Ch.6 Limits and Derivatives and Differentiation 函數的極限與導數及微分法

A1 解

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}$$
$$= \lim_{x\to 9}\frac{1}{\sqrt{x}+3}$$
$$= \frac{1}{6}$$

A2 解

$$\lim_{x\to -1}\frac{1-|x|}{1+x} = \lim_{x\to -1}\frac{1-(-x)}{1+x} \quad (\because \quad x<0)$$
$$= 1$$

A3 解

$$\lim_{x\to 0^-}\frac{x}{|x|}\sin x = \lim_{x\to 0^-}\frac{x}{-x}\sin x$$
$$= \lim_{x\to 0^-}(-\sin x)$$
$$= 0$$
$$\lim_{x\to 0^+}\frac{x}{|x|}\sin x = \lim_{x\to 0^+}\frac{x}{x}\sin x$$
$$= 0$$
$$\therefore \quad \lim_{x\quad 0}\frac{x}{|x|} = 0$$

A4 解

$$\lim_{x\to\infty}\frac{x^3+2x-3}{4x^3-8x^2+6} = \lim_{x\to\infty}\frac{1+\frac{2}{x^2}-\frac{3}{x^3}}{4-\frac{8}{x}+\frac{6}{x^3}} = \frac{1}{4}$$

A5 解

$$\Delta y = 2\sqrt{x+\Delta x}-2\sqrt{x}$$
$$= \frac{(2\sqrt{x+\Delta x}-2\sqrt{x})(2\sqrt{x+\Delta x}+2\sqrt{x})}{2\sqrt{x+\Delta x}+2\sqrt{x}}$$
$$= \frac{4\Delta x}{2\sqrt{x+\Delta x}+2\sqrt{x}}$$
$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{2}{\sqrt{x+\Delta x}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}}$$

A6 解

$$\Delta y = \frac{(t+\Delta t)+1}{(t+\Delta t)-1}-\frac{t+1}{t-1}$$
$$= \frac{[(t+\Delta t)+1](t-1)-(t+1)[(t+\Delta t)-1]}{[(t+\Delta t)-1](t-1)}$$
$$= \frac{-2\Delta t}{[(t+\Delta t)-1](t-1)}$$
$$\therefore \quad \frac{dy}{dx} = \frac{-2}{(t-1)^2}$$

A7 解

(a) $\lim_{x\to 1^-} f(x) = 2, \quad \lim_{x\to 1^+} f(x) = 3$

$\therefore \quad \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

故此， $\lim_{x\to 1} f(x)$ 並不存在。

(b) $\lim_{x\to 3} f(x) = 3$

事實上， $f(x)$ 在 $x = 3$ 時是連續的。

(c) $\lim_{x\to 4} f(x) = 1$

A8 解

$$\begin{aligned}\lim_{x\to 0}\frac{(k+x)^3-k^3}{x} &= \lim_{x\to 0}\frac{(k^3+3k^2x+3kx^2+x^3)-k^3}{x}\\ &= \lim_{x\to 0}\frac{3k^2x+3kx^2+x^3}{x}\\ &= \lim_{x\to 0}(3k^2+3kx+x^2)\\ &= 3k^2\end{aligned}$$

A9 解

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^3 3x}{x\sin^2 5x} &= \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^3\cdot\left(\frac{5x}{\sin 5x}\right)^2\cdot\frac{27}{25}\\ &= \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^3\cdot\lim_{x\to 0}\left(\frac{5x}{\sin 5x}\right)^2\cdot\frac{27}{25}\\ &= 1\times 1\times\frac{27}{25}\\ &= \frac{27}{25}\end{aligned}$$

A10 **解**

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} &= \lim_{h\to 0}\frac{-2\sin\frac{(x+h)+x}{2}\sin\frac{(x+h)-x}{2}}{h}\\ &= \lim_{h\to 0}\frac{-2\sin(x+\frac{h}{2})\sin\frac{h}{2}}{h}\\ &= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\\ &= -\sin x\times 1\\ &= -\sin x\end{aligned}$$

A11 解

$$\begin{aligned}\lim_{x\to\infty}\frac{4x^3-3x^2+2x-8}{(x+1)(5x-3)^2} &= \lim_{x\to\infty}\frac{4-\frac{3}{x}+\frac{2}{x^2}-\frac{8}{x^3}}{(1+\frac{1}{x})(5-\frac{3}{x})^2}\\ &= \frac{4}{1\times 5^2}\\ &= \frac{4}{25}\end{aligned}$$

A12 解

(a)

Δx	1	0.1	0.01	0.001	0.000 1

$x_1 + \Delta x$	3	2.1	2.01	2.001	2.000 1
$\Delta y = f(x_1+\Delta x) - f(x_1)$	7	0.52	0.050 2	0.005 002	0.000 500 02
$\frac{\Delta y}{\Delta x}$	7	5.2	5.02	5.002	5.000 2

(b)

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(2+\Delta x)-f(2)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{[2(2+\Delta x)^2-3(2+\Delta x)+7]-[2(2)^2-3(2)+7]}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\{2[4+4\Delta x+(\Delta x)^2]-6-3\Delta x+7\}-9}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{5\Delta x+2(\Delta x)^2}{\Delta x}\\ &= \lim_{\Delta x\to 0}(5+2\Delta x)\\ &= 5\end{aligned}$$

A13 解

(a)

$$\begin{aligned}\Delta y &= (x+\Delta x+5)^2-(x+5)^2\\ &= (x+\Delta x)^2+10(x+\Delta x)+25-(x^2+10x+25)\\ &= x^2+2x(\Delta x)+(\Delta x)^2+10x+10\Delta x+25-x^2-10x-25\\ &= (2x+10)\Delta x+(\Delta x)^2\end{aligned}$$

(b)

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{(2x+10)\Delta x+(\Delta x)^2}{\Delta x}\\ &= 2x+10+\Delta x\end{aligned}$$

(c)

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}(2x+10+\Delta x)\\ &= 2x+10\end{aligned}$$

A14 解

(a)

$$\begin{aligned}\Delta y &= f(x+\Delta x)-f(x)\\ &= [2(x+\Delta x)^3+7(x+\Delta x)]-(2x^3+7x)\\ &= 2[x^3+3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3]+7x+7\Delta x-2x^3-7x\\ &= (6x^2+7)\Delta x+6x(\Delta x)^2+2(\Delta x)^3\end{aligned}$$

$$\begin{aligned}f'(x) &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}[6x^2+7+6x(\Delta x)+2(\Delta x)^2]\\ &= 6x^2+7\end{aligned}$$

(b)

$$\begin{aligned}f'(-1) &= 6(-1)^2+7\\ &= 13\end{aligned}$$

此即曲線 $y = f(x)$ 在 $x = -1$ 的斜率為 13。

A15 解

(a)

$$\begin{aligned}\Delta y &= 20(t+\Delta t)+(t+\Delta t)^2-(20t+t^2)\\ &= 20\Delta t+2t(\Delta t)+(\Delta t)^2\end{aligned}$$

$$\begin{aligned}\therefore\quad \frac{dy}{dt} &= \lim_{\Delta t\to 0}(20+2t+\Delta t)\\ &= 20+2t\end{aligned}$$

(b) 所求的增長率 $= 20 + 2(10)$

$= 40$ 細菌/小時

B1 解

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{4}\left(\frac{x+1}{2x+3}\right)^{-\frac{3}{4}}\cdot\frac{(2x+3)-2(x+1)}{(2x+3)^2}\\ &= \frac{1}{4}(x+1)^{-\frac{3}{4}}(2x+3)^{-\frac{5}{4}}\end{aligned}$$

B2 解

$$
\begin{aligned}
x^2 + xy + y^5 &= 3 \\
2x + y + x\frac{dy}{dx} + 5y^4\frac{dy}{dx} &= 0 \\
\therefore \quad \frac{dy}{dx} &= -\frac{2x+y}{x+5y^4} \\
\left.\frac{dy}{dx}\right|_{(1,1)} &= -\frac{2+1}{1+5(1)^4} \\
&= -\frac{1}{2}
\end{aligned}
$$

B3 解

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{2}{3}(\operatorname{cosec} 2x)^{-\frac{1}{3}}(-\operatorname{cosec} 2x \cot 2x)(2) \\
&= -\frac{4}{3}(\operatorname{cosec} 2x)^{\frac{2}{3}}\cot 2x
\end{aligned}
$$

B4 解

$$
\begin{aligned}
f'(x) &= \sin\frac{\pi}{x} + x\cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right) \\
&= \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x} \\
f''(x) &= \cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right) - \left[-\frac{\pi}{x^2}\cos\frac{\pi}{x} + \frac{\pi}{x}\sin\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right)\right] \\
&= \frac{\pi^2}{x^3}\sin\frac{\pi}{x} \\
f''(6) &= \frac{\pi^2}{6^3}\sin\frac{\pi}{6} \\
&= \frac{\pi^2}{432}
\end{aligned}
$$

B5 解

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\
&= \frac{3\sec^2\theta}{2\sec\theta\tan\theta} \\
&= \frac{3\sec\theta}{2\tan\theta} \\
\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} &= \frac{3\sec\frac{\pi}{4}}{2\tan\frac{\pi}{4}} \\
&= \frac{3\sqrt{2}}{2}
\end{aligned}
$$

B6 解

$$
\begin{aligned}
x &= y + \cos y \\
\therefore \quad 1 &= \frac{dy}{dx} - \sin y\frac{dy}{dx} \\
\therefore \quad \frac{dy}{dx} &= \frac{1}{1-\sin y} \\
\frac{d^2y}{dx^2} &= \frac{-(-\cos y)}{(1-\sin y)^2}\cdot\frac{dy}{dx}
\end{aligned}
$$

$$= \frac{\cos y}{(1-\sin y)^2} \cdot \frac{1}{1-\sin y}$$
$$= \frac{\cos y}{(1-\sin y)^3}$$

B7 解

(a)
$$f(x) = \frac{6x^2}{\sqrt{x^3}}$$
$$= 6x^{\frac{1}{2}}$$
$$\therefore \quad f'(x) = 6\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$
$$= 3x^{-\frac{1}{2}}$$

(b)
$$f'(4) = 3(4^{-\frac{1}{2}})$$
$$= \frac{3}{2}$$

B8 解

(a)
$$y = \frac{k}{x}$$
$$\frac{dy}{dx} = -kx^{-2}$$
依題意,
$$-2 = -k(1^{-2})$$
$$\therefore \quad k = 2$$

(b) 當 $x = 1$ 時, $y = \frac{2}{1} = 2$。

$\therefore$ 所求的切線方程為
$$y - 2 = -2(x - 1)$$
即 $2x + y - 4 = 0$。

(c) 若 $\frac{dy}{dx} = -\frac{1}{2}$,

則
$$-\frac{1}{2} = -2x^{-2}$$
$$x^2 = 4$$
$$x = \pm 2$$
$\therefore$ 所求的點為 $(-2, -1)$ 或 $(2, 1)$。

B9 解
$$\frac{dy}{dx} = (x^{\frac{1}{3}} - 4)\cdot\frac{d}{dx}(x^2+3x) + (x^2+3x)\cdot\frac{d}{dx}(x^{\frac{1}{3}}-4)$$
$$= (x^{\frac{1}{3}} - 4)(2x+3) + (x^2+3x)(\frac{1}{3}x^{-\frac{2}{3}})$$
$$= 2x^{\frac{4}{3}} - 8x + 3x^{\frac{1}{3}} - 12 + \frac{1}{3}x^{\frac{4}{3}} + x^{\frac{1}{3}}$$
$$= \frac{7}{3}x^{\frac{4}{3}} - 8x + 4x^{\frac{1}{3}} - 12$$

B10 解
$$\frac{dy}{dx} = 3x^2 - 5x + 2$$
$$= (3x-2)(x-1)$$
切線為水平, 當
$$\frac{dy}{dx} = 0$$

即 $x = \frac{2}{3}$ 或 1

當 $x = \frac{2}{3}$ 時, $y = y = \left(\frac{2}{3}\right)^3 - \frac{5}{2}\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right) + 1 = \frac{41}{27}$

當 $x = 1$ 時, $y = 1^3 - \frac{5}{2}(1)^2 + 2(1) + 1 = \frac{3}{2}$

即 所求的點為 $\left(\frac{2}{3}, \frac{41}{27}\right)$ 及 $\left(1, \frac{3}{2}\right)$。

B11 解

$$\frac{dy}{dt} = \frac{(t^{-1}-2)\cdot\frac{d}{dt}(t^{-2}+3) - (t^{-2}+3)\cdot\frac{d}{dt}(t^{-1}-2)}{(t^{-1}-2)^2}$$

$$= \frac{(t^{-1}-2)(-2t^{-3}) - (t^{-2}+3)(-t^{-2})}{(t^{-1}-2)^2}$$

$$= \frac{-2t^{-4}+4t^{-3}+t^{-4}+3t^{-2}}{(t^{-1}-2)^2}$$

$$= \frac{-t^{-4}+4t^{-3}+3t^{-2}}{(t^{-1}-2)^2}$$

$$\left.\frac{dy}{dt}\right|_{t=2} = \frac{-\frac{1}{16}+\frac{4}{8}+\frac{3}{4}}{(\frac{1}{2}-2)^2}$$

$$= \frac{19}{36}$$

B12 解

$$y = \frac{\sqrt{x}}{(2x+1)(x-3)} = \frac{\sqrt{x}}{2x^2-5x-3}$$

$$\frac{dy}{dx} = \frac{(2x^2-5x-3)\cdot\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\cdot\frac{d}{dx}(2x^2-5x-3)}{(2x^2-5x-3)^2}$$

$$= \frac{(2x^2-5x-3)(\frac{1}{2}x^{-\frac{1}{2}}) - \sqrt{x}(4x-5)}{(2x^2-5x-3)^2}$$

$$= \frac{x^{\frac{3}{2}} - \frac{5}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}} - 4x^{\frac{3}{2}} + 5x^{\frac{1}{2}}}{(2x^2-5x-3)^2}$$

$$= \frac{-3x^{\frac{3}{2}} + \frac{5}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}}{(2x^2-5x-3)^2}$$

$$= \frac{-6x^2+5x-3}{2x^{\frac{1}{2}}(2x^2-5x-3)^2}$$

B13 解

$$\frac{dy}{dx} = (\sqrt{x}+1)\cdot\frac{d}{dx}(\sqrt{x}-1)^{\frac{3}{2}} + (\sqrt{x}-1)^{\frac{3}{2}}\cdot\frac{d}{dx}(\sqrt{x}+1)$$

$$= (\sqrt{x}+1)\cdot\frac{3}{2}(\sqrt{x}-1)^{\frac{1}{2}}(\frac{1}{2}x^{-\frac{1}{2}}) + (\sqrt{x}-1)^{\frac{3}{2}}(\frac{1}{2}x^{-\frac{1}{2}})$$

$$= \frac{1}{4}(\sqrt{x}-1)^{\frac{1}{2}}x^{-\frac{1}{2}}[3(\sqrt{x}+1)+2(\sqrt{x}-1)]$$

$$= \frac{(\sqrt{x}-1)^{\frac{1}{2}}(5\sqrt{x}+1)}{4\sqrt{x}}$$

B14 解

$$F'(t) = \frac{d(\frac{3t+4}{2t-7})^5}{d(\frac{3t+4}{2t-7})} \cdot \frac{d(\frac{3t+4}{2t-7})}{dt}$$

$$= 5\left(\frac{3t+4}{2t-7}\right)^4 \cdot \frac{(2t-7)(3)-(3t+4)(2)}{(2t-7)^2}$$

$$= \frac{-145(3t+4)^4}{(2t-7)^6}$$

B15 解

$x^3 - x^2y + 3xy^2 + y^3 = 0$

對 x 微分，得

$$\frac{d}{dx}(x^3) - x^2\left(\frac{dy}{dx}\right) - y\frac{d}{dx}(x^2) + 3x\frac{d}{dx}(y^2) + 3y^2\frac{d}{dx}(x) + \frac{d}{dx}(y^3) = 0$$

$$3x^2 - x^2 \cdot \frac{dy}{dx} - 2xy + 3x \cdot 2y \cdot \frac{dy}{dx} + 3y^2 + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$(3x^2 - 2xy + 3y^2) - (x^2 - 6xy - 3y^2)\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2 - 2xy + 3y^2}{x^2 - 6xy - 3y^2}$$

B16 解

$$\frac{dy}{dx} = \frac{d\cos(\frac{2x+3}{2x-3})}{d(\frac{2x+3}{2x-3})} \cdot \frac{d(\frac{2x+3}{2x-3})}{dx}$$

$$= -\sin\left(\frac{2x+3}{2x-3}\right) \cdot \frac{(2x-3)(2)-(2x+3)(2)}{(2x-3)^2}$$

$$= \frac{12}{(2x-3)^2}\sin\left(\frac{2x+3}{2x-3}\right)$$

B17 解

(a)

$$y = x\cos 7x$$

$$\frac{dy}{dx} = -7x\sin 7x + \cos 7x$$

$$\frac{d^2y}{dx^2} = -7(7x\cos 7x + \sin 7x) - 7\sin 7x$$

$$= -49x\cos 7x - 14\sin 7x$$

(b)

$$\frac{d^2y}{dx^2} + 49y = (-49x\cos 7x - 14\sin 7x) + 49x\cos 7x$$

$$= -14\sin 7x$$

$$f'(x) = \frac{\sec^2 x}{\sqrt{x^2 + A}} + \frac{-x}{(x^2 + A)^{\frac{3}{2}}}\tan x$$

$$f'(0) = \frac{1}{\sqrt{A}} = \frac{1}{2}$$

$$\therefore \quad A = 4$$

B19 解

$$\frac{dx}{dt} = -2\sin t - 2\sin 2t$$

$$\frac{dy}{dt} = 2\cos 2t - 2\cos t$$

故此，$$\frac{dy}{dx} = \frac{2\cos 2t - 2\cos t}{-2\sin t - 2\sin 2t}$$

$$= \frac{2\cos^2 t - 1 - \cos t}{-\sin t - 2\sin t\cos t}$$

$$= \frac{(2\cos t + 1)(\cos t - 1)}{-\sin t(1 + 2\cos t)}$$

$$= \frac{1 - \cos t}{\sin t}$$

$$= \frac{2\sin^2 \frac{t}{2}}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \tan\frac{t}{2}$$

B20 解

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3+2x^2}{5-x^2}\right)^{-\frac{1}{2}} \cdot \frac{4x(5-x^2)+2x(3+2x^2)}{(5-x^2)^2}$$

$$= y \cdot \frac{1}{2}\left(\frac{5-x^2}{3+2x^2}\right) \cdot \frac{4x(5-x^2)+2x(3+2x^2)}{(5-x^2)^2}$$

$$= y\left(\frac{2x}{3+2x^2} + \frac{x}{5-x^2}\right)$$

∴ A = 2, B = 1

16 微分法的應用 (A)

B21 解

$$dH = \frac{2u^2\cos 2\theta}{g}\Delta\theta$$

$$= \frac{2(20)^2\cos 2\left(\frac{\pi}{6}\right)}{10}(0.005)$$

$$= 0.2$$

B22 解

$$\frac{dy}{dx} = 3x^2 + 2x + 1$$

$$\frac{d^2y}{dx^2} = 6x + 2$$

$$\frac{d^3y}{dx^3} = 6 > 0$$

故此，當 $x = -\frac{1}{3}$ 時，斜率 $\frac{dy}{dx}$ 為最小。

$$\left.\frac{dy}{dx}\right|_{x=-\frac{1}{3}} = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) + 1 = \frac{2}{3}$$

所求切線方程為

$$y - \frac{20}{27} = \frac{2}{3}\left(x + \frac{1}{3}\right)$$

即 $18x - 27y + 26 = 0$。

B23 解

(a)

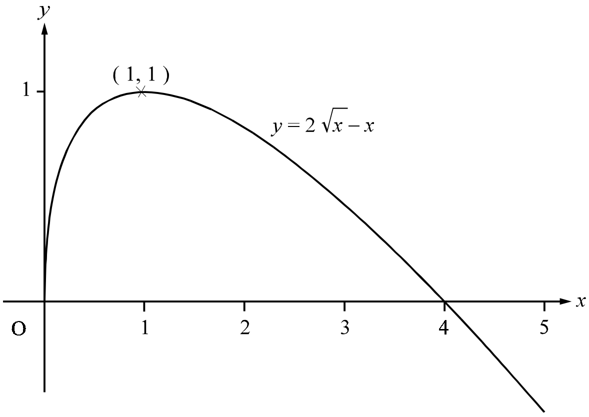
$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} - 1$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}}$$

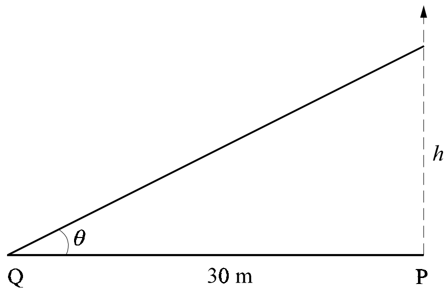
$$\frac{dy}{dx} = 0 \quad 當 \quad x = 1$$

由此, (1, 1) 是一極大點。

(b)



B24 解



$$\tan\theta = \frac{h}{30}$$

$$= \frac{30t - 5t^2}{30}$$

$$\sec^2\theta\frac{d\theta}{dt} = 1 - \frac{1}{3}t$$

當 $t = 4$,

$$h = 30(4) - 5(4)^2 = 40$$

$$\tan\theta = \frac{4}{3}$$

$$\therefore \quad \left[1 + \left(\frac{4}{3}\right)^2\right]\frac{d\theta}{dt} = 1 - \frac{1}{3}(4)$$

$$\frac{d\theta}{dt} = -\frac{3}{25}$$

即 所求變率為 $-\frac{3}{25}$ 弧度/s。

B25 解

(a) 設其餘兩邊的長度為 a cm 及 b cm。

則 $a = 10\sin\theta$

$b = 10\cos\theta$

(b) 三角形的面積為

$$A = \frac{1}{2}ab$$

$$= 50\sin\theta\cos\theta$$

$$= 25\sin 2\theta$$

$\therefore$ 最大面積為 25 cm^2, 當

$\sin 2\theta = 1$, 即 $\theta = \frac{\pi}{4}$。

B26 解

$$s = 48t - t^3$$

$$v = \frac{ds}{dt} = 48 - 3t^2$$

$$a = \frac{d^2s}{dt^2} = -6t$$

若 $v = 48 - 3t^2 = 0$

$t = \pm 4$

$\therefore$ 所求加速度 $= -6(4)$

$= -24$ m/s^2

B27 解

$$y^2 = x^2y + 5 \quad \ldots\ldots\ldots\ldots (1)$$

當 $x = 2$,

$$y^2 = 4y + 5$$
$$(y-5)(y+1) = 0$$
$$y = 5 \quad 或 \quad -1$$

將 (1) 對 x 微分,

$$2y\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2xy}{2y - x^2}$$
$$\left.\frac{dy}{dx}\right|_{(2,5)} = \frac{2(2)(5)}{2(5)-2^2} = \frac{10}{3}$$

$\therefore$ 在 (2, 5) 的切線方程為

$$y - 5 = \frac{10}{3}(x-2)$$

即 $10x - 3y - 5 = 0$

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{2(2)(-1)}{2(-1)-2^2} = \frac{2}{3}$$

$\therefore$ 在 (2, –1) 的切線方程為

$$y + 1 = \frac{2}{3}(x-2)$$

即 $2x - 3y - 7 = 0$

B28 解

$$x^2 + y^2 = 100$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2x(2y) + 2y\frac{dy}{dt} = 0$$
$$\therefore \quad \frac{dy}{dt} = -2x$$

B29 解

設當太陽的仰角為 θ 時，影子的長度為 x m。

依題意,

$$\frac{d\theta}{dt} = -15\times\frac{\pi}{180} \text{ 弧度/h}$$
$$= \frac{-\pi}{12} \text{ 弧度/h}$$

由於 $x = 20\cot\theta$,

$$\frac{dx}{dt} = 20(-\csc^2\theta)\cdot\frac{d\theta}{dt}$$

當 $x = 25$,

$$25 = 20\cot\theta$$
$$\cot\theta = \frac{5}{4}$$
$$\therefore \quad \csc^2\theta = 1 + \cot^2\theta = \frac{41}{16}$$

因此,

$$\frac{dx}{dt} = 20\left(-\frac{41}{16}\right)\left(-\frac{\pi}{12}\right)$$
$$= \frac{205\pi}{48} \text{ m/h}$$

B30 解

(a) $y = x^4 - 4x^3$

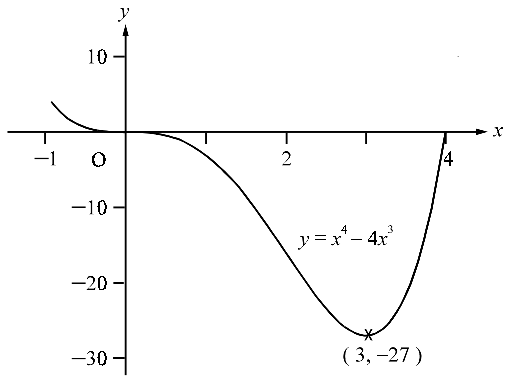
$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$\frac{dy}{dx} = 0 \quad \text{當} \quad 4x^3 - 12x^2 = 0$$

即 $x = 0$ 或 3

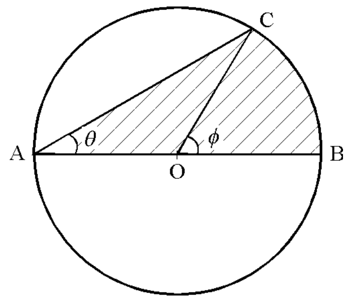
$\therefore$ 駐點為 $(0, 0)$ 及 $(3, -27)$。

(b)



B31 解

(a) 連接 OC 及設 $\angle BOC = \phi$。



$\phi = 2\theta$ (圓心角 = 2 圓周角)

$$\Delta AOC \text{ 的面積} = \frac{1}{2} \times 6 \times 6 \times \sin(\pi - \phi)$$

$$= 18 \sin \phi$$

$$= 18 \sin 2\theta \quad \text{cm}^2$$

$$\text{扇形 BOC 的面積} = \frac{1}{2} \times 6 \times 6 \times \phi$$

$$= 36\theta \quad \text{cm}^2$$

$\therefore$ $S = 18 \sin 2\theta + 36\theta$ . . . . (1)

(b) $\theta = 45° = \frac{\pi}{4}$ 弧度

$0.5° = 0.5 \times \frac{\pi}{180} = 0.008\ 73$ 弧度

將 (1) 微分,

$dS = [18 \cos 2\theta (2) + 36]d\theta$

$\therefore$ $dS = 36(\cos 2\theta + 1)d\theta$

當 $\theta = \frac{\pi}{4}$ 及 $d\theta = 0.008\ 73$

$$dS = 36\left(\cos\frac{\pi}{2} + 1\right) \times 0.008\ 73$$

$= 0.314$ (準確至 3 位有效數字)

B32 解

(a) $R = u\sqrt{\frac{2h}{g}}$

$$\therefore \quad dR = u \cdot d\left(\sqrt{\frac{2h}{g}}\right) + \sqrt{\frac{2h}{g}} \cdot du$$

$$= u \cdot \frac{1}{2}\sqrt{\frac{2}{g}} \cdot h^{-\frac{1}{2}} dh + \sqrt{\frac{2h}{g}} du$$

$$= \frac{u}{2}\sqrt{\frac{2}{gh}} dh + \sqrt{\frac{2h}{g}} du$$

(b) $\frac{dR}{R} = \frac{1}{u\sqrt{\frac{2h}{g}}}\left(\frac{u}{2}\sqrt{\frac{2}{gh}}dh+\sqrt{\frac{2h}{g}}du\right)$

$= \frac{1}{2}\cdot\frac{dh}{h}+\frac{du}{u}$

$= \frac{1}{2}\times 2\%+1\%$

$= 2\%$

即　在計算 R 時的可能百分誤差為 2%。

14　函數的極限與導數 (B)

B33 解

(a) 設 S(n) 為命題

"$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$"。

當 $n = 1$ 時,

左方 $= 1^2 = 1$,

右方 $= \frac{1}{6}(1)(2)(3) = 1$。

∴ S(1) 成立。

假設 S(k) 成立。

即 $1^2 + 2^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$。

等式兩邊同時加上 $(k+1)^2$,

$$\begin{aligned}1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2\\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]\\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6)\\ &= \frac{1}{6}(k+1)(k+2)(2k+3)\\ &= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]\end{aligned}$$

∴ S($k + 1$) 成立。

根據數學歸納法，對於所有自然數 n，S(n) 成立。

(b)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\ \ldots\ +\frac{n^2}{n^3}\right) &= \lim_{n\to\infty}\left(\frac{1^2+2^2+3^2+\ \ldots\ +n^2}{n^3}\right)\\ &= \lim_{n\to\infty}\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3}\\ &= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\\ &= \frac{1}{6}\times 1\times 2\\ &= \frac{1}{3}\end{aligned}$$

B34 解

(a) $\Delta M = G(4) - G(3)$

$= \frac{18}{4+1} - \frac{18}{3+1}$

$= -0.9$

∴ 所求的平均率 $= \frac{\Delta M}{\Delta t}$

$$= \frac{-0.9}{4-3}$$

$$= -0.9 \text{ g/s}$$

(b) $\Delta M = G(3.2) - G(3)$

$$= \frac{18}{3.2+1} - \frac{18}{3+1}$$

$$= -0.214\ 3$$

∴ 所求的平均率 $= \frac{-0.214\ 3}{3.2-3}$

$= -1.07$ g/s (準確至 3 位有效數字)

(c) **(i)**

$$\lim_{\Delta t \to 0} \frac{G(3+\Delta t) - G(3)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{18}{(3+\Delta t)+1} - \frac{18}{3+1}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{18[\frac{4-(4+\Delta t)}{4(4+\Delta t)}]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-9}{2(4+\Delta t)}$$

$$= -\frac{9}{8}$$

(ii) 從 **(i)** 的答案顯示當 $t = 3$ 秒時，該反應物之質量的瞬時下降率為 $\frac{9}{8}$ g/s。

15 微分法的法則 (B)

B35 解

(a)

$$x = t + \frac{1}{t}$$

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}$$

$$y = t - \frac{1}{t}$$

$$\frac{dy}{dt} = 1 + \frac{1}{t^2}$$

∴ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$= \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{1+\frac{1}{4}}{1-\frac{1}{4}} = \frac{5}{3}$$

(b) **(i)**

$$x^2 - y^2 = \left(t+\frac{1}{t}\right)^2 - \left(t-\frac{1}{t}\right)^2$$

$$= \left(t^2+2+\frac{1}{t^2}\right) - \left(t^2-2+\frac{1}{t^2}\right)$$

∴ $x^2 - y^2 = 4$ . . . . (*)

(ii) 上式方程 (*) 對 x 微分，得

$$2x - 2y\frac{dy}{dx} = 0$$

∴ $\frac{dy}{dx} = \frac{x}{y}$

當 $t = 2$ 時，可得 $x = 2 + \frac{1}{2} = \frac{5}{2}$ 及 $y = 2 - \frac{1}{2} = \frac{3}{2}$。

$$\therefore \quad \left.\frac{dy}{dx}\right|_{t=2} = \frac{\frac{5}{2}}{\frac{3}{2}} = \frac{5}{3}$$

B36 解

(a) 利用鏈式法則,

$$\frac{dx}{dx} = \frac{dx}{dt}\cdot\frac{dt}{dx}$$

$$\therefore \quad 1 = \frac{dx}{dt}\cdot\frac{dt}{dx}$$

故此，若 $\frac{dx}{dt} \neq 0$，則

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$ 。

(b)

$$\frac{dy}{dx} = f(t)$$

$$\begin{aligned}\therefore \quad \frac{d^2y}{dx^2} &= \frac{d\ f(t)}{dx} \\ &= \frac{d\,f(t)}{dt}\cdot\frac{dt}{dx} \qquad \text{(鏈式法則)} \\ &= \frac{\frac{d\,f(t)}{dt}}{\frac{dx}{dt}} \qquad \text{(由 (a))}\end{aligned}$$

(c)

$$\begin{aligned} x &= \cos^3\theta - 3\cos\theta \\ \frac{dx}{d\theta} &= 3\cos^2\theta(-\sin\theta) - 3(-\sin\theta) \\ &= 3\sin\theta(-\cos^2\theta + 1) \\ &= 3\sin^3\theta \\ y &= 3\sin\theta - \sin^3\theta \\ \frac{dy}{d\theta} &= 3\cos\theta - 3\sin^2\theta\cos\theta \\ &= 3\cos\theta(1 - \sin^2\theta) \\ &= 3\cos^3\theta \end{aligned}$$

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{3\cos^3 q}{3\sin^3 q} \\ &= \cot^3\theta\end{aligned}$$

由此，從 **(b)** 的結果可得

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d\cot^3\theta}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{3\cot^2\theta(-\csc^2\theta)}{3\sin^3\theta} \\ &= -\cot^2\theta\csc^5\theta\end{aligned}$$

B37 解

(a)

$$\begin{aligned} y &= ax^2 + bx \\ \frac{dy}{dx} &= 2ax + b \\ \frac{d^2y}{dx^2} &= 2a \end{aligned}$$

$$\begin{aligned}\therefore \quad x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y &= x^2(2a) - 2x(2ax + b) + 2(ax^2 + bx) \\ &= 0\end{aligned}$$

(b) 當 $x = 1$, $y = 3$ 及 $\frac{dy}{dx} = 8$。

$\therefore \quad 3 = a(1)^2 + b(1)$

即 $\quad 3 = a + b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

及 $\quad 8 = 2a(1) + b$

即 $\quad 8 = 2a + b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

(2) – (1),

$a = 5$

$\therefore \quad b = -2$

B38 解

(a)

$$y = \sqrt{a^2 - x^2}$$

$$\frac{dy}{dx} = \frac{d\sqrt{a^2 - x^2}}{d(a^2 - x^2)} \cdot \frac{d(a^2 - x^2)}{dx}$$

$$= \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= -x(a^2 - x^2)^{-\frac{1}{2}}$$

(b)

$$\frac{d^2y}{dx^2} = -x\frac{d(a^2 - x^2)^{-\frac{1}{2}}}{dx} - (a^2 - x^2)^{-\frac{1}{2}}\frac{dx}{dx}$$

$$= -x \cdot \left(-\frac{1}{2}\right)(a^2 - x^2)^{-\frac{3}{2}}(-2x) - (a^2 - x^2)^{-\frac{1}{2}}$$

$$= (a^2 - x^2)^{-\frac{3}{2}}[-x^2 - (a^2 - x^2)]$$

$$= -a^2(a^2 - x^2)^{-\frac{3}{2}}$$

(c)

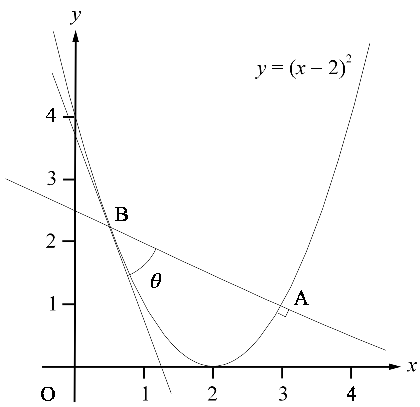
$$x(a^2 - x^2)\frac{d^2y}{dx^2} - a^2\frac{dy}{dx} = x(a^2 - x^2)\cdot[-a^2(a^2 - x^2)^{-\frac{3}{2}}] - a^2[-x(a^2 - x^2)^{-\frac{1}{2}}]$$

$$= -a^2x(a^2 - x^2)^{-\frac{1}{2}} + a^2x(a^2 - x^2)^{-\frac{1}{2}}$$

= 016 微分法的應用 (B)

B39 解

(a)



$$y = (x - 2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{dy}{dx} = 2(x - 2)$$

$$\left.\frac{dy}{dx}\right|_{x=3} = 2(3 - 2)$$

$= \quad 2$

∴ 所求的法線方程為

$$y - 1 = \frac{-1}{2}(x - 3)$$

即 $x + 2y - 5 = 0$ ....(2)

(b) 由 (2), $x = 5 - 2y$。 ....(3)

把 (3) 代入 (1),

$$y = (5 - 2y - 2)^2$$
$$y = 9 - 12y + 4y^2$$
$$4y^2 - 13y + 9 = 0$$
$$(4y - 9)(y - 1) = 0$$
$$y = \frac{9}{4} \quad 或 \quad 1$$

當 $y = \frac{9}{4}$,

$$x = 5 - 2\left(\frac{9}{4}\right) = \frac{1}{2}。$$

∴ B 的坐標 $= \left(\frac{1}{2}, \frac{9}{4}\right)$

(c) 在 B 的切線的斜率 $= m_1 = \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}}$

$$= 2\left(\frac{1}{2} - 2\right)$$
$$= -3$$

弦 AB 的斜率 $= m_2 = \dfrac{\frac{9}{4} - 1}{\frac{1}{2} - 3}$

$$= -\frac{1}{2}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
$$= \left|\frac{-3 + \frac{1}{2}}{1 + (-3)(-\frac{1}{2})}\right|$$
$$= 1$$

∴ $\theta = 45°$

即 所求的角為 45°。

(4 分)

B40 解

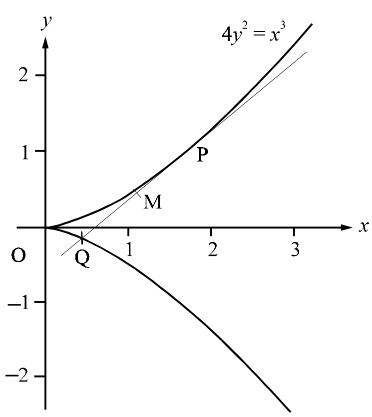
(a) $4y^2 = x^3$ ....(1)

當 $x = 4t^2$,

$$4y^2 = (4t^2)^3$$
$$y^2 = 16t^6$$

∴ $y = \pm 4t^3$

因此, $(4t^2, 4t^3)$ 在曲線上。



(b) 將 (1) 對 x 微分,

$$8y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{8y}$$

$$\left.\frac{dy}{dx}\right|_{(4t^2,\,4t^3)} = \frac{3(4t^2)^2}{8(4t^3)}$$

$$= \frac{3t}{2}$$

∴ 在 P 的切線方程為

$$y - 4t^3 = \frac{3t}{2}(x - 4t^2)$$

即 $3tx - 2y - 4t^3 = 0$ .... (2)

(c) 由 (2), $y = \frac{1}{2}(3tx - 4t^3)$ .... (3)

把 (3) 代入 (1),

$$4\left[\frac{1}{2}(3tx - 4t^3)\right]^2 = x^3$$

$$9t^2x^2 - 24t^4x + 16t^6 = x^3$$

$$x^3 - 9t^2x^2 + 24t^4x - 16t^6 = 0$$

$$(x - 4t^2)^2(x - t^2) = 0$$

∴ $x = t^2$ 或 $4t^2$

把 $x = t^2$ 代入 (3),

$$y = \frac{1}{2}[3t(t^2) - 4t^3]$$

$$= -\frac{1}{2}t^3$$

故此, Q 的坐標為 $\left(t^2,\ -\frac{1}{2}t^3\right)$。

(d) 設 M 的坐標為 (x, y)。

則 $x = \frac{1}{2}(4t^2 + t^2)$

$= \frac{5}{2}t^2$ .... (4)

及 $y = \frac{1}{2}\left[4t^3 + \left(-\frac{1}{2}t^3\right)\right]$

$= \frac{7}{4}t^3$ .... (5)

由 (4) 及 (5), $\left(\frac{2x}{5}\right)^3 = \left(\frac{4y}{7}\right)^2 = t^6$。

由此, M 的軌跡方程為 $y^2 = \frac{49}{250}x^3$。

B41 解

(a)

$$x = t^3 - 7t^2 + 8t$$

$$\frac{dx}{dt} = 3t^2 - 14t + 8$$

∴ 在 $t = 3$ 時的水平速度 $= \left.\frac{dx}{dt}\right|_{t=3}$

$= 3(3^2) - 14(3) + 8$

$= -7$

$$y = \sqrt{2t+1}$$

$$\frac{dy}{dt} = \frac{1}{2}(2t+1)^{-\frac{1}{2}}(2)$$

$$= (2t+1)^{-\frac{1}{2}}$$

∴ 在 $t = 3$ 時的垂直速度 $= \left.\dfrac{dy}{dt}\right|_{t=3}$

$$= [2(3)+1)]^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{7}}$$

$$= \frac{\sqrt{7}}{7}$$

(b) 設 v 為質點在 $t = 3$ 時的速度。

$$v \text{ 的模} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-7)^2 + \left(\frac{1}{\sqrt{7}}\right)^2}$$

$$= \sqrt{\frac{344}{7}}$$

$= 7.01$ (準確至 3 位有效數字)

$$v \text{ 與正水平方向的角度} = \tan^{-1}\left(\frac{\frac{1}{\sqrt{7}}}{-7}\right)$$

$$= -3.09°$$

即 質點在 $t = 3$ 的速度為 7.01 並與正水平方向成 176.91°。

(c)

$$\frac{dx}{dt} = 3t^2 - 14t + 8 < 0$$

$$(3t-2)(t-4) < 0$$

$$\frac{2}{3} < t < 4$$

即 所求的時間區間為 $\left(\frac{2}{3}, 4\right)$。

(d)

$$\frac{d^2y}{dt^2} = -\frac{1}{2}(2t+1)^{-\frac{3}{2}}(2)$$

$$= -(2t+1)^{-\frac{3}{2}}$$

∴ 在 $t = 4$ 時的垂直加速度 $= -[2(4)+1]^{-\frac{3}{2}}$

$$= \frac{-1}{27}$$

B42 解

(a) 圓錐體的底半徑 $= 24 \sin \theta$

圓錐體的高 $= 24 \cos \theta$

$$\therefore \quad V = \frac{1}{3}\pi(24\sin\theta)^2(24\cos\theta)$$

$$= 4608\pi \sin^2\theta \cos\theta$$

(b) **(i)**

$$\frac{dV}{dt} = 4680\pi[\sin^2\theta(-\sin\theta) + (2\sin\theta\cos\theta)\cos\theta]\frac{d\theta}{dt}$$

$$= 4680\pi \sin\theta\,(-\sin^2\theta + 2\cos^2\theta)\frac{d\theta}{dt}$$

$$= 4680\pi \sin\theta\,(2 - 3\sin^2\theta)\frac{d\theta}{dt}$$

當 $\theta = \frac{\pi}{6}$ 及 $\frac{d\theta}{dt} = \frac{\pi}{30}$,

$$\frac{dV}{dt} = 4680\pi \sin\frac{\pi}{6}\left(2 - 3\sin^2\frac{\pi}{6}\right)\cdot\frac{\pi}{30}$$

$$= 97.5\pi^2$$

即　　圓錐體的體積遞增率為 $97.5\pi^2$ cm³/s。

(ii)　V 遞增當

$$\sin\theta(2-3\sin^2\theta) > 0 \quad (\because \quad 0<\theta<\frac{\pi}{2})$$
$$2-3\sin^2\theta > 0$$
$$\sin^2\theta < \frac{2}{3}$$
$$0 < \sin\theta < \sqrt{\frac{2}{3}}$$

即　$0 < \theta < \sin^{-1}\sqrt{\frac{2}{3}}$ 。

B44 解

(a)
$$f(x) = \frac{x^2+9}{2x}$$
$$= \frac{x}{2}+\frac{9}{2x}$$
$$f'(x) = \frac{1}{2}-\frac{9}{2x^2}$$
$$f''(x) = \frac{9}{x^3}$$

(b)
$$f'(x) = \frac{1}{2}-\frac{9}{2x^2} > 0$$
$$\frac{x^2-9}{2x^2} > 0$$
$$x^2-9 > 0$$
$$(x-3)(x+3) > 0$$

$\therefore$　$x < -3$　或　$x > 3$

即　當 $x<-3$ 或 $x>3$，$f(x)$ 遞增。

(c)　$f'(x)=0$ 當 $x=-3$ 或 $x=3$。

$$f''(-3)=\frac{9}{(-3)^3}=\frac{-1}{3}<0$$

$\therefore$　$f(x)$ 在 $x=-3$ 有極大值

$$f(x)\text{的極大值} = f(-3)=\frac{(-3)^2+9}{2(-3)}=-3$$

$$f''(3)=\frac{9}{3^3}=\frac{1}{3}>0$$

$\therefore$　$f(x)$ 在 $x=3$ 有極小值

$$f(x)\text{的極小值} = f(3)=\frac{3^2+9}{2(3)}=3$$

B45 解

(a)
$$y = 4\sin x + k\cos 2x$$
$$\frac{dy}{dx} = 4\cos x - 2k\sin 2x$$
$$= 4\cos x - 2k(2\sin x\cos x)$$
$$= 4\cos x(1-k\sin x)$$

由此，$\frac{dy}{dx} = 0$

當　$\cos x = 0$　或　$\sin x = \frac{1}{k}$

$x = \frac{\pi}{2}$　或　$x = \sin^{-1}\frac{1}{k}$

$\because$　函數在 $x=\frac{\pi}{6}$ 有駐點,

$\therefore \quad \frac{\pi}{6} = \sin^{-1}\frac{1}{k}$

$\frac{1}{k} = \frac{1}{2}$

$k = 2$

(b) $\frac{dy}{dx} = 4\cos x - 4\sin 2x$ (由 **(a)**)

$\frac{d^2y}{dx^2} = -4\sin x - 8\cos 2x$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}} = -4\sin\frac{\pi}{2} - 8\cos\left(2\cdot\frac{\pi}{2}\right)$$

$= 4$

> 0

$\therefore \quad \left(\frac{\pi}{2},\ 2\right)$ 是一極小點。

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{6}} = -4\sin\frac{\pi}{6} - 8\cos\left(2\cdot\frac{\pi}{6}\right)$$

$= -6$

< 0

$\therefore \quad \left(\frac{\pi}{6},\ 3\right)$ 是一極大點。

B46 解

(a) $f(x) = \frac{x}{x^2+4}$

$f(-x) = \frac{-x}{(-x)^2+4}$

$= -\frac{x}{x^2+4}$

$\therefore \quad f(x)$ 是奇函數。

(b) $f'(x) = \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2}$

$= \frac{4-x^2}{(x^2+4)^2}$

$= \frac{(2-x)(2+x)}{(x^2+4)^2}$

$\therefore \quad f'(x) = 0$ 當 $x = -2$ 或 2

由於 $f'(x) < 0$ 當 $x < -2$ 或 $x > 2$

及 $f'(x) > 0$ 當 $-2 < x < 2$

$\therefore \quad f(x)$ 在 $x = -2$ 時是極小而在 $x = 2$ 時是極大。

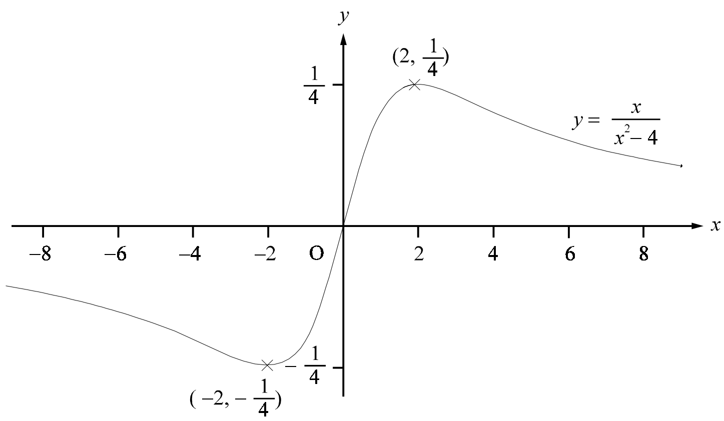
$f(-2) = \frac{-2}{(-2)^2+4} = -\frac{1}{4}$

$f(2) = \frac{2}{2^2+4} = \frac{1}{4}$

$\therefore \quad \left(-2,\ -\frac{1}{4}\right)$ 是一極小點。

$\left(2,\ \frac{1}{4}\right)$ 是一極大點。

(c) 下圖為 $y = f(x)$ 的圖像。



B47 解

(a) $6x + 4y = 1200$

$\therefore \quad y = 300 - \frac{3x}{2}$

(b)
$$\begin{aligned} A &= 3xy \\ &= 3x\left(300 - \frac{3x}{2}\right) \\ &= 900x - \frac{9x^2}{2} \end{aligned}$$

(c)
$$\frac{dA}{dx} = 900 - 9x$$
$$\frac{d^2A}{dx^2} = -9$$
$$\therefore \quad \frac{dA}{dx} = 900 - 9x = 0$$

當 $x = 100$,

$$\left.\frac{d^2A}{dx^2}\right|_{x=100} = -9 < 0$$

$\therefore$ A 為最大當 $x = 100$。

$$\text{A 的最大值} = 900(100) - \frac{9(100^2)}{2} = 45\,000$$

即 他能圍繞的最大面積為 $45\,000$ m²。

B48 解

(a) 圓錐體的高度 $= h = 3 + 3\csc\theta$

圓錐體的底半徑 $= h\tan\theta$
$= (3 + 3\csc\theta)\tan\theta$

$$\therefore \quad V = \frac{1}{3}\pi[(3 + 3\csc\theta)\tan\theta]^2(3 + 3\csc\theta)$$
$$= 9\pi(1 + \csc\theta)^3\tan^2\theta$$

(b)
$$\begin{aligned} \frac{dV}{d\theta} &= 9\pi[3(1 + \csc\theta)^2(-\csc\theta\cot\theta)\tan^2\theta + (1 + \csc\theta)^3(2\tan\theta \quad \sec^2\theta)] \\ &= 9\pi(1 + \csc\theta)^2\tan\theta[-3\csc\theta + 2(1 + \csc\theta)\sec^2\theta] \\ &= 9\pi(1 + \csc\theta)^2 \cdot \frac{\sin\theta}{\cos\theta}\left[\frac{-3}{\sin\theta} + 2\left(1 + \frac{1}{\sin\theta}\right)\cdot\frac{1}{\cos^2\theta}\right] \\ &= 9\pi(1 + \csc\theta)^2\left(\frac{-3\cos^2\theta + 2\sin\theta + 2}{\cos^3\theta}\right) \\ &= 9\pi(1 + \csc\theta)^2\left(\frac{3\sin^2\theta + 2\sin\theta - 1}{\cos^3\theta}\right) \\ &= 9\pi(1 + \csc\theta)^2 \cdot \frac{(3\sin\theta - 1)(\sin\theta + 1)}{\cos^3\theta} \end{aligned}$$

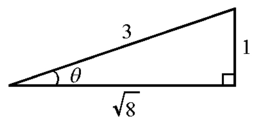
對於 $0 < \theta < \frac{\pi}{2}$，$(1 + \csc\theta)^2$、$(\sin\theta + 1)$ 及 $\cos^3\theta$ 均為正值。

$\therefore \quad \frac{dV}{d\theta} \le 0$ 當 $3\sin\theta - 1 \ge 0$

即 $\theta \ge \sin^{-1}\left(\frac{1}{3}\right)$

由此，當 $0 < \theta \le \sin^{-1}\left(\frac{1}{3}\right)$，V 遞減。

(c)



$\frac{dV}{d\theta} \le 0$ 當 $3\sin\theta - 1 \ge 0$

即 $\theta \le \sin^{-1}\left(\frac{1}{3}\right)$

$\therefore$ 當 $\sin^{-1}\left(\frac{1}{3}\right) \le \theta < \frac{\pi}{2}$，V 遞增。

因此，當 $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ 時，V 是最小。

當 $\theta = \sin^{-1}\left(\frac{1}{3}\right)$，$\csc\theta = 3$ 及 $\tan\theta = \frac{1}{\sqrt{8}}$。

$$\therefore \quad \text{圓錐體的最小體積} = 9\pi(1+3)^3\left(\frac{1}{\sqrt{8}}\right)^2$$
$$= 72\pi \text{ cm}^3$$

注意：圓錐體的最小體積是球體體積的兩倍。

B49 解

(a) 底半徑 $= 6\sin\theta$

高 $= 6\cos\theta$

$$\therefore \quad V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(6\sin\theta)^2(6\cos\theta)$$
$$= 72\pi\sin^2\theta\cos\theta$$

(b)
$$\frac{dV}{d\theta} = 72\pi(2\sin\theta\cos^2\theta - \sin^3\theta)$$
$$= 72\pi\sin\theta(2\cos^2\theta - \sin^2\theta)$$

$\therefore \quad \frac{dV}{d\theta} = 0$ 當 $\theta = 0$ 或 $\tan\theta = \sqrt{2}$

$\therefore$ 當 $\theta = \tan^{-1}\sqrt{2}$ 時，V 是最大。

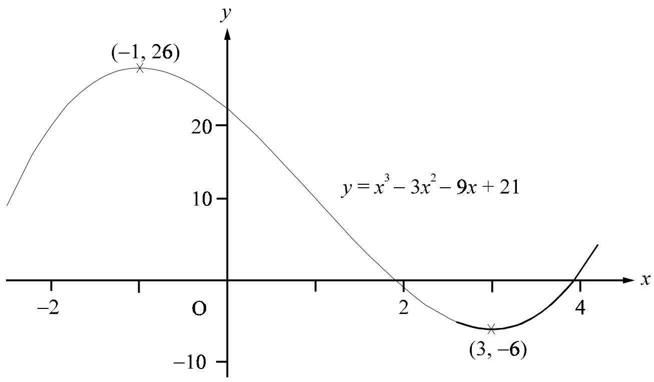
$$\text{最大的體積} = 72\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 \cdot \left(\frac{1}{\sqrt{3}}\right)$$
$$= 16\sqrt{3}\pi$$

B50 解

(a)
$$f(x) = x^3 + ax^2 + bx + c$$
$$f'(x) = 3x^2 + 2ax + b$$
$$f'(-1) = 3 - 2a + b = 0 \ldots\ldots\ldots\ldots (1)$$
$$f'(3) = 27 + 6a + b = 0 \ldots\ldots\ldots\ldots (2)$$
$$f(3) = 27 + 9a + 3b + c = -6 \ldots\ldots\ldots (3)$$

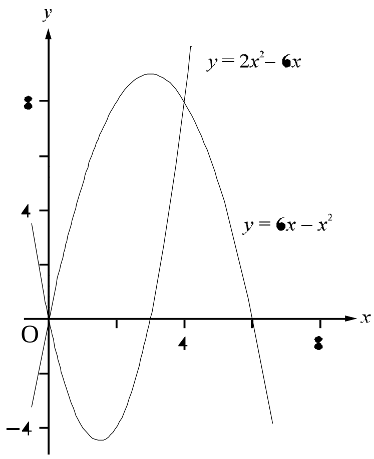
解 (1)、(2) 及 (3)，得 $a = -3$，$b = -9$，$c = 21$。

(b)



B51 解

(a)



(b)

$y = 6x - x^2$ .... (1)

$y = 2x^2 - 6x$ .... (2)

把 (1) 代入(2),

$$6x - x^2 = 2x^2 - 6x$$

$$3x^2 - 12x = 0$$

$$3x(x - 4) = 0$$

$x = 0$ 或 4. .... (3)

把 (3) 代入(1),

∴ 交點為 (0, 0) 及 (4, 8) 。

對於 (1), $\frac{dy}{dx} = 6 - 2x$

∴ $\left.\frac{dy}{dx}\right|_{x=0} = 6$ 及 $\left.\frac{dy}{dx}\right|_{x=4} = -2$

對於 (2), $\frac{dy}{dx} = 4x - 6$

∴ $\left.\frac{dy}{dx}\right|_{x=0} = -6$ 及 $\left.\frac{dy}{dx}\right|_{x=4} = 10$

在 (0, 0) 的交角 $= \tan^{-1}\left|\frac{6-(-6)}{1+6(-6)}\right|$

$= \tan^{-1}\frac{12}{35}$

$= 18.9°$

在 (4, 8) 的交角 $= \tan^{-1}\left|\frac{-2-10}{1+(-2)(10)}\right|$

$= \tan^{-1}\frac{12}{19}$

$= 32.3°$