}$. Find the value of x.

Answer

1. $\overrightarrow{AB} = -6\vec{j}$, $\overrightarrow{CD} = \vec{i} - 4\vec{j}$, $\overrightarrow{EF} = 3\vec{i}$.

2.

   (1) $3(2\vec{i} + 3\vec{j}) - (\vec{i} - 4\vec{j}) = 6\vec{i} + 9\vec{j} - \vec{i} + 4\vec{j} = 5\vec{i} + 13\vec{j}$

   (2) $4(\vec{i} - 3\vec{j}) + 3(2\vec{i} + \vec{j}) = 4\vec{i} - 12\vec{j} + 6\vec{i} + 3\vec{j} = 10\vec{i} - 9\vec{j}$

   (3) $6(\vec{i} + 3\vec{j}) - 4(2\vec{i} + 4\vec{j}) = 6\vec{i} + 18\vec{j} - 8\vec{i} - 16\vec{j} = -2\vec{i} + 2\vec{j}$

3.

   (1) $(n + 2)\vec{i} + 4\vec{j} = 5\vec{i} + m\vec{j}$

   $n + 2 = 5$

   $n = 3$

   $4 = m$

$\therefore\ m = 4, n = 3$

(2) $(4 - m)\vec{i} - (5 - n)\vec{j} = 6\vec{j}$

$4 - m = 0$

$m = 4$

$-(5 - n) = 6$

$-5 + n = 6$

$n = 11$

$\therefore\ m = 4, n = 11$

(3) $(20 - 3m)\vec{i} + (30 - 5n)\vec{j} = (m - 4)\vec{i} + 10\vec{j}$

$20 - 3m = m - 4$

$4m = 24$

$m = 6$

$30 - 5n = 10$

$5n = 20$

$n = 4$

$\therefore\ m = 6, n = 4$

4. $\vec{a} + 2\vec{b} = (\vec{i} + \vec{j}) + 2(x\vec{i} + \vec{j}) = (1 + 2x)\vec{i} + 3\vec{j}$

$2\vec{a} - b = 2(\vec{i} + \vec{j}) - (x\vec{i} + \vec{j}) = (2 - x)\vec{i} + \vec{j}$

$\because$ $\vec{a} + 2\vec{b}$ and $2\vec{a} - \vec{b}$ are parallel to each other.

$\therefore$ $\vec{a} + 2\vec{b} = k(2\vec{a} - \vec{b})$

$(1 + 2x)\vec{i} + 3\vec{j} = (2 - x)k\vec{i} + k\vec{j}$

$k = 3$

$1 + 2x = 3(2 - x)$

$5x = 5$

$\therefore\ x = 1$

5. $\because$ $\vec{a} // \vec{b}$

$2\vec{i} + 3\vec{j} = k(x\vec{i} - 6\vec{j})$

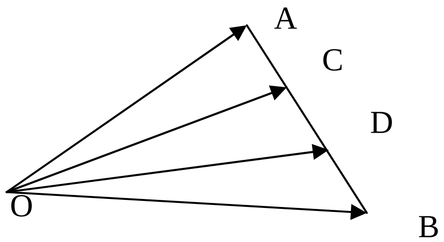
$2 = kx$

$3 = -6k$ $\qquad\therefore\ k = -0.5$

$2 = -0.5x \quad \therefore \; x = -4$

Practice 7.2G

1. Write down the unit vector of :
   $3\vec{i} + 3\vec{j}$
   $-8\vec{i} + 6\vec{j}$
   $-5\vec{i} - 12\vec{j}$
2. Write down the position vector of A(–3, 5), B(3, 6) and C(4, 0).
3. Given point P is the midpoint of line AB. If the coordinates of point A and point P are (–4, –7) and (6, 11) respectively, find the position vector of point B.
4. Given $\overrightarrow{OA} = 3\vec{e_1}$, $\overrightarrow{OB} = 3\vec{e_2}$, C and D divide AB into 3 equal parts, express $\overrightarrow{OC}$ and $\overrightarrow{OD}$ in terms of $\vec{e_1}$ and $\vec{e_2}$.



5. Given P is a point on line AB, $\overrightarrow{AP} = 2\overrightarrow{PB}$, if the coordinates of point A and point B are (–3, –6) and (6, 18) respectively, find the position vector of point B.
6. Given M(3, 2), N(–5, –1), and $2\overrightarrow{MP} = \overrightarrow{MN}$, find the position vector of point P.

Answer

1. (1) $\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$ (2) $-\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$ (3) $-\frac{5}{13}\vec{i} - \frac{12}{13}\vec{j}$
2. $\overrightarrow{OA} = -3\vec{i} + 5\vec{j}$, $\overrightarrow{OB} = 3\vec{i} + 6\vec{j}$, $\overrightarrow{OC} = 4\vec{i}$
3. $\overrightarrow{OA} = -4\vec{i} - 7\vec{j}$

   $\overrightarrow{OP} = 6\vec{i} + 11\vec{j} \; -\vec{i} - \frac{2}{3}\vec{j}$

   By mid-point theorem,

   $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

$\therefore\ \overrightarrow{OB} = 2\overrightarrow{OP} - \overrightarrow{OA} = 2(6\vec{i} + 11\vec{j}) - (-4\vec{i} - 7\vec{j}) = 16\vec{i} + 29\vec{j}$

4. By section formula,

$$\overrightarrow{OC} = \frac{2(3\overrightarrow{e_1}) + 3\overrightarrow{e_2}}{2+1}$$

$$\therefore\ \overrightarrow{OC} = 2\overrightarrow{e_1} + \overrightarrow{e_2}$$

$$\overrightarrow{OD} = \frac{3\overrightarrow{e_1} + 2(3\overrightarrow{e_2})}{1+2}$$

$$\therefore\ \overrightarrow{OD} = \overrightarrow{e_1} + 2\overrightarrow{e_2}$$

5. $\overrightarrow{OA} = -3\vec{i} - 6\vec{j}$

$\overrightarrow{OP} = 6\vec{i} + 18\vec{j}$

$\because\ \overrightarrow{AP} = 2\overrightarrow{PB}$

$\therefore$ AP : PB = 2 : 1

By section formula,

$$\overrightarrow{OP} = \frac{2\overrightarrow{OB} + \overrightarrow{OA}}{2+1}$$

$$\therefore\ \overrightarrow{OB} = \frac{1}{2}(3\overrightarrow{OP} - \overrightarrow{OA})$$

$$\overrightarrow{OB} = \frac{1}{2}[3(6\vec{i} + 18\vec{j}) - (-3\vec{i} - 6\vec{j})] = \frac{21}{2}\vec{i} + 27\vec{j}$$

6. $\overrightarrow{OM} = 3\vec{i} + 2\vec{j}$

$\overrightarrow{ON} = -5\vec{i} - \vec{j}$

$\because\ 2\overrightarrow{MP} = \overrightarrow{MN}$

$\therefore$ MP : PN = 1 : 1 (P is the mid-point of MN)

By mid-point formula,

$$\overrightarrow{OP} = \frac{\overrightarrow{OM} + \overrightarrow{ON}}{2} = -\vec{i} + \frac{1}{2}\vec{j}$$